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Multivariate semi-logistic distributions

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ABSTRACT

Three new multivariate semi-logistic distributions (denoted by $MSL^{(1)}$, $MSL^{(2)}$, and GMSL respectively) are studied in this paper. They are more general than Gumbel's (1961) [1] and Arnold's (1992) [2] multivariate logistic distributions. They may serve as competitors to these commonly used multivariate logistic distributions. Various characterization theorems via geometric maximization and geometric minimization procedures of the three $MSL^{(1)}$, $MSL^{(2)}$ and GMSL are proved. The particular multivariate logistic distribution used in the multiple logistic regression model is introduced. Its characterization theorem is also studied. Finally, some further research work on these MSL is also presented. Some probability density plots and contours of the bivariate $MSL^{(1)}$, $MSL^{(2)}$ as well as Gumbel's and Arnold's bivariate logistic distributions are presented in the Appendix.

1. Introduction and motivation

Research work on multivariate logistic distribution (denoted by ML) in recent decades is rather scarce compared to the voluminous work that has been carried out on the univariate logistic distribution (see the books of Johnson, Kotz and Balakrishnan's [3] ch. 23 and Kotz, Balakrishnan and Johnson's [4] ch. 51 and the references therein). Gumbel [1] made the first attempt to define bivariate logistic distribution. A lucid review of ML was prepared by Arnold [2].

Numerous applications of univariate and Gumbel's [1] multivariate logistic distributions can be found in many literatures, such as the logit regression in categorical data analysis [5], the logistic growth model and the proportional hazard rate model in biomedical sciences [6,7]. The problem of medical diagnosis through the logistic discriminant function was introduced first by Cox [8]. The univariate logistic function was also used in studies on physiochemical phenomenon by Reed and Berkson [9]. Moreover, the univariate and Gumbel's [1] multivariate logistic distributions are also in the parametric families of the univariate and multivariate extreme value distributions respectively ([10], [4, ch. 53] and the references therein). Many other fields of applications of the univariate and multivariate logistic distributions in the recent twenty years can be found in the volume of [11].

Owing to the plentiful applications of the logistic distributions including univariate and multivariate variables, the importance of the logistic distributions is evident. Pillai [12] was the pioneer to study the univariate semi-distributions including the semi-Pareto and semi-Weibull distributions. Some special bivariate semi-Pareto distributions were studied by Balakrishna and Jayakumar [13] and Thomas and Jose [14]. Yeh [15–17] extends their results to more general cases and studies many properties of the multivariate semi-Pareto distribution. Arnold, Robertson and Yeh [18] study the

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characterizations of univariate Pareto and univariate logistic distributions. Arnold [2,19] briefly introduces the univariate semi-logistic distribution. In the past decade, few properties of the multivariate semi-logistic distribution (denoted by MSL) have been discussed in the existent literatures since [19].

The relative few literatures about the characterizations of the existent ML and the MSL drives the author to study the properties of the MSL in this paper.

Three general MSL distributions denoted by $\text{MSL}^{(1)}$, $\text{MSL}^{(2)}$, and GMSL respectively are introduced in Section 2. Various characterization theorems of $\text{MSL}^{(1)}$, $\text{MSL}^{(2)}$ and GMSL are studied in Sections 3 and 4. The most well-known application of ML is the multiple logistic regression model, the particular ML distribution in logit regression is introduced in Section 5, and its characterization is also studied. Some further research work of these multivariate semi-logistic distributions is given in Section 6. Some probability density plots of the bivariate $\text{MSL}^{(1)}$, $\text{MSL}^{(2)}$ as well as Gumbel's and Arnold's bivariate logistic distributions and their corresponding contour plots are presented in the Appendix.

