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Fredholm Alternatives for Nonlinear A-Proper Mappings with Applications to Nonlinear Elliptic Boundary Value Problems*

W. V. PETRYSHYN†

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

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Let X and Y be real Banach spaces with a projectionally complete scheme $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ and let $T: X \to Y$ be an asymptotically linear mapping which is A-proper with respect to Γ and whose asymptotic derivative $T_\infty \in L(X, Y)$ is also A-proper with respect to Γ. Necessary and sufficient conditions are given in order that the equation $T(x) = f$ be solvable for a given f in Y. Under certain additional conditions it is shown that solutions can be constructed as strong limits of finite dimensional Galerkin type approximates $x_n \in X_n$. Theorems 1 and 2 include as special cases the recent results of Kachurovskii, Hess, Nečas, and the author. The abstract results for A-proper mappings are then applied to the (constructive) solvability of boundary value problems for quasilinear elliptic equations of order $2m$ with asymptotically linear terms of order $2m - 1$.

INTRODUCTION

Let (X, Y) be a pair of real Banach spaces with a projectionally complete system $\Gamma = \{X_n, Y_n, P_n, Q_n\}$, X^* and Y^* the adjoints of X and Y, respectively, and (u, x) the value of $u \in X^*$ at $x \in X$. For a bounded linear operator $T: X \to Y$ let $N(T) \subset X$ and $R(T) \subset Y$ denote the null space and the range of T, respectively, and let $T^*: Y^* \to X^*$ denote the adjoint of T. We introduce the usual notation:

$$N(T)^\perp = \{u \in X^* \mid (u, x) = 0 \text{ for } x \in N(T)\},$$
$$N(T^*)^\perp = \{y \in Y \mid (w, y) = 0 \text{ for } w \in N(T^*).$$

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If $C: X \to X$ is a compact linear operator, then the classical Fredholm alternative asserts that

$$\dim N(I - C) = \dim N(I - C^*) < \infty$$

and the equation $x - Cx = f$ *is solvable if and only if* $f \in N(I - C^*)^\perp$. In [20, 21] the writer obtained a constructuve generalization of the above alternative which we state here in the following form to be used below.

THEOREM A. (a) *Suppose* $T: X \to Y$ *is bounded, linear, and A-proper with respect to* Γ. *Then either the equation* $Tx = y$ *is uniquely approximation-solvable for each* y *in* Y, *or* $N(T) \neq \{0\}$. *In the latter case,* T *is Fredholm of index* $i(T) \geqslant 0$ *and* $Tx = y$ *is solvable if and only if* $y \in N(T^*)^\perp$.

(b) *If one assumes additionally that the adjoint system* Γ^* *is projectionally complete for the pair* (Y^*, X^*), *then* $i(T) = 0$ *if and only if* T^* *is A-proper with respect to* Γ^*.[1]

In [10] Kachurovskii extended (without proof) the classical Fredholm alternative to nonlinear equations in X of the form $x - Fx = y$, where F is compact and asymptotically linear, and gave applications to nonlinear integral and ordinary differential equations. In [25] the writer extended the results of [10] to the case when F is asymptotically linear and k-ball-contractive with $k < 1$, and used it to obtain the existence of classical solutions for nonlinear elliptic boundary-value problems. An alternative, which is analogous to [10], has been established by Hess [9] for the case when X is reflexive and $T: X \to X^*$ is a bounded, demicontinuous, and asymptotically linear map of type (S) with an asymptotic derivative which is also of type (S). In [16, 17] Nečas studied the surjectivity of the map $T: X \to X^*$ of type (S) and of the form $T = A + N$, where N is asymptotically zero and A is of type (S) and positively homogeneous with either A or T odd. Applications to the existence of weak solutions for Dirichlet elliptic boundary-value problems has been given in [9, 16, 17].

The purpose of this paper is twofold. In Section 1, we extend Theorem A to equations $T(x) = y$ involving nonlinear A-proper mappings $T: X \to Y$ which are either asymptotically linear (Theor. 1) or of the form $T = A + N$ with A an A-proper map and N asymptotically zero (Theor. 2). Since the above class of nonlinear A-proper

[1] In its present form Theorem A and other perturbation results were presented by the author at the Scientific Congress in 1973 in New York commemorating the one-hundredth Anniversary of the Shevchenko Scientific Society.

mappings includes the mappings studied by the abovementioned authors, the alternatives in [9, 10, 25] are deduced here as corollaries of our Theorem 1 while those in [16, 17] as corollaries of Theorem 2. We add, that unlike the results in [9, 10, 25, 16, 17], our Theorems 1 and 2 are essentially constructive. Indeed, if T is injective, then Theorems 1 and 2 assert that for each y in Y the solution $x \in X$ of equation $Tx = y$ is obtained as the strong limit of solutions $x_n \in X_n$ of finite-dimensional equations $Q_n T(x_n) = Q_n(y)$, i.e., the Galerkin type method converges. In case one is only interested in the existence of solutions, then Theorems 1 and 2 admit generalizations to pseudo-A-proper mappings (Theorem 2′).

In Section 2 we apply the results of Section 1 to the approximation-solvability and/or solvability of Dirichlet boundary-value problems for nonlinear elliptic equations involving differential operators T of the form $T = A + N$, where A is linear or nonlinear elliptic operator and N is nonlinear and asymptotically linear. Such elliptic operators have been recently treated either by the application of the theory of the operators of monotone type [9, 11, 16, 17] or the theory of k-ball-contractions [25, 8].

We add in passing that in recent years the abstract theory of operators of monotone type has been thoroughly developed and widely applied by Browder, Brezis, Dubinskii, Hess, Leray, Lions, Nečas, Pohodjayev, and many others (see [1, 2, 9, 14, 27] for further references) to obtain existence results for various classes of nonlinear elliptic and parabolic boundary-value problems. Although the abstract theory of A-proper mappings has also been developed recently to a considerable degree (see [23] for the survey of these results) the applicative aspect of this theory is only beginning to attract attention of the researchers working in the field of differential equations and other applications. In Section 2 of this paper we indicate how the theory of A-proper mappings can be used to obtain approximation-solvability and/or solvability results for partial differential equations.

1. Fredholm Alternative

In what follows we assume that (X, Y) is a pair of real separable Banach spaces with a projectionally complete scheme

$$\Gamma = \{X_n, Y_n, P_n, Q_n\},$$

where $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences of monotonically increasing finite dimensional subspaces with $\dim X_n = \dim Y_n$ and $\{P_n\}$ and

$\{Q_n\}$ are linear projections with $P_n(X) = X_n$ and $Q_n(Y) = Y_n$ such that $P_n x \to x$ and $Q_n y \to y$ as $n \to \infty$ for each x in X and y in Y. It is obvious that (X, Y) has a projectionally complete scheme Γ if X and Y have Schauder bases. We use the symbols "$\to$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively.

In trying to characterize the class of not necessarily linear mappings $T: X \to Y$ for which a solution $x \in X$ of the equation

$$T(x) = y, \qquad y \in Y, \tag{1}$$

can be obtained as a strong limit of solutions $x_n \in X_n$ of finite-dimensional Galerkin type approximate equations

$$T_n(x_n) = Q_n(y), \qquad T_n = Q_n T\,|_{X_n}, \tag{2}$$

the writer has been led to the notion of an A-proper mapping (i.e., mapping satisfying condition H [19, 22]) which is defined as follows.

DEFINITION 1. $T : \to Y$ is *Approximation-proper* (A-proper) with respect to Γ if $T_n: X_n \to Y_n$ is continuous for each n and if

$$\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$$

is any bounded sequence such that $T_{n_j}(x_{n_j}) \to g$ as $j \to \infty$ for some g in Y, then there exists an $x \in X$ such that

(i) $T(x) = g$,

(ii) $x \in \mathrm{cl}\{x_{n_j}\}$, where $\mathrm{cl}\{x_{n_j}\}$ denotes the closure of $\{x_{n_j}\}$ in X.

It turned out that in addition to the mappings $T: X \to X$ of the form $T = I - S - C$, where C is a not necessarily linear compact mapping and S is Lipschitzian with constant $l \in [0, 1)$, the class of A-proper mapping includes also the following general classes of mappings whose finitions will be given below.

(a) Bounded demicontinuous mappings T of a reflexive space X into X^* which are of type (S) and, in particular, of strongly montone type (see [2, 22]).

(b) P_1-compact mappings $T: X \to X$ and, in particular, mappings of the form $T = I - F$, where F is k-ball-contractive with $k < 1$ (see [23, 30]).

For other examples of A-proper mappings see [23, 5]. To state our results precisely and to indicate the intimate relationship of A-properness of T to the constructive solvability of Eq. (1) via the Galerkin type method we will need the following definition.

DEFINITION 2. (a) Equation (1) is *uniquely approximation-solvable* with respect to Γ if Eq. (2) has a unique solution $x_n \in X_n$ for each large n such that $x_n \to x$ and x is the unique solution of Eq. (1).

(b) Equation (1) is *strongly* (respectively *feebly*) *approximation-solvable* with respect to Γ if Eq. (2) has a solution $x_n \in X_n$ for each large n such that $x_n \to x$ (respectively, $x_{n_j} \to x$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}$) and $T(x) = y$.

We call the reader's attention to the difference between the solvability (i.e., the existence of solutions) of Eq. (1) and the approximation-solvability of Eq. (1) (i.e., the constructive solvability of (Eq. (1)).

Following Krasnoselskii [12] we call a nonlinear map $T: X \to Y$ *asymptotically linear* if there exists a bounded linear map (i.e., $T_\infty \in L(X, Y)$), called the *asymptotic derivative* of T, such that for all x in X

$$T(x) = T_\infty(x) + N(x) \qquad \text{with} \quad \| N(x)\|/\| x \| \to 0 \quad \text{as} \quad \| x \| \to \infty. \quad (3)$$

It was shown in [12] that if T is asymptotically linear and compact, then T_∞ is also compact and this fact was then used in the study of the existence of nontrivial solutions, bifurcation theory, and in other applications. Kachurovskii also utilized this fact in [10].

Our first basic result in this paper is the following alternative extending Theorem A to nonlinear maps, which in a sense provides a constructive generalization and the unification of the results in [9, 10, 25].

THEOREM 1. *Let $T: X \to Y$ be an asymptotically linear A-proper mapping with an A-proper asymptotic derivative $T_\infty \in L(X, Y)$. Then either Eq.* (1) *is feebly approximation-solvable for each y in Y* (*and strongly approximation-solvable if T is also one-to-one*), *or $N(T_\infty) \neq \{0\}$. In the latter case, assuming additionally that* $\dim N(T_\infty) = \operatorname{codim} R(T_\infty)$ *and that*

$$R(N) \subset N(T_\infty{}^*)^\perp \, (= R(T_\infty)), \quad (4)$$

or equivalently that

$$R(T) \subset R(T_\infty), \quad (4')$$

Eq. (1) *is solvable if and only if $y \in N(T_\infty{}^*)^\perp$ $(=R(T_\infty))$; in case of solvability, a solution x of Eq.* (1) *is obtained as a limit point of a constructable sequence $\{x_n \mid x_n \in X_n\}$.*

Proof. The proof of Theorem 1 is based upon Theorem A, the finite dimensional Brouwer degree theory, and the following proposition which includes Proproposition 1 in [9].

PROPOSITION 1. *Let $T: X \to Y$ be A-proper with the A-proper asymptotic derivative T_∞. If $N(T_\infty) = \{0\}$, then Eq.* (1) *is feebly approximation-solvable for each y in Y. If we additionally assume that T is-one-to-one, then Eq.* (1) *is strongly approximation-solvable for each y in Y.*

Since Proposition 1 will be deduced as a special case of Theorem 2 below we omit its proof.

To continue with the proof of Theorem 1, we note that, in view of Proposition 1, we need only consider the case when $N(T_\infty) \neq \{0\}$. It follows from Theorem A that $\alpha = \dim N(T_\infty) < \infty$, $R(T_\infty)$ is closed, $\beta = \operatorname{codim} R(T_\infty) < \infty$ with $\alpha \geqslant \beta$ and, by our hypothesis, $\alpha = \beta$. Hence there exists a closed subspace X_1 of X and a subspace N_1 of Y with $\dim N_1 = \alpha$ such that $X = N(T_\infty) \oplus X_1$, $T_\infty(X_1) = R(T_\infty)$, T_∞ is injective on X_1, and $Y = Y_1 \oplus N_1$ with $Y_1 = R(T_\infty)$. Let Π be an isomorphism of $N(T_\infty)$ onto N_1, let P be a bounded linear projection of X onto $N(T_\infty)$, and let U be a linear mapping of X into Y defined by $U = T_\infty + C$, where $C = \Pi P$. Since C is compact, U is A-proper. Moreover, U is one-to-one. Indeed, if

$$U(x) = T_\infty(x) + C(x) = 0,$$

then $T_\infty(x) = -C(x)$ and since $R(T_\infty) \cap N_1 = \{0\}$ it follows that $T_\infty(x) = 0$ and $C(x) = 0$. Hence $\Pi P(x) = 0$, i.e., $Px = 0$ since Π is one-to-one. Consequently, $x \in X_1$ with $T_\infty(x) = 0$. Thus $x = 0$ since T_∞ is one-to-one on X_1, i.e., U is injective and, in fact, bijective by Theorem A.

It is now easy to see that, under the additional condition (4), the Eq. (1) is solvable if and only if $y \in N(T_\infty{}^*)^\perp$. Indeed, suppose first that $y \in N(T_\infty{}^*)^\perp$, i.e., $y \in R(T_\infty)$ since $R(T_\infty) = N(T_\infty{}^*)^\perp$ for an A-proper mapping T_∞. Let $U = T_\infty + C$ with $C = \Pi P$ and note U is a one-to-one bounded linear A-proper mapping which is the asymptotic derivative of the A-proper mapping $T_1 = U + N$. Hence, by Proposition 1, the equation $T_1(x) = y$ is feebly approximation solvable, i.e., we can construct a sequence $\{x_n \mid x_n \in X_n\}$ and $\bar{x} \in \operatorname{cl}\{x_n\}$ such that $T_1(\bar{x}) = T_\infty(\bar{x}) + C(\bar{x}) + N(\bar{x}) = y$. Since $y \in R(T_\infty)$ and $N(\bar{x}) \in R(T_\infty)$ by hypothesis (4), it follows that

$$C(\bar{x}) = y - T_\infty(\bar{x}) - N(\bar{x}) \in R(T_\infty).$$

But $C(\bar{x})$ also lies in N_1 and $R(T_\infty) \cap N_1 = \{0\}$; therefore, $C(\bar{x}) = 0$ and thus $T_\infty(\bar{x}) + N(\bar{x}) = y$, i.e., Eq. (1) is solvable with $\bar{x}$ obtained as a limit point of a constructable sequence $\{x_n \mid x_n \in X_n\}$.

To prove the converse, suppose that y in Y is such that $T(x) = y$ for some x in X. Since, by (4), $T(x) - T_\infty(x) \in N(T_\infty{}^*)^\perp = R(T_\infty)$, it follows that $y = (T(x) - T_\infty(x)) + T_\infty(x) \in R(T_\infty) = N(T_\infty{}^*)^\perp$. It is obvious that (4) is equivalent to (4'). Q.E.D.

Remark 1. In connection with the second part of Theorem 1, it should be noted that when $N(T_\infty)$ and $N(T_\infty{}^*)$ are known and $\dim N(T_\infty) = \dim N(T_\infty{}^*)$, then one can construct an operator C so that a solution x_n of the equation $Q_n T_1(x_n) = Q_n y$ can be constructed by the first part of Theorem 1 and thus a solution of Eq. (1) can be obtained as a strong limit point of the known sequence $\{x_n \mid x_n \in X_n\}$ even when $N(T_\infty) \neq \{0\}$.

Remark 2. To solve Eq. (1) constructively for a given y in $R(T_\infty) = Y_1$ when $N(T_\infty) \neq \{0\}$ and condition (4) holds, it is probably worth noting that for certain specific pairs (X, Y) and/or A-proper maps $T: X \to Y$ one may be able to construct a projectionally complete scheme Γ_1 for the pair (X_1, Y_1), verify the A-properness of T: $X_1 \to Y_1$ with respect to Γ_1, and apply the approximation-solvability results to Eq. (1) in (X_1, Y_1) in which the linear part is one-to-one. This approach, for example, always works when $T: X \to X^*$ is of the form $T = T_\infty + N$ with N compact and $T_\infty = L + C$, where $(Lx, x) \geqslant c \| x \|^2 \, \forall x \in X$ and C linear and compact.

Remark 3. If we are only interested in the existence of solutions of Eq. (1), then under somewhat stronger assumption on T, namely that T be of the form $T = T_\infty + N$ with T_∞ A-proper and N compact, it can be shown by different arguments that the second assertion of Theorem 1 in the form "Eq. (1) is solvable if and only if $y \in N(T_\infty{}^*)^\perp$" is valid without the assumption that $\alpha = \beta$. This will be deduced from a more general result which will be published elsewhere.

Special Cases

We now deduce the Fredholm Alternatives obtained in [9, 10, 25] as corollaries of our Theorem 1 under the assumption that X has a projectionally complete scheme $\Gamma_0 = \{X_n, P_n\}$. In this case $Y_n = X_n$, $Q_n = P_n$ and $T_n = P_n T|_{X_n}$.

(a) *k-ball-contraction with* $k < 1$. In the discussion of (a), we assume that Γ_0 is such that $\| P_n \| = 1$ for all n. We begin with some definitions. Following [13, 6], for any bounded set $D \subset X$, we define

$\chi(D)$, the ball-measure of noncompactness of D, to be $\inf\{r > 0 \mid D$ can be covered by a finite number of balls with centers in X and radius $r\}$. The above terminology is justified by the fact that $\chi(D) = 0$ if $\bar{D}$ is compact. For the various properties of χ and the companion notion of a set-measure of noncompactness $\gamma(D)$ see [28, 18, 26]. Closely associated with χ is the notion of a "k-ball-contraction" defined to be a continuous bounded mapping $F: X \to X$ such that $\chi(F(D) \leqslant k\chi(D))$ for any bounded set D in X and some constant $k \geqslant 0$. It follows immediately that $C: X \to X$ is compact iff C is 0-ball-contractive. If $S: X \to X$ is such that $\| Sx - Sy \| \leqslant l \| x - y \|$ for all x and y in X, then S is l-ball contractive and $F = S + C$ is also l-ball contractive. For some more complicated examples of k-ball contractions see [28, 26].

As a first corollary of Theorem 1 we deduce the following feebly constructive version of the alternative in [25].

COROLLARY 1. *Let $F: X \to X$ be k-ball-contractive with $k \in [0, 1)$ and with $F_\infty \in L(X)$ as its asymptotic derivative. Then either the equation*

$$x - F(x) = y, \qquad y \in X, \tag{5}$$

is feebly approximation-solvable for each y in X (and strongly approximation-solvable if $T = I - F$ is also qne-to-one), or $N(I - F_\infty) \neq \{0\}$. In the latter case, assuming additionally that

$$F(x) - F_\infty(x) \in N((I - F_\infty)^*)^\perp \qquad \text{for all} \quad x \in X, \tag{6}$$

Eq. (5) is solvable if and only if $y \in N((I - F_\infty)^*)^\perp$.