From the results of this paper. It is discerned that Gumbel's [1] ML is in the class of $\text{MSL}^{(1)}$, and Arnold's [2] ML is in the class of $\text{MSL}^{(2)}$, and the logit multivariate logistic distribution (denoted by LML) introduced in Section 5 is also in the class of $\text{MSL}^{(1)}$. Therefore, all the characterization theorems proved in this paper can be applied directly to the existent univariate and multivariate logistic distributions. On the other hand, since all the characteristic theorems are located on the borderline between probability theory and mathematical statistics, so the $\text{MSL}^{(1)}$, and $\text{MSL}^{(2)}$ can be identified by Theorems 3.1 and 3.2 respectively. Analogously, Theorem 4.1 is used to identify the GMSL, Theorems 5.1–5.3 are the characterizations of the logit multivariate logistic distributions, LML and GLML in multiple logistic regression models. The three proposed multivariate semi-logistic distributions, $\text{MSL}^{(1)}$, $\text{MSL}^{(2)}$, and GMSL may serve as competitors to the commonly used Gumbel's [1] and Arnold's [2] multivariate logistic distributions. The physical reasons for the particular generalizations are drawn from the idea of Pillai [12] who was the first to study the univariate semi-Pareto distribution through a functional equation. Later Arnold [2,19] extends Pillai's result to univariate semi-logistic distribution through some functional equations analogous to Eqs. (2.2), (2.5), (2.8) and (2.10) in this paper. Yeh [20] studies the multivariate semi-Weibull distribution through the functional Eq. (2.1) in [20] which is more closely related to the multivariate semi-logistic distributions developed in this paper. The moments, the covariance structures and the estimation problems of the various multivariate semi-logistic distributions studied in this paper may be the author's further research topics in the near future.

2. Three multivariate semi-logistic distributions

Three more general MSL than those proposed by Gumbel [1]. Arnold [2], as well as Kotz, Balakrishnan and Johnson [4] (ch. 51 and the references therein) are introduced in this section.

Definition 2.1. A random vector $\underline{X} = (X_1, X_2, \ldots, X_k)$ is said to have a k-variate semi-logistic distribution with parameters $p \in (0, 1)$, $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k) > \underline{0}$, if its joint cumulative distribution function (cdf) is of the form

$$F(\underline{x}) = \frac{1}{1 + \varphi(\underline{x})}, \quad \underline{x} \in R^k, \tag{2.1}$$

where $\varphi(\cdot)$ is nonincreasing and right continuous k-valued real function and satisfies

$$\varphi(\underline{x}) = \frac{1}{p}\varphi(\underline{x} - \underline{\sigma}\ \ln\ p) \tag{2.2}$$

with the point $\underline{x} - \underline{\sigma}\ \ln\ p = (x_1 - \sigma_1\ \ln\ p, \ldots, x_k - \sigma_k\ \ln\ p)$, and $\underline{X}$ is denoted by $\underline{X} \sim \text{MSL}^{(1)}(p, \underline{\sigma})$, where ln represents nature logarithm and throughout this paper.

Remark 1. The simplest and best behaved solution to Eq. (2.2) is $\varphi(\underline{x}) = \sum_{i=1}^{k} e^{-x_i/\sigma_i}$ and thus the joint cdf of $\underline{X}$ is

$$F(\underline{x}) = \frac{1}{1 + \sum_{i=1}^{k} e^{-x_i/\sigma_i}}, \quad \underline{x} \in R^k, \tag{2.3}$$

which reduces back to Gumbel's [1] ML distribution.

Remark 2. The general solution of the functional equation (2.2) is given by

$$\varphi(\underline{x}) = \sum_{i=1}^{k} e^{-x_i/\sigma_i} h_i(x_i),$$

where $h_i(x_i)$ are periodic functional in x_i with period $\sigma_i(\ln p)$ respectively for $1 \leqslant i \leqslant k$. A proof of this result can be found in [21] p. 163.

As an example initiated by Pillai [12], if we consider the bivariate case, take $p = e^{-2\pi}$, $h_i(x_i) = e^{\beta(\text{Cos}(x_i/\sigma_i))}$ $i = 1, 2$, then we see that it satisfies Eq. (2.2) for bivariate vector $\underline{x} = (x_1, x_2)$. Fig. 2.1 presents the probability density function (denoted by pdf) and contours of the bivariate semi-logistic distribution (denoted by $\text{BSL}^{(1)}$) ($p = e^{-2\pi}$, $\underline{\sigma} = (\sigma_1, \sigma_2)$) for various values of σ_1, σ_2 and β. The cdf of $\text{BSL}^{(1)}$ is

$$F(x_1, x_2) = \frac{1}{1 + e^{-x_1/\sigma_1} h_1(x_1) + e^{-x_2/\sigma_2} h_2(x_2)},$$

where $h_i(x_i) = e^{\beta(\text{Cos}(x_i/\sigma_i))}$, $i = 1, 2$, $\underline{x} = (x_1, x_2) \in R^2$.