Proof. Since $F: X \to X$ is k-ball contractive with $k < 1$ and $\| P_n \| = 1$ for all n, the result of Nussbaum and Webb implies that $T = I - F$ is A-proper with respect to Γ_0. Since, by Proposition 1 in [25], F_∞ is also k-ball-contractive, the mapping $T_\infty \equiv I - F_\infty$, which is the asymptotic derivative of T, is also A-proper with respect to Γ_0. Moreover, the results in [6, 18] imply that T_∞ is Fredholm of index zero and, in particular, $\alpha(T_\infty) = \beta(T_\infty)$. Finally, (6) implies that $T(x) - T_\infty(x) \in N(T_\infty^*)^\perp$ for $x \in X$, i.e., (4) also holds. In view of the above discussion, the validity of Corollary 1 follows from Theorem 1. Q.E.D.

We add in passing that since every compact map $C: X \to X$ is k-ball contractive for $k = 0$, the alternative in [10] also follows from Theorem 1 since it is a special case of Corollary 1 for $k = 0$.

(b) *Mappings of type* (S). Let X be a separable reflexive Banach space. Following Browder [2] we say that a map $T: X \to X^*$ is of type (S) is for any sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x$ in X and $(Tx_n - Tx, x_n - x) \to 0$ as $n \to \infty$ it follows that $x_n \to x$ as $n \to \infty$. It was shown in [2] that if $T: X \to X^*$ is a bounded, continuous, and of type (S) mapping, then T is A-proper with respect to an injective scheme. Similar results hold for projective schemes with continuity replaced by demicontinuity. Recall that T is said to be demicontinuous if T is continuous from the strong topology in X to the weak topology in X^*. It is known [1, 9, 16] that the notion of a mapping of type (S) is very useful in obtaining the existence of weak solutions for elliptic boundary value problems.

As another corollary of Theorem 1 we deduce the feebly constructive version of the alternative in [9].

COROLLARY 2. *Suppose X is reflexive and $\Gamma_2 = \{X_n, Y_n, P_n, Q_n\}$ is a projectionally complete system for (X, X^*), where $Y_n = Q_n(X^*)$ and $Q_n = P_n^*$ for each n. Suppose $T: X \to X^*$ is an asymptotically linear, bounded, demicontinuous, and of type (S) map with the asymptotic derivative $T_\infty \in L(X, X^*)$ which is also of type (S). Then either Eq. (1) is feebly approximation-solvable for each y in X^* (and strongly approximation-solvable if T is also injective), or $N(T_\infty) \neq \{0\}$. In the latter case, assuming additionally that $T(x) - T_\infty(x) \in N(T_\infty^*)^\perp$ for $x \in X$, Eq. (1) is solvable if and only if $y \in N(T_\infty^*)^\perp$.*

Proof. It is easy to see that T is A-proper with respect to Γ_2. Indeed, $T_n = P_n^*T|_{X_n}$ is continuous for each n since T is demicontinuous and the strong and weak convergence coincide in finite dimensional spaces; moreover, if $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ is any bounded sequence such that $P_{n_j}T(x_{n_j}) \to g$ for some g in X^*, then assuming without loss of generality that $x_{n_j} \rightharpoonup x_0$ in X we see that

$$(Tx_{n_j} - Tx_0, x_{n_j} - x_0) = (P_{n_j}^*Tx_{n_j} - P_{n_j}^*Tx_0, x_{n_j} - P_{n_j}x_0) + (Tx_{n_j} - Tx_0, P_{n_j}x_0 - x_0) \to 0,$$

since $P_{n_j}^*T(x_{n_j}) - P_{n_j}^*Tx_0 \to g - Tx_0$, $x_{n_j} - P_{n_j}x_0 \rightharpoonup 0$, and

$$(Tx_{n_j} - Tx_0, P_{n_j}x_0 - x_0) \to 0$$

by the boundedness of $\{Tx_{n_j}\}$. Thus, since T is of type (S), $x_{n_j} \to x_0$ as $j \to \infty$. Moreover, since $P_{n_j}^*f \to f$ as $j \to \infty$ for each f in X^* and $Tx_{n_j} \rightharpoonup Tx_0$ in X^* we see that for each f in X^* we have

$$(g, f) = \lim_j (P_{n_j}Tx_{n_j}, f) = \lim_j (Tx_{n_j}, P_{n_j}^*f) = (Tx_0, f).$$

Hence $Tx_0 = g$, i.e., T is A-proper with respect to Γ_2. By the same argument one shows that T_∞ is also A-proper.

Now, since X is reflexive, $T_\infty: X \to X^*$ is of type (S) if and only if $T_\infty^*: X \to X^*$ is of type (S). Hence T_∞^* is also A-proper with respect to Γ_2 and, therefore, by the second part of Theorem A, T_∞ is Fredholm of index zero, i.e., $R(T_\infty)$ is closed and dimensions of $N(T_\infty)$ and $B(T_\infty^*)$ are finite and equal. Consequently, Corollary 2 follows from Theorem 1.

Remark 4. Since, for a reflexive X, $L \in L(X, X^*)$ is of type (S) if and only if L^* is of type (S), Theorem 1 in Hess [9], which asserts that L is Fredholm of index zero, is a special case of the writer's Theorem 5 in [20] (i.e., Theorem A above) at least when X has a Schauder basis since in that case there is a natural scheme Γ_2 which is projectionally complete for the pair (X, X^*).

Remark 5. Concerning the second part of Theorem A, it should be added that the adjoint scheme $\Gamma^* = \{Q_n^*(Y^*), P_n^*(X^*), Q_n^*, P_n^*\}$ is certainly projectionally complete for the pair (Y^*, X^*) if X and Y have shrinking Schauder bases (and, in particular, if X and Y are reflexive and have Schauder bases), where the latter is defined to be a Schauder basis, say $\{\phi_i\} \subset X$, such that for each f in X^* and any $c > 0$ one has the relation $\alpha_n(f) = \sup\{|(x - P_n x, f)| \mid \|x\| \leqslant c\} \to 0$ as $n \to \infty$. Thus, for example, the space c_0 is not reflexive since $c_0^* = l_1$ but c_0 has a shrinking basis and so the pair $\{c_0, c_0^*\}$ has a projectionally complete scheme.

A-Properness in Terms of General Approximation Schemes

To complete the proof of Theorem 1 we must establish the validity of Proposition 1. This we do by proving Theorem 2 below for A-proper mappings defined in terms of general approximation schemes used in [3, 22] and then show that it includes not only Proposition 1 but also unites and extends the two alternatives of Nečas [16, 17]. Theorem 2 also establishes the convergence of the Galerkin type methods for the class of nonlinear equations treated here.

Definition 3. Let $\{E_n\}$ and $\{F_n\}$ be two sequences of oriented finite dimensional spaces with $\dim E_n = \dim F_n$ and let $\{V_n\}$ and $\{W_n\}$ be two sequences of continuous linear mappings with V_n mapping E_n into X and W_n mapping Y onto F_n such that $\{V_n\}$ and $\{W_n\}$ are uniformly bounded, $\operatorname{dist}(x, V_n E_n) \to 0$ as $n \to \infty$ for each x in X and for any given $r > 0$ the set

$$V_n^{-1}(B(0, r)) = \{x \in E_n \mid V_n x \in B(0, r)\}$$

is bounded for each n. We call $\Gamma_3 = \{E_n, F_n, V_n, W_n\}$ an *admissible* approximation scheme for mappings $T: X \to Y$.

For the sake of simplicity we use $\|\ \|$ to denote all norms $\|\ \|_X$, $\|\ \|_Y$, $\|\ \|_{E_n}$, $\|\ \|_{F_n}$ in respective spaces X, Y, E_n and F_n. Note that Definition 3 does not require that E_n and F_n be subspaces of X and Y, respectively, nor that V_n and W_n be linear projections. Consequently, in addition to projective scheme considered above, the schemes Γ_3 given by Definition 3 include injective schemes as well as abstract finite difference schemes (see [3, 22]). The following examples of Γ_3 will illustrate the difference even when $\{E_n\}$ are subspaces of X.

Let $\{X_n\}$ be a monotonically increasing sequence of oriented finite-dimensional subspaces of X such that $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each x in X and let V_n be a linear injection of X_n into X for each n.

(i) If $Y = X$, $Y_n = X_n$ and $W_n = P_n$, where $P_n: X \to X_n$ is a linear projection such that $P_n x \to x$ for each x in X, then $\Gamma_p \equiv \{X_n, Y_n, V_n, W_n\}$ is an admissible projection scheme for $T: X \to X$.

(ii) If $Y = X^*$, $Y_n = P_n^*(X^*)$ and $W_n = P_n^*$, then $\Gamma_p^* = \{X_n, Y_n, V_n, W_n\}$ is an admissible projection scheme for $T: X \to X^*$.

(iii) If $Y = X^*$, $Y_n = X_n^*$, and $W_n = V_n^*$, then $\Gamma_I = \{X_n, Y_n, V_n, W_n\}$ is an admissible injection scheme for $T: X \to X^*$.

We add in passing that the scheme (iii) is particularly useful (see e.g. [14]) in the approximate solvability of differential equations. Example (ii) shows that a projectional scheme could be admissible for $T: X \to X^*$ without being projectionally complete for the pair (X, X^*).

DEFINITION 4. $T: X \to Y$ is said to be A-proper with respect to Γ_3 if $T_n \equiv W_n T V_n: E_n \to F_n$ is continuous and if for any $\{x_{n_j} \mid x_{n_j} \in E_{n_j}\}$ such that $\{V_{n_j} x_{n_j}\}$ is bounded in X and

$$\| T_{n_j}(x_{n_j}) - W_{n_j} y \| \to 0$$

as $j \to \infty$ for some y in Y, there exists $\{x_{n_{j(k)}}\}$ and x in X such that $V_{n_{j(k)}} x_{n_{j(k)}} \to x$ and $T(x) = y$.

It is known that most (but not all) of the results obtained for A-proper maps given by Definition 1 carry over to maps given by Definition 4 (see also a series of papers by Grigorieff who studies A-proper maps in terms of general discrete schemes introduced by

Stummel (e.g. [7, 29])). In this paper we restrict ourselves to the following.