In particular if we choose $h_i(x_i) = 1$, $i = 1, 2$, then $\text{BSL}^{(1)}(p, \underline{\sigma})$ reduces to Gumbel's [1] bivariate logistic distribution with cdf as

$$F(\underline{x}) = \frac{1}{1 + e^{-x_1/\sigma_1} + e^{-x_2/\sigma_2}}, \quad \underline{x} \in R^2.$$

Density plots and contours of Gumbel's bivariate logistic distribution for various values of σ_1 and σ_2 are presented in Fig. 2.2.

The second MSL is given below:

Definition 2.2. A random vector $\underline{X}$ is said to have the MSL of the second type if its joint survival function is of the form

$$\overline{F}(\underline{x}) = \frac{1}{1 + \psi(\underline{x})}, \quad \underline{x} \in R^k, \tag{2.4}$$

where $\psi(\cdot)$ is nondecreasing, right continuous k-valued real function and satisfies

$$\psi(\underline{x}) = \frac{1}{p} \psi(\underline{x} + \underline{\sigma} \ln p) \tag{2.5}$$

for some $\underline{\sigma} \geqslant \underline{0}$ and the point $\underline{x} + \underline{\sigma} \ln p = (x_1 + \sigma_1 \ln p, \ldots, x_k + \sigma_k \ln p)$, then $\underline{X}$ is denoted by $\underline{X} \sim MSL^{(2)}(p, \underline{\sigma})$.

Remark 3. The simplest solution to Eq. (2.5) is of the form $\psi(\underline{x}) = \sum_{i=1}^{k} e^{x_i/\sigma_i}$; then the joint survival function of $\underline{X}$ is

$$\overline{F}(\underline{x}) = \frac{1}{1 + \sum\limits_{i=1}^{k} e^{x_i/\sigma_i}}, \quad \underline{x} \in R^k \tag{2.6}$$

which is the special case of Arnold's [2] ML distribution.

Remark 4. Analogous discussions as in Remark 2, the general solution of the functional equation (2.5) is given by $\psi(\underline{x}) = \sum_{i=1}^{k} e^{x_i/\sigma_i} h_i(x_i)$, where $h_i(x_i)$ are the same as in Remark 2. If we take $p = e^{-2\pi}$, $h_i(x_i) = e^{\beta(\text{Cos}(x_i/\sigma_i))} i = 1, 2$, then it is clear that the particular choice of $h_i(\cdot)$ satisfies Eq. (2.5). Fig. 2.3 is the probability density function and contours of the bivariate semi-logistic distribution (denoted by $\text{BSL}^{(2)}$) ($p = e^{-2\pi}$, $\underline{\sigma}$) for various values of σ_1, σ_2 and β.

The survival function of $\text{BSL}^{(2)}$ is

$$\overline{F}(\underline{x}) = \frac{1}{1 + e^{x_1/\sigma_1} h_1(x_1) + e^{x_2/\sigma_2} h_2(x_2)},$$

where $h_i(x_i) = e^{\beta(\text{Cos}(x_i/\sigma_i))}$, $i = 1, 2$, $\underline{x} = (x_1, x_2) \in R^2$.

In particular if we choose $h_i(x_i) = 1$, $i = 1, 2$, then $\text{BSL}^{(2)}(p, \underline{\sigma})$ reduces to Arnold's [2] bivariate logistic distribution with survival function as

$$\overline{F}(\underline{x}) = \frac{1}{1 + e^{x_1/\sigma_1} + e^{x_1/\sigma_2}}, \quad \underline{x} \in R^2.$$

Density plots and contours of Arnold's bivariate logistic distribution for various values of σ_1 and σ_2 are presented in Fig. 2.4.

Figs. 2.1–2.4 are presented in the Appendix.

Remark 5. (i) According to Arnold's [19] definition, if Eqs. (2.2) and/or (2.5) hold for one particular value of $p \in (0, 1)$, then this two types of distribution (2.1) and (2.4) are the so called multivariate semi-logistic distributions, $\text{MSL}^{(1)}$ and $\text{MSL}^{(2)}$ respectively (Pillai [12]) introduced closely related semi-Pareto distributions as well as for the bivariate semi-Pareto distributions [13].

(ii) If Eq. (2.2) and/or (2.5) hold for every $p \in (0, 1)$, or for two distinct values of p, say p_1 and p_2 provided that $\{p_1^j/p_2^k; j = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots\}$ is dense in R^+, then $\text{MSL}^{(1)}$ will reduce to Gumbel's [1] ML as in Eq. (2.3) and $\text{MSL}^{(2)}$ will reduce to Arnold's [2] ML as in Eq. (2.6) respectively.

(iii) The result of (ii) is cited from Arnold [19] p. 140. The main idea is drawn from Galambos and Kotz [22] p.38 or from a well-known result in number theory that the set $E = \{v | v = i \ln n_1 + j \ln n_2, i, j = 0, \pm 1, \pm 2, \dots\}$ is dense in the real line **R**, if $n_1, n_2 \in N, n_1 \neq n_2, \ln n_1 / \ln n_2$ is irrational ([23] p. 4), then the set $F \triangleq \exp(E) = \{e^v = n_1^i \cdot n_2^j | i, j \in$ **I** (integer set)$\}$ is dense in the positive real line $\mathbf{R}^+$.

These two types of MSL distributions $\mathrm{MSL}^{(1)}$ and $\mathrm{MSL}^{(2)}$ have the following property.

Property 2.1. *Suppose that* $\underline{X} \sim MSL^{(1)}(p, \underline{\sigma})$*, then the necessary and sufficient condition for* $\underline{X} \sim MSL^{(2)}(p, \underline{\sigma})$ *is that the two functional equations (2.2) and (2.5) have the radial symmetric property about* $\underline{0} \in R^k$*, i.e.,* $\varphi(\underline{x}) = \psi(-\underline{x})$ *for* $\underline{x} \in R^k$.

Note: The proof of this property is straightforward and hence is omitted.

Remark 6. Both of the $MSL^{(1)}(p, \underline{\sigma})$ and $\mathrm{MSL}^{(2)}(p, \underline{\sigma})$ have the univariate semi-logistic distributions [19] as marginals and are denoted by $\mathrm{SL}^{(1)}(p, \sigma_i)$, $\mathrm{SL}^{(2)}(p, \sigma_i)$ respectively $i = 1, 2, \dots, k$, i.e., the ith marginal d.f of Eq. (2.1) is

$$F_i(x_i) = \frac{1}{1 + \varphi_i(x_i)}, \quad x_i \in R, \tag{2.7}$$

where $\varphi_i(x_i) = \varphi(\infty, \dots, \infty, x_i, \infty, \dots, \infty)$ and $\varphi_i(\cdot)$ satisfies

$$\varphi_i(x_i) = \frac{1}{p}\varphi_i(x_i - \sigma_i \ln p) \quad \text{for some } p \in (0, 1), \tag{2.8}$$

and the X_i in $\underline{X}$ is denoted by $X_i \sim \mathrm{SL}^{(1)}(p, \sigma_i)$.

Analogously, the ith survival function of Eq. (2.4) is

$$\overline{F}_i(x_i) = \frac{1}{1 + \psi_i(x_i)}, \quad x_i \in R, \tag{2.9}$$

where $\psi_i(x_i) = \psi(-\infty, \dots, -\infty, x_i, -\infty, \dots, -\infty)$ and $\psi_i(\cdot)$ satisfies

$$\psi_i(x_i) = \frac{1}{p}\psi_i(x_i + \sigma_i \ln p) \quad \text{for some } p \in (0, 1), \tag{2.10}$$

and X_i in $\underline{X}$ is denoted by $X_i \sim \mathrm{SL}^{(2)}(p, \sigma_i)$.

The univariate semi-logistic distribution was first briefly introduced by Arnold [19]. Its cdf is of the form Eqs. (2.9) and (2.10). Arnold et al. [18] studied the characterizations of the univariate Pareto and univariate logistic distributions. Some of their results can be parallelly extended to the MSL case, if we impose some conditions on the marginal distribution of X_i in $\underline{X}$, then a potentially rich and more general collection of MSL distribution is defined as follows which is the third MSL introduced in this section.