THEOREM 2. *Let* (X, Y) *be a pair of Banach spaces and let* T: $X \to Y$ *be A-proper with respect to an admissible scheme* Γ_3 *given by Definition* 3. *Assume that* $T = A + N$, *where* A *is A-proper with respect to* Γ_3 *and positively homogeneous of order* $\alpha > 0$ (*i.e.*, $A(tx) = t^\alpha A(x)$ *for all* x *in* X *and* $t > 0$) *and* N *is such that*

$$\| N(x)\|/\| x \|^\alpha \to 0 \qquad as \quad \| x \| \to \infty. \tag{7}$$

Suppose also that either A *or* T *is odd and that* $x = 0$ *whenever* $Ax = 0$.
Then, in either case, the equation

$$Ax + Nx = y \tag{8}$$

is feebly approximation-solvable with respect to Γ_3 *for each* y *in* Y. *If* T *is also one-to-one, then Eq.* (8) *is strongly approximation-solvable for each* y *in* Y (*i.e., the Galerkin type method when applied to Eq.* (8) *is convergent*).

Proof. Note first that for each n the set $B_r^n \equiv V_n^{-1}(B(0, r))$ is an open bounded set in E_n, $B_n^r \cap \partial B_r^n = \varnothing$ and $V_n^{-1}(\bar{B}(0, r))$ is a closed subset of E_n containing B_r^n. Hence $\mathrm{cl}(B_r^n) \subset V_n^{-1}(\bar{B}(0, r))$ and $\partial B_r^n \subset V_n^{-1}(\partial \bar{B}(0, r))$ for each n.

Suppose first that A is odd, i.e., $A(-x) = -A(x)$ for $x \in X$. Our conditions on A and N imply that for each given y in Y there exist a real number $R_y > 0$ and an integer $N_y \geqslant 1$ such that for all $t \in [0, 1]$ and $n \geqslant N_y$ we have

$$W_n A V_n(x) + (1 - t)\, W_n(NV_n(x) - y) \neq 0 \qquad \text{for all} \quad x \in \partial B_{R_y}^n. \tag{9}$$

Indeed, if (9) were not true for some y in Y, then there would exist sequences $\{n_j\}$, $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ and $\{t_j\} \subset [0, 1]$ such that $n_j \to \infty$, $\| V_{n_j} x_{n_j} \| \to \infty$ as $j \to \infty$ and

$$W_{n_j} A V_{n_j} x_{n_j} + (1 - t_j)\, W_{n_j}(NV_{n_j} x_{n_j} - y) = 0$$

for each j. Since $\{W_n\}$ are linear and uniformly bounded, A is positively homogeneous of order $\alpha > 0$, and $\| V_{n_j} x_{n_j} \| \to \infty$ as $j \to \infty$, it follows from the last equality and condition (7) that

$$W_{n_j} A V_{n_j}(z_{n_j}) = (t_j - 1)\, W_{n_j}(NV_{n_j} x_{n_j} - y)/\| V_{n_j} x_{n_j} \|^\alpha \to 0 \qquad \text{as} \quad j \to \infty,$$

where $z_{n_j} = \| V_{n_j} x_{n_j} \|^{-1} x_{n_j}$. Since $\{V_{n_j} z_{n_j}\} \subset X$ is bounded and

$W_{n_j}AV_{n_j}(z_{n_j}) \to 0$ as $j \to \infty$, the A-properness of A with respect to Γ_3 implies the existence of a subsequence $\{z_{n_{j(k)}}\}$ and a z in X such that $V_{n_{j(k)}}z_{n_{j(k)}} \to z$ in X and $Az = 0$ with $\| z \| = 1$, in contradiction to our condition on A. Thus, for each given y in Y, there exist $R_y > 0$ and $N_y \geqslant 1$ such that (9) holds for $n \geqslant N_y$.

Now, for each $n \geqslant N_y$, define the continuous homotopy H_t^n: $B_{R_y}^n \times [0, 1] \to Y_n$ by $H_t^n = A_n(x) + (1 - t)\, W_n(NV_n(x) - y)$. By (9), $H_t^n \neq 0$ for all $x \in \partial B_{R_y}^n$ and all $t \in [0, 1]$. Hence the Brouwer degree $\deg(H_t^n, B_{R_y}^n, 0)$ is well defined for $n \geqslant N_y$ and is independent of $t \in [0, 1]$ by the homotopy theorem. Since A_n is odd for each n, the classical Borsuk theorem then implies that

$$\deg(T_n - W_n y, B_{R_y}^n, 0) = \deg(A_n, B_{R_y}^n, 0) \neq 0 \qquad \text{for} \quad n \geqslant N_y .$$

Thus, for each $n \geqslant N_y$, there exists an $x_n \in B_{R_y}^n \subset E_n$ such that $T_n(x_n) - W_n(y) = 0$. Since $V_n x_n \in B(0, R_y)$ for each n and T is A-proper with respect to Γ_3 there exists a subsequence $\{x_{n_j}\}$ and an x in $\bar{B}(0, R_y)$ such that $V_{n_j}x_{n_j} \to x$ in X and $T(x) = y$, i.e., Eq. (8) is feebly approximation-solvable with respect to the admissible scheme Γ_3 for each y in Y.

Suppose now that instead of A it is assumed that T is odd. Then, as above, one shows that for each given y in Y there exist $R_y > 0$ and $N_y \geqslant 1$ such that for all $t \in [0, 1]$ and $n \geqslant N_y$ we have

$$W_nAV_n(x) + W_nNV_n(x) - tW_n y \neq 0 \qquad \text{for} \quad x \in \partial B_{R_y}^n . \tag{10}$$

As above, it follows from (10) and the oddness of T_n that

$$\deg(T_n - W_n y, B_{R_y}^n, 0) = \deg(T_n, B_{R_y}^n, 0) \neq 0 \qquad \text{for} \quad n \geqslant N.$$

Hence, for each $n \geqslant N_y$, there exists $x_n \in B_{R_y}^n \subset E_n$ such that $T_n(x_n) = W_n(y)$. The assertion that Eq. (8) is feebly approximation-solvable with respect to Γ_3 for each y in Y now follows as in the first case.

To prove the last assertion of Theorem 2 we note that, by what has been proved above, for each y in Y there exists a sequence $\{x_n \mid x_n \in E_n\}$ of solutions of $T_n(x_n) = W_n y$ and a strong limit point x_0 of $\{V_n x_n\}$ in X such that $T(x_0) = y$. Since T is one-to-one and A-proper with respect to Γ_3, $x_0 = \lim_n V_n x_n$. Indeed, if not, then there would exist a subsequence $\{x_{n_j}\}$ such that $\| V_{n_j}x_{n_j} - x_0 \| \geqslant \epsilon$ for all j and some $\epsilon > 0$. But $T_{n_j}(x_{n_j}) = W_{n_j}(y)$ for each j and therefore, by the A-properness of T with respect to Γ_3, there exists a subsequence $\{x_{n_{j(k)}}\}$ and x_0' in X such that $V_{n_{j(k)}}x_{n_{j(k)}} \to x_0'$ as

$k \to \infty$ and $T(x_0') = y$ with $x_0 \neq x_0'$. This, however, contradicts the one-to-one property of T.

Q.E.D.

It is obvious that Proposition 1 follows from Theorem 2 for the case when A is odd with $A = T_\infty$ and we take $E_n = X_n$, $F_n = Y_n$, $V_n: X_n \to X$ a linear injection and $W_n = Q_n$.

As a corollary of Theorem 2 we deduce the following feebly constructive versions of the alternatives of Nečas (see [16, Theor. 2] and [17, Theor. 2]).

COROLLARY 3. *Suppose that X is a separable reflexive Banach space with an injective scheme Γ_I given by* (iii) *for $T: X \to X^*$.*

(a) *Let A be an odd, bounded, and demicontinuous map of X into X^* such that A is of type* (S) *and positively homogeneous of order $\alpha > 0$. Suppose $N: X \to X^*$ is compact and satisfies* (7) *of Theorem* 2. *If $x = 0$ whenever $Ax = 0$, then Eq.* (8) *is feebly approximation-solvable with respect to Γ_I for each y in Y.*

(b) *Suppose $T = A + N: X \to X^*$ is odd, bounded, and of type* (S). *Suppose further that A is demicontinuous, of type* (S) *and positively homogeneous of order $\alpha > 0$, while N is demicontinuous and satisfies* (7) *of Theorem* 2. *If $x = 0$ whenever $Ax = 0$, then Eq.* (8) *is feebly approximation-solvable with respect to Γ_I.*

If T is also one-to-one, then in both cases Eq. (8) *is strongly approximation-solvable with respect to Γ_I for each y in Y* (*i.e., the Galerkin method converges*).

Proof. It was shown by Browder [2] that if $A: X \to X^*$ is a bounded continuous map of type (S), then A is A-proper with respect to Γ_I. Similar argument shows that A is A-proper with respect to Γ_I if the continuity of A is replaced by demicontinuity. For the sake of completeness we outline this argument.

Let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}$ be any bounded sequence so that

$$\| W_{n_j} A x_{n_j} - W_{n_j} y \| \to 0$$

as $j \to \infty$ for some y in Y, where $W_n = V_n^*$. Since $\{x_{n_j}\}$ is bounded and X is reflexive we may assume that $x_{n_j} \rightharpoonup x$ in X. Let $v \in \bigcup_n X_n$; then $v \in X_n$ for some n and therefore for sufficiently large j, $x_{n_j} - v \in X_{n_j}$. Hence

$$|(Ax_{n_j} - y, x_{n_j} - v)| = |(W_{n_j} A x_{n_j} - W_{n_j} y, x_{n_j} - v)| \leqslant \| W_{n_j} A x_{n_j} - W_{n_j} y \| \| x_{n_j} - v \| \to 0.$$

Since $(y, x_{n_j} - v) \to (y, x - v)$, the above relation implies that $(Ax_{n_j}, x_{n_j} - v) \to (y, x - v)$ for each v in $\bigcup_n X_n$. In view of this and the fact that $\bigcup X_n$ is dense in X and $\{Ax_{n_j}\}$ is bounded in X, it follows that $(Ax_{n_j}, x_{n_j} - v) \to (y, x - v)$ for each v in X. In particular, $(Ax_n, x_{n_j} - x) \to (y, x - x) = 0$. Consequently,

$$(Ax_{n_j} - Ax, x_{n_j} - x) \to (y - Ax, x - x) = 0$$

from which, since A is of type (S), it follows that $x_{n_j} \to x$ as $j \to \infty$. Hence $Ax_{n_j} \rightharpoonup Ax$ in X^* by the demicontinuity of A and therefore, since $x_{n_j} - v \to x - v$ for each v in X, we see that

$$(Ax_{n_j}, x_{n_j} - v) \to (Ax, x - v).$$

Thus, $(Ax, x - v) = (y, x - v)$ for each v in X. This implies that $Ax = y$ and shows that A is A-proper with respect to the injective scheme Γ_I.