Definition 2.3. A random vector $\underline{X} = (X_1, X_2, \dots, X_k)$ is said to have a general multivariate semi-logistic (denoted by GMSL) distribution if its k marginals of each X_i in $\underline{X}$ are both $\mathrm{SL}^{(1)}(p, \sigma_i)$ and $\mathrm{SL}^{(2)}(p, \sigma_i)$ distributed, i.e., the cdf of each X_i is of the form

$$F_i(x_i) = \frac{1}{1 + \varphi_i(x_i)} = 1 - \overline{F}_i(x_i) = 1 - \frac{1}{1 + \psi_i(x_i)}, \quad x_i \in R, \tag{2.11}$$

or equivalently $\varphi_i(x_i) = 1/\psi_i(x_i), i = 1, \dots, k$, where $\varphi_i(\cdot)$ satisfies Eq. (2.8), and $\psi_i(\cdot)$ satisfies Eq. (2.10), then $\underline{X}$ is denoted by $\underline{X} \sim \mathrm{GMSL}(p, \underline{\sigma})$ for some $p \in (0, 1)$, and $\underline{\sigma} = (\sigma_1, \dots, \sigma_k) > \underline{0}$.

3. Geometric maxima and minima

In this section, we consider the first two types of MSL, i.e., $\mathrm{MSL}^{(1)}$ and $\mathrm{MSL}^{(2)}$ and study their geometric maxima and geometric minima respectively. Conversely, the closure property under geometric maximization and geometric minimization can be utilized to characterize the $\mathrm{MSL}^{(1)}$ and $\mathrm{MSL}^{(2)}$ respectively.

Theorem 3.1. *Let* $\{\underline{X}^i = (X_1^i, X_2^i, \dots, X_k^i)\}_1^\infty$ *be a sequence of i.i.d. random vectors with common joint cdf as* $F(\cdot)$*. Suppose that N is a geometric random variable with pmf as* $P(N = i) = p(1-p)^{i-1}, i = 1, 2, \dots,$ *and N is independent of* $\underline{X}^i$*'s. Let* $\underline{M} = (X^{(1)}, \dots, X^{(k)})$ *be the componentwise k-variate geometric maxima with* $X^{(j)} = \max\{X_j^1, X_j^2, \dots, X_j^N\}$, $1 \leqslant j \leqslant k$*. Then the following two statements are equivalent.*

$$(1)\ \underline{M} + \underline{\sigma} \ln p \overset{d}{=} \underline{X}^1. \tag{3.1}$$

(2) $\underline{X}^1 \sim \mathrm{MSL}^{(1)}(p, \underline{\sigma})$.

Proof. (1)⇒(2):

Suppose $G(\cdot)$ is the joint cdf of $\underline{M}+\underline{\sigma}\ln p$, then

$$G(\underline{x}) = P(\underline{M}+\underline{\sigma}\ln p \leqslant \underline{x}) = P(\underline{M} \leqslant \underline{x}-\underline{\sigma}\ln p)$$
$$= \sum_{i=1}^{\infty}(F(\underline{x}-\underline{\sigma}\ln p))^{i} p(1-p)^{i-1} = \frac{pF(\underline{x}-\underline{\sigma}\ln p)}{1-(1-p)F(\underline{x}-\underline{\sigma}\ln p)} = F(\underline{x}). \tag{3.2}$$

Express $\varphi(\underline{x}) = (1-F(\underline{x}))/F(\underline{x})$, then $F(\underline{x}) = 1/(1+\varphi(\underline{x}))$, substitute in Eq. (3.2) then

$$\frac{p\,\frac{1}{1+\varphi(\underline{x}-\underline{\sigma}\ln p)}}{1-(1-p)\,\frac{1}{1+\varphi(\underline{x}-\underline{\sigma}\ln p)}} = \frac{1}{1+\varphi(\underline{x})},$$

after simplification, we have

$$\frac{1}{1+\frac{1}{p}\varphi(\underline{x}-\underline{\sigma}\ln p)} = \frac{1}{1+\varphi(\underline{x})},$$

thus $\varphi(\cdot)$ satisfies Eq. (2.2), i.e., $\varphi(\underline{x}) = (1/p)\,\varphi(\underline{x}-\underline{\sigma}\ln p)$ for $0<p<1$, and $\forall \underline{x}\in R^k$. This is just the definition of $MSL^{(1)}(p,\underline{\sigma})$ in Definition 2.1. Thus (2) follows.