Now, since A is A-proper and N is compact, it follows easily that $T = A + N$ is also A-proper with respect to Γ_I. In view of this, Corollary 3 follows from Theorem 2. Q.E.D.

We add in passing that the argument used in [16] to establish the solvability aspect of Corollary 3(b) is not applicable if instead of T one assumes only that A is odd.

It was observed by the writer [24] that if one is only interested in the existence theorems for Eq. (1), then under certain conditions the requirement that T be A-proper can be replaced by the weaker hypothesis, namely, that T be pseudo-A-proper; the latter notion is obtained when in Definition 1 we drop the requirement (ii). It was shown in [24] that, in addition to A-proper maps, the class of pseudo-A-proper maps defined in terms of projectional schemes Γ contains monotone and pseudomonotone maps $T: X \to X^*$, K-monotone, weakly continuous maps and others (see [24]). In terms of the admissible schemes $\Gamma_n = \{E_n, F_n, V_n, W_n\}$ we say that $T: X \to Y$ is *pseudo-A-proper with respect to* Γ_3 if $T_n = E_n \to Y_n$ is continuous and if $\{x_{n_j} \mid x_{n_j} \in E_{n_j}\}$ is any sequence such that $\{V_{n_j}(x_{n_j})\}$ is bounded in X and $\| T_{n_j}(x_{n_j}) - W_{n_j} y \| \to 0$ as $j \to \infty$ for some y in Y, then there exists an x in X such that $Tx = y$.

We note that a careful examination of the proof of Theorem 2 reveals that if instead of the approximation-solvability of Eq. (8) we are only interested in the solvability of Eq. (8), then the first part of Theorem 2 admits the following generalization.

THEOREM 2′. *Suppose that all the conditions of Theorem 2 are satisfied except for the requirement that $T = A + N$ be A-proper with respect to Γ_3. Then Eq. (8) is solvable for each y in Y provided T is pseudo-A-proper with respect to Γ_3.*

Finally, we remark that an analogous fact holds for Theorem 1, i.e., one obtains the corresponding solvability results if the A-properness of T with respect to Γ is replaced by the pseudo-A-properness with all other conditions on A and N remaining the same.

2. APPROXIMATION-SOLVABILITY OF QUASILINEAR ELLIPTIC EQUATIONS

In this section we apply the results of Section 1 to the approximation-solvability and/or solvability of generalized boundary value problems for quasilinear elliptic equations of order $2m$ with asymptotically linear terms of order $2m - 1$. We shall indicate later the relation of our results to those obtained by other authors for similar type of equations.

To define our problem we must first introduce some basic notation and definitions. Let Q be a bounded domain in R^n with smooth boundary ∂Q so that the Sobolev Imbedding Theorem holds on Q. For any fixed p with $1 < p < \infty$ let $L_p \equiv L_p(Q)$ denote the usual Banach space of real-valued functions $u(x)$ on Q with norm $\| u \|_p$. We use the standard notation for derivatives

$$D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of nonnegative integers with the order of D^α being written as $| \alpha | = \alpha_1 + \cdots + \alpha_n$. If m is a nonnegative integer, we denote by $W_p^m \equiv W_p^m(Q)$ the real Sobolev space of all u in L_p whose generalized derivatives $D^\alpha u$, $| \alpha | \leqslant m$, also lie in L_p. W_p^m is a separable uniformly convex Banach space with respect to the norm

$$\| u \|_{m,p} = \Big(\sum_{|\alpha| \leqslant m} \| D^\alpha u \|_p^p \Big)^{1/p}.$$

In case $p = 2$ we get the Hilbert space W_2^m. Let $C_c^\infty(Q)$ be the family of infinitely differentiable functions with compact support in Q considered as a subset of W_p^m and let $\mathring{W}_p^m$ be the closure in W_p^m of $C_c^\infty(Q)$. Let $\langle u, v \rangle = \int_Q uv\, dx$ denote the natural pairing between u in L_p and v in L_q with $q = p(p - 1)^{-1}$, and let R^{s_m} denote the vector space whose elements are $\xi_m = \{\xi_\alpha \mid |\alpha| \leqslant m\}$ with $|\xi_m|^k = \sum_{|\alpha| \leqslant m} |\xi_\alpha|^k$ for an integer $k \geqslant 0$ and set $\xi_m(u) = \{D^\alpha u \mid |\alpha| \leqslant m\}$.

(a) *Nonlinear Elliptic Equations with Asymptotically Zero Perturbation*

We first consider a quasilinear elliptic formal partial differential equation of the form

$$\mathscr{T}(u) \equiv \mathscr{A}(u) + \mathscr{N}(u) = f(x) \qquad (f \in L_q), \tag{11}$$

where

$$\mathscr{A}(u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^m u), \tag{12}$$

and

$$\mathscr{N}(u) = \sum_{|\beta| \leqslant m-1} (-1)^{|\beta|} D^\beta N_\beta(x, u, \ldots, D^m u) \tag{13}$$

with $\mathscr{N}(u)$ being asymptotically zero in a sense to be defined below. To define the "generalized boundary-value problem for Eq. (11)" let us suppose initially that $A_\alpha(x, \xi_m(u)(x))$ and $N_\beta(x, \xi_m(u)(x))$ are measurable functions of x on Q for each u in W_p^m and lie in L_q. Then it follows from Hölder's inequality that the generalized Dirichlet forms $a(u, v)$ and $n(u, v)$ associated with the formal differential operators $\mathscr{A}(u)$ and $\mathscr{N}(u)$ are well defined on W_p^m by the equations

$$a(u, v) = \sum_{|\alpha| \leqslant m} \langle A_\alpha(x, \xi_m(u)), D^\alpha v \rangle, \tag{14}$$

$$n(u, v) = \sum_{|\beta| \leqslant m-1} \langle N_\beta(x, \xi_m(u)), D^\beta v \rangle, \tag{15}$$

and so the problem can formally be described by the following (see [1,14]).

Generalized Boundary-Value Problem. *Let V be a closed subspace of W_p^m with $\mathring{W}_p^m \subset V$, and let $a(u, v)$ and $n(u, v)$ be the generalized Dirichlet forms as defined in equations* (14) *and* (15) *above. Let f be a given element in L_q. By the generalized boundary value problem for Eq.* (11) *corresponding to V we mean the problem of finding an element u in V, called the weak solution of Eq.* (11), *which satisfies the equation*

$$a(u, v) + n(u, v) = \langle f, v \rangle \qquad \textit{for all } v \textit{ in } V. \tag{16}$$

Remark 6. As was pointed out in [1], Eq. (16) together with the restriction that u lies in V has the force not only of requiring that u should satisfy Eq. (11) (at least in generalized sense) but also of imposing boundary conditions upon u. The requirement that u lies in V may impose a boundary condition if V is significantly smaller than $W_p{}^m$ (as in the homogeneous Dirichlet problem when $V = \mathring{W}_p{}^m$). Equation (16) imposes boundary conditions as soon as V is significantly larger than $\mathring{W}_p{}^m$ (see [1] for further discussion).

Now, under suitable condtions on $A_\alpha(x, \xi_m(u))$ and $N_\beta(x, \xi_m(u))$, for each fixed u in V, $a(u, v)$ and $n(u, v)$ are continuous linear functionals of v in V which we denote by $A(u)$ and $N(u)$, respectively, so that $A(u)$ and $N(u)$ are elements of V^* and A and N are mappings of V into V^* determined by

$$a(u, v) = (Au, v) \qquad \text{for all } v \text{ in } V, \tag{17}$$

$$n(u, v) = (Nu, v) \qquad \text{for all } v \text{ in } V, \tag{18}$$

where (Au, v) and (Nu, v) denote the value of the functionals $A(u)$ and $N(u)$ in V^* at v in V. Similarly, for each f in L_q there exists a unique w_f in V^* such that $\langle f, v\rangle = (w_f, v)$ for all v in V. Consequently, in view of (17) and (18), Eq. (16) is equivalent to the operator equation

$$Tu \equiv Au + Nu = w_f \tag{19}$$

for a given w_f in V^* and the mapping $T = A + N\colon V \to V^*$. Thus the generalized boundary value problem for Eq. (11) corresponding to V is equivalent to the solvability of the operator Eq. (19).

In order to apply Theorem 2 to Eq. (16) or Eq. (19) we have first to select an admissible approximation scheme Γ_3 for mappings $T\colon V \to V^*$. Since V is a separable reflexive Banach space we may find a sequence $\{\phi_i\} \subset V$ which is linearly independent and complete in V and then construct an increasing sequence $\{X_n\}$ of finite-dimensional subspaces of V by taking $X_n = \text{sp}\{\phi_1, \ldots, \phi_n\}$ so that $\bigcup_n X_n$ is dense in V. For a given f in L_q (or any $w_f \in V^*$), we associate with Eq. (16) a sequence of finite dimensional nonlinear algebraic equations

$$a(u_n, \phi_j) + n(u_n, \phi_j) = \langle f, \phi_j\rangle \qquad (1 \leqslant j \leqslant n), \tag{20}$$

for the determination of an approximate solution $u_n = \sum_{i=1}^n a_i{}^n \phi_i \in X_n$, i.e., we determine the unknowns $\{a_1{}^n, \ldots, a_n{}^n\}$ from the Galerkin type system:

$$\sum_{|\alpha| \leqslant m} \int_Q (A_\alpha(x, \xi_m(u_n))\, D^\alpha \phi_j\, dx - \sum_{|\beta| \leqslant m-1} \int N_\beta(x, \xi_m(u_n))\, D^\beta \phi_j\, dx = \int_Q f\phi_j\, dx \qquad (1 \leqslant j \leqslant n). \tag{21}$$

If for each n we let V_n be the linear injection of X_n into V and V_n^*, its dual, the corresponding projection of V^* onto X_n^*, then Eq. (20) is equivalent to the operator equation

$$T_n(u_n) \equiv A_n(u_n) + N_n(u_n) = W_n(w_f), \tag{22}$$

where A_n and N_n are maps of X_n into $Y_n \equiv X_n^*$ given by $A_n = W_nA\,|_{X_n}$, $N_n = W_nN\,|_{X_n}$ and $T_n \equiv A_n + N_n$ with $W_n \equiv V_n^*$. It was mentioned in Section 1 that $\Gamma_I = \{X_n, Y_n, V_n, W_n\}$ (i.e., $\Gamma_3 = \Gamma_I$) forms an admissible approximation scheme for mappings from V to V^* in the sense of Definition 3. The above discussion suggests the following definition concerning Eq. (11).

DEFINITION 5. The generalized boundary value problem for Eq. (11) corresponding to a given closed subspace V of W_p^m with $\mathring{W}_p^m \subset V$ is *strongly* (respectively, *feebly*) *approximation-solvable* with respect to the admissible scheme Γ_I for maps from V to V^* if Eq. (20) has a solution $\{a_1^n,..., a_n^n\}$ for each large n with $u_n = \sum_{i=1}^n a_i^n\phi_i$ such that $u_n \to u$ in V (respectively, $u_{n_j} \to u$ in V) and u satisfies Eq. (16), (i.e., iff Eq. (19) is strongly (respectively, feebly) approximation-solvable with respect to Γ_I in the sense of Definition 2).

Basic Analytic Problem

The preceding discussion indicates that the basic problem in the approximation-solvability and/or solvability of the generalized boundary value problem for Eq. (11) corresponding to a given subspace V via the theory of A-proper mappings lies in the following: What concrete analytic (and practically meaningful and verifiable) assumptions one should impose on the functions $A_\alpha(x, \xi_m)$ and $N_\beta(x, \xi_m)$ defining our nonlinear problem with respect to V so that the operators A, N: $V \to V^*$ are well-defined by the corresponding generalized Dirichlet forms $a(u, v)$ and $n(u, v)$ and are such that A and $T = A + N$ are A-proper with respect to Γ_I and either N is asymptotically zero as in Theorem 2 or T is asymptotically linear as in Theorem 1.

In what follows we consider some simple analytic hypotheses on $A_\alpha(x, \xi_n)$ and $N_\beta(x, \xi_m)$ which are sufficient for the mappings A, N and T to have the abovementioned properties. Our primary aim in this section is to show how the theory of A-proper mappings can be used in the approximation-solvability of generalized boundary value problems but not to strive for most general results. Nevertheless our hypotheses are general enough so as to include and strengthen some (existence) results obtained earlier in [9, 11, 17] for asymptotically linear elliptic equations by different methods. The application of the general theory of A-proper mappings to more general elliptic and parabolic equations as well as the treatment of more concrete examples will be given elsewhere.

We start with the following simple conditions on A_α and N_β.

(A) *$A_\alpha(x, \xi_m)$ satisfies the Caratheodory conditions, i.e., $A_\alpha(x, \xi_m)$ is measurable in x on Q for each ξ_m in R^{S_m} and continuous in ξ_m for almost all x in Q. In addition, we assume that:*

(1) *There exist a constant $k_0 > 0$ and a function h_0 in L_q such that $| A_\alpha(x, \xi_m)| \leqslant k_0 | \xi_m |^{p-1} + h_0$ for $| \alpha | \leqslant m$.*

(2) *There exist constants $c_0 \geqslant 0$ and $c_1 > 0$ such that for each $x \in Q$, each pair ξ_m and ξ_m' in R^{S_m} and some integer $k \leqslant m - 1$ we have*

$$\sum_{|\alpha|\leqslant m} [A_\alpha(x, \xi_m) - A_\alpha(x, \xi_m')][\xi_\alpha - \xi_\alpha'] \geqslant c_1\Big(\sum_{|\alpha|\leqslant m} | \xi_\alpha - \xi_\alpha' |^p\Big) - c_0\Big(\sum_{|\alpha|\leqslant k} | \xi_\alpha - \xi_\alpha' |^p\Big).$$

(B) *$N_\beta(x, \xi_m)$ satisfies the Caratheodory conditions for each $| \beta | \leqslant m - 1$. In addition we assume that for $| \beta | \leqslant m - 1$, we have*

(1) *$| N_\beta(x, \xi_m)| \leqslant k_1 | \xi_m |^{p-1} + h_1$ for some $k_1 > 0$ and $h_1 \in L_q$.*

We are now in a position to state our first new approximation-solvability result concerning the generalized boundary value problem for a quasilinear elliptic equation with an asymptotically zero perturbation.

THEOREM 3. *Let $\mathscr{A}(u)$ and $\mathscr{N}(u)$ be the quasilinear formal differential operators given by* (12) *and* (13) *for which assumptions* (A) *and* (B) *hold. Let V be a closed subspace of $W_p{}^m$ with $\mathring{W}_p{}^m \subset V$ such that*

(C) *The linear imbedding of V into W_p^{m-1} is compact. Suppose also that the following additional hypotheses hold:*

(D) *$\mathscr{A}(u)$ is odd and positively homogeneous, i.e., $A_\alpha(x, -\xi_m) = -A_\alpha(x, \xi_m)$ and $A_\alpha(x, t\xi_m) = t^d A_\alpha(x, \xi_m)$ for all x, ξ_m, $t > 0$ and some $d > 0$.*

(E) *There exists a continuous function* $k: R^+ \to R^+$ *with* $k(t)\, t^{-d} \to 0$ *as* $t \to \infty$ *such that for u in V*

$$| N_\beta(x, \xi_m(u))| \leqslant k(\| u \|_{m,p}) \qquad \textit{for} \quad | \beta | \leqslant m - 1.$$

(F) *If* $a(u, v) = 0$ *for some u in V and all v in V, then* $u = 0$.

Under the above conditions the generalized boundary-value problem for Eq. (11) *corresponding to V is feebly approximation-solvable with respect to* Γ_I *for each f in* L_q, *and it is strongly approximation-solvable if* $T = A + N$ *is also injective. In particular, this includes the Dirichlet* ($V = \mathring{W}_p{}^m$) *and the Neumann* ($V = W_p{}^m$) *boundary value problem for Eq.* (11).

Proof. It follows from assumptions (A1) and (B1) and the standard results about Nemytskii operators (e.g., [15, Chap. 2.2]) that the generalized Dirichlet forms $a(u, v)$ and $n(u, v)$ are well defined on $W_p{}^m$ and that for a given closed subspace V of $W_p{}^m$ with $\mathring{W}_p{}^m \subset V$ one can associate with $a(u, v)$ and $n(u, v)$ in a unique way bounded continuous mappings A and N of V into V^* such that (17) and (18) hold. Thus, to prove Theorem 3, it suffices to show that A, N and $T = A + N$ satisfy all the hypotheses of Theorem 2. To verify this we first establish the following lemma.

LEMMA 1. *Under assumptions* (A) *and* (B) *and the hypothesis* (C) *of Theorem* 3, *the mappings A and* $T = A + N$ *are A-proper with respect to the admissible scheme* $\Gamma_I = \{X_n, Y_n, V_n, W_n\}$.

Proof of Lemma 1. We first show that A is A-proper with respect to Γ_I. Let u and v be any element of V. Then (A2) implies that for for some $k \leqslant m - 1$

$$\begin{aligned} a(u, u - v) &- a(v, u - v) \\ &= \sum_{|\alpha| \leqslant m} \langle A_\alpha(x, \xi_m(u)) - A_\alpha(x, \xi_m(v)), D^\alpha u - D^\alpha v \rangle \\ &\geqslant c_1 \Big(\sum_{|\alpha| \leqslant m} | D^\alpha u - D^\alpha v |_p^p \Big) - c_0 \Big(\sum_{|\alpha| \leqslant k} | D^\alpha u - D^\alpha v |_p^p \Big). \end{aligned}$$

Putting this in terms of A we find that for all u and v in V

$$(Au - Av, u - v) \geqslant c_1(\| u - v \|_{m,p}^p) - c_0(\| u - v \|_{k,p}^p). \tag{23}$$

We claim that (23) implies the A-properness of A: $V \to V^*$ with respect to Γ_I.

Let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ be any bounded sequence such that $\| W_{n_j}A(x_{n_j}) - W_{n_j}y \| \to 0$ for some y in V^*. Since V is reflexive and $\{x_{n_j}\}$ is bounded we may assume that $x_{n_j} \rightharpoonup x_0$ in V for some x_0 in V.