(2)⇒(1):

Let $G(\cdot)$ be the joint cdf of $\underline{M}+\underline{\sigma}\ln p$, then $G(\underline{x}) = P(\underline{M}+\underline{\sigma}\ln p\leqslant\underline{x}) = \sum_{i=1}^{\infty}(F(\underline{x}-\underline{\sigma}\ln p))^{i-1}p(1-p)^{i-1} = \frac{pF(\underline{x}-\underline{\sigma}\ln p)}{1-(1-p)F(\underline{x}-\underline{\sigma}\ln p)}$, by the assumption of (2), then

$$\frac{p\,\frac{1}{1+\varphi(\underline{x}-\underline{\sigma}\ln p)}}{1-(1-p)\,\frac{1}{1+\varphi(\underline{x}-\underline{\sigma}\ln p)}} = \frac{1}{1+\frac{1}{p}\varphi(\underline{x}-\underline{\sigma}\ln p)} = \frac{1}{1+\varphi(\underline{x})} = F(\underline{x}).$$

Hence $\underline{M}+\underline{\sigma}\ln p \overset{d}{=} \underline{X}^1$ follows. ■

Remark 1. Follow similar discussions as Arnold [2], if Eq. (3.1) holds for every $p\in(0,1)$, then the common joint distribution of $\underline{X}^1$ must be Gumbel's [1] ML as in Eq. (2.3), if Eq. (3.1) holds for two distinct values of p, say p_1 and p_2 such that $\{p_1^j/p_2^k; j,k=0,1,2,\ldots\}$ is dense in R^+ then the common distribution of $\underline{X}^1$ is Gumbel's ML.

As for the geometric minima, there is an analogous characterization theorem developed as follows:

Theorem 3.2. *Let $\{\underline{X}^i = (X_1^i, X_2^i,\ldots,X_k^i)\}_1^\infty$ be a sequence of i.i.d. random vectors with common joint survival function as $\overline{F}(\cdot)$, suppose N is a geometric random variable with pmf $P(N=i)=p(1-p)^{i-1}$, $i=1,2,\ldots$, and N is independent of $\underline{X}^i$'s. Let $\underline{m}=(X_{(1)},X_{(2)},\ldots,X_{(k)})$ be the componentwise k-variate geometric minima with $X_{(j)}=\min\{X_j^1,X_j^2,\ldots,X_j^N\}$, $j=1,2,\ldots,k$. Then the following two statements are equivalent.*

(1) $\underline{m}-\underline{\sigma}\ln p \overset{d}{=} \underline{X}^1$. (3.3)

(2) $\underline{X}^1 \sim MSL^{(2)}(p,\underline{\sigma})$.

Note: The proof of Theorem 3.2 is similar to that of Theorem 3.1 by just considering the common survival function of $\underline{X}^1$ as $\overline{F}(\underline{x}) = 1/(1+\psi(\underline{x}))$, where $\psi(\underline{x}) = (1/p)\,\psi(\underline{x}+\underline{\sigma}\ln p)$, and hence is omitted.

Remark 2. Similar comments in the Remark 1, if Eq. (3.3) holds for every $p\in(0,1)$, or for two distinct values of p, say p_1, p_2 such that $\{p_1^j/p_2^k; j,k=0,1,2,\ldots\}$ is dense in R^+, then the common joint survival function of $\underline{X}^1$ must be Arnold's [2] ML.

The explanations of Remarks 1 and 2 are given in Remark 5 of (iii) on p. 7 in Section 2.

Remark 3. (i) If compare Eq. (3.2) with the equation in Section 1 of Marshall and Olkin [24] (denoted by M–O). the one parameter α in M–O is restricted to $\alpha>0$ which is different from the parameter p with $0<p<1$ in this paper.

(ii) A more close relationship between Eqs. (3.1)–(3.3) of this paper is in Eq. (3.2) of Rachev and Resnick [25] (denoted by R–R) where the geometric maxima stability is defined. Yeh [17] extends their result to geometric minima stability.

(iii) From Eq. (3.1) of R–R and Eq. (3.1) of this paper, it is discerned that $MSL^{(1)}(p,\underline{\sigma})$ possesses the geometric maxima stability. From Eq. (5.1) of Yeh [17] and Eq. (3.3) of this paper, it is found that $\mathrm{MSL}^{(2)}(p,\underline{\sigma})$ possesses the geometric minima stable property. Marshall and Olkin [24] also briefly studied the geometric extreme stability for univariable and bivariate variables in Sections 5 and 6 of their paper.