Hence, since the sequence $\{Ax_{n_j}\}$ is bounded in V^*, the argument used in the proof of Corollary 3 shows that

$$(Ax_{n_j} - y, x_{n_j} - x_0) \to 0 \qquad \text{as} \quad j \to \infty.$$

This implies that

$$\begin{aligned}(Ax_{n_j} &- Ax_0 , x_{n_j} - x_0) \\ &= (Ax_{n_j} - y, x_{n_j} - x_0) + (y - Ax_0 , x_{n_j} - x_0) \to 0\end{aligned}$$

as $j \to \infty$. Now, by the inequality (23), for all j we have the relation

$$(Ax_{n_j} - Ax_0 , x_{n_j} - x_0) + c_0 \| x_{n_j} - x_0 \|_{k,p}^p \geqslant c_1 \| x_{n_j} - x_0 \|_{m,p}^p . \quad (24)$$

Since, by the hypothesis (C), the imbedding of V into W_p^k is compact for each $k \leqslant m - 1$ and $x_{n_j} \rightharpoonup x_0$ in W_p^m as $j \to \infty$, it follows that $\| x_{n_j} - x_0 \|_{k,p}^p \to 0$ as $j \to \infty$. The above discussion and the inequality (24) imply that $\| x_{n_j} - x_0 \|_{m,p} \to 0$ as $j \to \infty$. Hence, by the continuity of A, $Ax_{n_j} \to Ax_0$ in V^*. This together with the above shows that for each v in V

$$(y, x_0 - v) = \lim(Ax_{n_j} , x_{n_j} - v) = (Ax_0 , x_0 - v).$$

Thus $y = Ax_0$ and so A is A-proper with respect to Γ_I.

We next show that the map N: $V \to V^*$ given by (18) is compact. Indeed, let $\{u_n\}$ be any bounded sequence in V. Since N is bounded and V^* is reflexive, we may assume without loss of generality that Nu_n converges weakly to some element w in V^*. We claim that, in fact, $Nu_n \to w$ in V^* as $n \to \infty$. Indeed, if this were not the case, then there would exist a sequence $\{v_n\} \subset V$ and a constant $\epsilon > 0$ such that $\| v_n \|_{m,p} = 1$, $v_n \rightharpoonup v_0$ in V and $(Nu_n - w, v_n) \geqslant \epsilon > 0$ for each n. Since $(Nu_n - w, v_n) = (Nu_n - w, v_n - v_0) + (Nu_n - w, v_0)$ and $(Nu_n - w, v_0) \to 0$ as $n \to \infty$, to arrive at the contradiction it suffices to show that $(Nu_n - w, v_n - v_0) \to 0$ as $n \to \infty$ and that will be the case if we show that $(Nu_n , v_n - v_0) \to 0$ as $n \to \infty$. Now, by (15) and (18), $(Nu_n , v_n - v_0) = \sum_{|\beta| \leqslant m-1} \langle N_\beta(x, \xi_m(u_n)), D^\beta v_n - D^\beta v_0 \rangle$. Since $v_n \rightharpoonup v_0$ in W_p^m and, by condition (C), the imbedding of V into

W_p^{m-1} is compact, we see that $v_n \to v_0$ in W_p^{m-1} as $n \to \infty$. It follows from this and assumption (B1) or (E) that

$$\sum_{|\beta| \leqslant m-1} \int_Q N_\beta(x, \xi_m(u_n))(D^\beta v_n - D^\beta v_0)\, dx \to 0 \qquad \text{as} \quad n \to \infty.$$

This contradiction shows that N is compact.

Now, since A is A-proper and N is compact, it is easy to show that $T = A + N$ is A-proper with respect to Γ_I.

Proof of Theorem 3 *completed.* In view of Lemma 1, to complete the proof of Theorem 3, it remains to verify some further conditions which have to be satisfied by A and N for the abstract Theorem 2 to be applicable. Now the fact that A is odd and positively homogeneous follows immediately from condition (D). Moreover, condition (F) implies that if $Au = 0$ for some u in V, then $u = 0$. Finally, it follows from (15), (18) and condition (E) that for all u and v in V

$$|(Nu, v)| = | n(u, v)| \leqslant Mk(\| u \|_{m,p}) \| v \|_{m,p}$$

for some constant $M > 0$. Since $\| Nu \|_{V^*} \leqslant \sup_{v \in V} |(Nu, v)|/\| v \|_{m,p}$, the preceding inequality and the property of $k(t)$ imply that

$$\| Nu \|/\| u \|_{m,p}^{-d} \to 0 \qquad \text{as} \quad \| u \|_{m,p} \to \infty.$$

Thus the operators A, N: $V \to V^*$ defined by the generalized Dirichlet forms $a(u, v)$ in (14) and $n(u, v)$ in (15) satisfy all the conditions of the abstract Theorem 2. Consequently the validity of Theorem 3 follow from Theorem 2. Q.E.D.

The following Remarks concern various conditions imposed in Theorem 3.

Remark 7. Using the full power of the Sobolev Imbedding Theorem one can prove the boundedness and at least the demicontinuity of A given by (14) and N given by (15) under conditions which are much weaker than (A1) and (B1) used above (see, for example, Theorem 3.1 in [14] and Lemma 3 in [1]).

Remark 8. Condition (A2) can sometimes be verified from the following concrete analytic assumptions on A_α. Suppose $A_\alpha(x, \xi_m)$ is differentiable with respect to ξ_β and set $A_{\alpha\beta}(x, \xi_m) = (\partial A_\alpha/\partial \xi_\beta)(x, \xi_m)$

for $|\beta|, |\alpha| \leqslant m$ and $u_t = tu + (1-t)v$. Then it is not hard to see that

$$a(u, u-v) - a(v, u-v)$$
$$\int_0^1 \sum_{|\alpha| \leqslant m} \frac{d}{dt} \langle A_\alpha(x, \xi_m(u_t)), D^\alpha(u-v) \rangle \, dt$$
$$= \int_0^1 \sum_{|\alpha|, |\beta| \leqslant m} \langle A_{\alpha\beta}(x, \xi_m(u_t)) \, D^\beta(u-v), D^\alpha(u-v) \rangle \, dt.$$

Consequently, the inequalities from below for

$$a(u, u-v) - a(v, u-v)$$

can be obtained from the inequalities from below for the form $\sum_{|\alpha|, |\beta| < m} ((A_{\alpha\beta}(x, z) \, \eta_\alpha, \eta_\beta))$. This approach has been used by Browder, Nečas, and others.

Remark 9. Instead of condition (A2) which involves the dependence properties of $A_\alpha(x, \xi_m)$ on the lower-order derivatives and for $|\alpha| < m$ one could, following the approaches of Leray–Lions [14] and of Browder (e.g. [1]), impose conditions which involve only the highest order terms. Thus, for example, instead of (A2) one can use Browder's Assumption (B′) (see [1, p. 18]) which guarantees that A, determined by (17), is of type (S) and therefore A-proper with respect to the scheme Γ_I used here. Consequently the results of Nečas [17] for elliptic differential operators whose Dirichlet forms in $V = \mathring{W}_{m,p}$ give rise to mappings of type (S) follow from our Theorem 2.

Remark 10. It was noted by Pohodjayev [27] that $N(u)$ given by (13) induces a compact map $N\colon \mathring{W}_p^m \to (\mathring{W}_p^m)^*$ by means of (18) if $N_\beta(x, \xi_m)$ satisfies the Caratheodory conditions and the inequalities $|N_\beta(x, \xi_m)| \leqslant c\{k(x) + \sum_{|\alpha| \leqslant m} |\xi_\alpha|^{q_{\beta\alpha}}\}$, where

$$0 \leqslant q_{\beta\alpha} < ((n(p-1) + (m - |\beta|)p)/(n - (m - |\alpha|)p))$$

for $n > (m - |\alpha|)p$ and $q_{\beta\alpha}$ arbitrary nonnegative numbers if $n \leqslant (m - |\alpha|)p$.

Remark 11. It is known (e.g. [15]) that condition (C) always holds if $V = \mathring{W}_p^m$ or if $V = W_p^m$ and the boundary ∂Q has a "cone condition".

(b) *Asymptotically Linear Elliptic Equations*

If condition (F) or (E) of Theorem 3 fails to hold, then Theorem 3 can no longer be used to assert anything about the solvability of the generalized boundary value problem for Eq. (11) although, as we know from Theorem 1, the equivalent Eq. (19) may still be solvable under certain additional conditions on f and the operators $\mathscr{A}(u)$ and $\mathscr{N}(u)$ (i.e., A and N). In this section we apply Theorem 1 to the solvability of the generalized Dirichlet problem for Eq. (11) in the case when $p = 2$, $V = \mathring{W}_2^m$, $\mathscr{A}(u)$ is also asymptotically linear, and $\mathscr{N}(u)$ is asymptotically linear but not necessarily asymptotically zero, i.e., we assume that in addition to assumptions (A1)–(A2) and (B1) the following conditions hold.

(A3) *There exist functions $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ with $a_{\alpha\beta} \in L^\infty(Q)$ for $|\alpha| \leqslant m$ and $|\beta| \leqslant m$ and $a_{\alpha\beta}$ in $C(\bar{Q})$ for $|\alpha| = m$ and $|\beta| = m$, a constant $d_0 > 0$, and a continuous function $c: R^+ \to R^+$ with $c(t)/t \to 0$ as $t \to \infty$ such that*

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x)\eta^\alpha\eta^\beta \geqslant d_0 |\eta|^{2m} \text{ for all } \eta = (\eta_1, ..., \eta_n) \in R^n \textit{ and a.e. } x \in Q \quad (25)$$

and for each fixed α with $|\alpha| \leqslant m$ we have

$$\left| A_\alpha(x, u, ..., D^m u) - \sum_{|\beta| \leqslant m} a_{\alpha\beta}(x) D^\beta u \right| \leqslant c(\| u \|_{m,2}) \qquad \forall u \in \mathring{W}_2^m. \quad (26)$$

(B2) *There exist functions $b_{\beta\gamma}(x) \in L^\infty(\bar{Q})$ for $|\beta| \leqslant m - 1$ and $|\gamma| \leqslant m$ and a continuous function $d: R^+ \to R^+$ with $d(t)/t \to 0$ as $t \to \infty$ such that for each β with $|\beta| \leqslant m - 1$ we have*

$$\left| N_\beta(x_1 u, ..., D^m u) - \sum_{|\gamma| \leqslant m} b_{\beta\gamma}(x) D^\gamma u \right| \leqslant d(\| u \|_{m,2}) \qquad \forall u \in \mathring{W}_2^m. \quad (27)$$