Theorem 3.1 can be extended to any finite steps of repeated geometric maximization procedure. It is described as follows:

Suppose we have a sequence of i.i.d. k-variate random vectors with common joint cdf $F_1(\cdot)$, i.e., assuming $\{\underline{X}_i^{(1)}\}_1^\infty \overset{\text{i.i.d.}}{\sim} F_1(\cdot)$, let $N_1 \sim$ geometric (p_1), define $\underline{X}_{N_1}^{(1)} = \max_{1\leqslant j\leqslant N_1}\{\underline{X}_j^{(1)}\}$ as the k-dim. geometric maxima of $\{\underline{X}_i^{(1)}\}$, assume the joint cdf of $\underline{X}_{N_1}^{(1)}$ is $F_2(\cdot)$. Also let $\{\underline{X}_i^{(2)}\}_1^\infty \overset{\text{i.i.d.}}{\sim} F_2(\cdot)$, $N_2 \sim$ geometric (p_2), define $\underline{X}_{N_2}^{(2)} = \max_{1\leqslant j\leqslant N_2}\{\underline{X}_j^{(2)}\}$, suppose $\underline{X}_{N_2}^{(2)} \sim F_3(\cdot)$. In general, for any fixed $\ell = 2, 3, \ldots$, after $(\ell-1)$ steps of repeated geometric maximization procedures, let $\{\underline{X}_i^{(\ell-1)}\}_1^\infty \overset{\text{i.i.d.}}{\sim} F_{\ell-1}(\cdot)$, let $N_{\ell-1} \sim$ geometric $(p_{\ell-1})$, define $\underline{X}_{N_{\ell-1}}^{(\ell-1)} = \max_{1\leqslant j\leqslant N_{\ell-1}}\{\underline{X}_j^{(\ell-1)}\}$. Suppose $\underline{X}_{N_{\ell-1}}^{(\ell-1)} \sim F_\ell(\cdot)$, then the following theorem characterizes the $\text{MSL}^{(1)}$ distribution via any finite steps of repeated geometric maximization.

Theorem 3.3. *Let $\{\underline{X}_i^{(1)}\}_1^\infty$ be a sequence of i.i.d. k-variate random vectors with common joint cdf $F_1(\cdot)$. For each $\ell = 2, 3, \ldots$, define $F_\ell(\cdot)$ sequentially in such manner that $F_\ell(\cdot)$ is the joint cdf of a geometric $(p_{\ell-1})$ maxima $(0 < p_{\ell-1} < 1)$ of a random sample of $\{\underline{X}_i^{(\ell-1)}\}_1^\infty \overset{\text{i.i.d.}}{\sim} F_{\ell-1}(\cdot)$, if there exists a parameter vector $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k) > \underline{0}$ as the scale parameter of $F_1(\cdot)$, then the following two statements are equivalent:*

(1) *For each finite $\ell = 2, 3, \ldots$, we have*

$$F_\ell\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right)\underline{\sigma}\right) = F_1(\underline{x}) \tag{3.4}$$

for any $\underline{x} \in R^k$, where $(\underline{x} - (\sum_{j=1}^{\ell-1} \ln p_j)\underline{\sigma}) \overset{\Delta}{=} (x_1 - (\sum_{j=1}^{\ell-1} \ln p_j)\sigma_1, \ldots, x_k - (\sum_{j=1}^{\ell-1} \ln p_j)\sigma_k)$, or equivalently,

$$\underline{X}_{N_{\ell-1}}^{(\ell-1)} + \left(\sum_{j=1}^{\ell-1} \ln\, p_j\right)\underline{\sigma} \overset{d}{=} \underline{X}^{(1)} \sim F_1(\cdot). \tag{3.5}$$

(2) *The common joint cdf of $\{\underline{X}_i^{(1)}\}$, $F_1(\cdot)$ is a multivariate semi-logistic distribution of the first type, i.e., $\underline{X}^{(1)} \sim \text{MSL}^{(1)}(p, \underline{\sigma})$ with $p = \prod_{j=1}^{\ell-1} p_j$.*

Proof. (1)⇒(2):