Along with the generalized Dirichlet problem in $\mathring{W}_2^m$ for the asymptotically linear equation

$$\mathscr{T}(u) \equiv \mathscr{A}(u) + \mathscr{N}(u) = f(x) \qquad (x \in Q), \quad (f \in L_2), \quad (28)$$

where $\mathscr{A}(u)$ and $\mathscr{N}(u)$ are formal differential operators given by (12) and (13), respectively, with A_α and N_β satisfying conditions (A1)–(A3) and (B1)–(B2), we consider the generalized Dirichlet problem in $\mathring{W}_m^2$ for the linear equation

$$\mathscr{T}(u) \equiv \mathscr{L}(u) + \mathscr{B}(u) = 0, \quad (29)$$

where

$$\mathscr{L}(u) = \sum_{|\alpha|,|\beta|\leqslant m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u), \tag{30}$$

$$\mathscr{B}(u) = \sum_{|\beta|\leqslant m-1,|\gamma|\leqslant m} (-1)^{|\beta|} D^\beta(b_{\beta\gamma}(x)\, D^\gamma u). \tag{31}$$

The conditions (A3) and (B2) imply that the bilinear forms

$$l(u, v) = \sum_{|\alpha|,|\beta|\leqslant m} \langle a_{\alpha\beta}(x)\, D^\beta u, D^\alpha v\rangle, \tag{32}$$

$$b(u, v) = \sum_{|\beta|\leqslant m-1,|\gamma|\leqslant m} \langle b_{\beta\gamma}(x)\, D^\gamma h, D^\beta v\rangle, \tag{33}$$

associated with $\mathscr{L}(u)$ and $\mathscr{B}(u)$, respectively, determine bounded linear mappings L, B: $\mathring{W}_2{}^m \to \mathring{W}_2{}^m$ such that

$$l(u, v) = (Lu, v), b(u, v) = (Bu, v) \qquad \text{for all} \quad u, v \in \mathring{W}_2{}^m, \tag{34}$$

where (,) denotes the inner product in $\mathring{W}_2{}^m$. By (25) of (A3), a Gärding inequality

$$l(u, u) \geqslant a_1 \| u \|_{m,2}^2 - a_0 \| u \|_{0,2}^2 \qquad (a_1 > 0, a_0 \geqslant 0) \tag{35}$$

is satisfied for all u in $\mathring{W}_2{}^m$. Since the imbedding of $\mathring{W}_2{}^m$ into L_2 is compact, in view of (34), it follows from (35) (see [23] or Lemma 1 above) that the operator L: $\mathring{W}_2{}^m \to \mathring{W}_2{}^m$ is A-proper with respect to any given projectionally complete scheme $\Gamma_0 = \{X_n, P_n\}$ in $\mathring{W}_2{}^m$. The latter exist since $\mathring{W}_2{}^m$ is a separable Hilbert space. Now, the linear operator B: $\mathring{W}_2{}^m \to \mathring{W}_2{}^m$ given by (34) is also compact. This follows from Lemma 1 or directly from an easily established inequality:

$$| b(u, u)| \leqslant m_0 \| u \|_{m,2} \| u \|_{m-1,2} \qquad \text{for all} \quad u \in \mathring{W}_2{}^m \quad \text{and some} \quad m_0 > 0.$$

Thus the operator $T_\infty \equiv L + B$: $\mathring{W}_2{}^m \to \mathring{W}_2{}^m$, associated with the bilinear form $t(u, v) = l(u, v) + b(u, v)$ corresponding to $\mathscr{T}_\infty(u)$, is linear and A-proper with respect to Γ_0. Note that the mapping $T_\infty{}^* = L^* + B^*$: $\mathring{W}_2{}^m \to \mathring{W}_2{}^m$, which is the adjoint of T_∞, is also A-proper with respect to Γ_0 since B^* is compact and, in view of (34) and (35), the mapping L^* also satisfies the inequality

$$(u, L^*u) \geqslant a_1 \| u \|_{m,2}^2 - a_0 \| u \|_{0,2}^2 \qquad \text{for all } u \text{ in } \mathring{W}_2{}^m.$$

Consequently, by Theorem 5 in [20] (or Theorem A above), $i(T_\infty) = 0$, i.e., $\dim N(T_\infty) = \operatorname{codim} R(T_\infty) = \dim N(T_\infty^*)$. Moreover, since $\mathscr{T}(u)$ induces the bounded nonlinear continuous operator T: $\mathring{W}_2^m \to \mathring{W}_2^m$ given by $T = A + N$, where A and N are determined by (17) and (18), respectively, it follows from (26) of condition (A3) and from condition (B2) that T is asymptotically linear with T_∞ as its asymptotic derivative, i.e.,

$$\| T(u) - T_\infty(u)\|_{m,2}/\| u \|_{m,2} \to 0 \quad \text{as} \quad \| u \|_{m,2} \to \infty \qquad (u \in \mathring{W}_2^m) \tag{36}$$

Indeed, since for all u and v in $\mathring{W}_2^m$ we have

$$\begin{aligned}(T(u) - T_\infty(u), v) &= \{a(u, v) - l(u, v)\} + \{n(u, v) - b(u, v)\} \\ &= \sum_{|\alpha| \leqslant m} \langle A_\alpha(x, \xi_m(u)) - \sum_{|\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\beta u, D^\alpha v\rangle \\ &\quad + \sum_{|\beta| \leqslant m-1} \langle N_\beta(x, \xi_m(u)) - \sum_{|\gamma| \leqslant m} b_{\beta\gamma} D^\gamma u, D^\beta v\rangle,\end{aligned} \tag{37}$$

it follows from (37), the compactness of the imbedding of $\mathring{W}_2^m$ into W_2^k for each $k \leqslant m - 1$, the condition (26) of (A3), and (B2) that there exist a constant $K > 0$ such that for all u and v in $\mathring{W}_2^m$

$$|(Tu - T_\infty u, v)| \leqslant K\{c(\| u \|_{m,2}) + d(\| u \|_{m,2})\}\, \| v \|_{m,2}\,. \tag{38}$$

Since $c(t)/t \to 0$ and $d(t)/t \to 0$ as $t \to \infty$ and $\| Tu - T_\infty u \| = \sup_{v \in \mathring{W}_2^m} |(Tu - T_\infty u, v)|/\| v \|_{m,2}$ for each fixed $u \in \mathring{W}_2^m$, we obtain (36) from (38).

We recall that, for a given f in L_2, the generalized Dirichlet problem for Eq. (28) admits a weak solution if there exists a function u in $\mathring{W}_2^m$ such that

$$t(u, v) \equiv a(u, v) + n(u, v) = \langle f, v\rangle \qquad \text{for all } v \text{ in } \mathring{W}_2^m. \tag{39}$$

Equation (39) is equivalent to the operator equation

$$T(u) \equiv A(u) + N(u) = w_f\,, \tag{40}$$

where w_f is the element in $\mathring{W}_2^m$ corresponding to f by $\langle f, v\rangle = (w_f, v)$ for all v in $\mathring{W}_2^m$.

In view of the above discussion and the A-properness of T: $\mathring{W}_2^m \to \mathring{W}_2^m$ with respect to Γ_0, the operators T and T_∞ satisfy all the conditions of Theorem 1 and therefore the validity of the following new results for the generalized Dirichlet problem for Eq. (28) follow from Theorem 1.

THEOREM 4. *Suppose* (28) *is an asymptotically linear elliptic equation for which conditions* (A1)–(A3) *and* (B1)–(B2) *hold.*

(a) *If* $N(T_\infty) = \{0\}$ (*i.e., if the generalized Dirichlet problem for linear Eq.* (29) *has only null solutions in* $\mathring{W}_2{}^m$), *then the generalized Dirichlet problem for the nonlinear Eq.* (28) *is feebly approximation-solvable with respect to* Γ_0 *for each* f *in* L_2, (*and, in particular, it is solvable for each* $f \in L_2$).

If we assume additionally that the generalized Dirichlet problem for Eq. (28) *has at most one weak solution in* $\mathring{W}_2{}^m$ *for a given* $f \in L_2$, *then it is strongly approximation-solvable with respect to* Γ_0 (*i.e., the Galerkin method converges in* $\mathring{W}_2{}^m$ *for asymptotically linear elliptic equations of type* (28)).

(b) *If* $N(T_\infty) \neq \{0\}$ *and if* $\{v_1, ..., v_k\}$ *is a basis for* $N(T_\infty{}^*) \subset \mathring{W}_2{}^m$ *with* $k = \dim N(T_\infty)$, *then under the additional condition*

$$\sum_{|\alpha| \leqslant m} \int_Q (A_\alpha(x, u, ..., D^m u) + N_\alpha(x, u, ..., D^m u)) D^\alpha v_j \, dx = 0, \qquad 1 \leqslant j \leqslant k, \tag{41}$$

where $N_\alpha(x_1 u, ..., D^m u) \equiv 0$ *for* $|\alpha| = m$, *the generalized Dirichlet problem for Eq.* (28) *is solvable in* $\mathring{W}_2{}^m$ *for a given* f *in* L_2 *if and only if* $\int_Q f v_j \, dx = 0$ *for* $1 \leqslant j \leqslant k$.

Remark 12. Theorem 4 includes the existence Theorems 6 and 7 of Kačurovskii [11] obtained by him by other methods for the case when $\mathscr{A}(u) \equiv \mathscr{L}(u)$ and N_β depends on x and D^α with $|\alpha| \leqslant m - 1$ but not $D^\alpha u$ with $|\alpha| = m$ and where $l(u, u)$ is assumed to be positive definite, i.e., $l(u, u) \geqslant c_1 \| u \|_{m,2}^2$ for all $u \in \mathring{W}_2{}^m$ and some $c_1 > 0$. Theorem 4 also includes the existence Theorem 3 of Hess [9] obtained by him for the case when $m = 1$, $\mathscr{A}(u) = \mathscr{L}(u)$, and $\mathscr{N}(u) = N_0(x, u, Du)$.

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