If Eqs. (3.4) and (3.5) hold, then the joint cdf of the geometric $(p_{\ell-1})$ maxima $\underline{X}_{N_{\ell-1}}^{(\ell-1)}$, $F_\ell(\cdot)$ is derived as

$$\begin{aligned} F_\ell(\underline{x}) &= P(\underline{X}_{N_{\ell-1}}^{(\ell-1)} \leqslant \underline{x}) = \sum_{n=1}^{\infty} P(\max_{1\leqslant i\leqslant n} \underline{X}_i^{(\ell-1)} \leqslant \underline{x})P(N_{\ell-1} = n) \\ &= \frac{p_{\ell-1}F_{\ell-1}(\underline{x})}{1-(1-p_{\ell-1})F_{\ell-1}(\underline{x})}. \end{aligned} \tag{3.6}$$

From Eq. (3.4)

$$F_\ell(\underline{x}) = F_1\left(\underline{x} + \left(\sum_{j=1}^{\ell-1} \ln\, p_j\right)\underline{\sigma}\right), \tag{3.7}$$

let $\varphi_\ell(\underline{x}) = \frac{1-F_\ell(\underline{x})}{F_\ell(\underline{x})}$, then $F_\ell(\underline{x}) = \frac{1}{1+\varphi_\ell(\underline{x})}$ for each $\ell \geqslant 1$, substitute back to Eq. (3.6), we have

$$\frac{1}{1+\varphi_\ell(\underline{x})} = \frac{p_{\ell-1}\cdot\frac{1}{1+\varphi_{\ell-1}(\underline{x})}}{1-(1-p_{\ell-1})\cdot\frac{1}{1+\varphi_{\ell-1}(\underline{x})}}, \tag{3.8}$$

after straightforward simplification and iteration, we conclude that for all $\underline{x} \in R^k$,

$$\varphi_\ell(\underline{x}) = \left(\prod_{j=1}^{\ell-1} p_j\right)^{-1}\varphi_1(\underline{x}). \tag{3.9}$$

Refer to Eq. (3.5), we have

$$\frac{1}{1+\varphi_\ell(\underline{x})} = \frac{1}{1+\varphi_1\left(\underline{x} + \left(\sum_{j=1}^{\ell-1} \ln\, p_j\right)\underline{\sigma}\right)}, \textit{by}\ \text{Eq. (3.9)},$$

$$= \frac{1}{1 + \left(\prod_{j=1}^{\ell-1} p_j\right)^{-1} \varphi_1(\underline{x})}. \tag{3.10}$$

Thus the function $\varphi_1(\cdot)$ satisfies

$$\varphi_1(\underline{x} + (\ln p)\underline{\sigma}) = \frac{1}{p}\varphi_1(\underline{x}), \quad \text{or } \varphi_1(\underline{x}) = \frac{1}{p}\varphi_1(\underline{x} - (\ln p)\underline{\sigma}), \tag{3.11}$$

where $p \stackrel{\Delta}{=} \prod_{j=1}^{\ell-1} p_j$, and the joint cdf of $\underline{X}^{(1)}$ is $F_1(\underline{x}) = \frac{1}{1+\varphi_1(\underline{x})}$, where $\varphi_1(\cdot)$ satisfies Eq. (3.11). Therefore, $\underline{X}^{(1)} \sim MSL^{(1)}(p, \underline{\sigma})$ follows.

(2)⇒(1):

If $\underline{X}^{(1)} \sim MSL^{(1)}(p, \underline{\sigma})$, then the joint cdf of $\underline{X}^{(1)}$ is

$$F_1(\underline{x}) = \frac{1}{1 + \varphi(\underline{x})}, \quad \text{where } \varphi(\underline{x}) = \frac{1}{p}\varphi(\underline{x} - \ln p\underline{\sigma}) \tag{3.12}$$

for all $\underline{x} \in R^k$ with $p = \prod_{j=1}^{\ell-1} p_j$. The joint cdf of the shifted geometric maxima $\underline{X}_{N_{\ell-1}}^{(\ell-1)} + (\sum_{j=1}^{\ell-1} \ln p_j)\,\underline{\sigma}$ is for any $\underline{x} \in R^k$,

$$P(\underline{X}_{N_{\ell-1}}^{(\ell-1)}) + \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma} \leqslant (\underline{x}) = P\left(\underline{X}_{N_{\ell-1}}^{(\ell-1)} \leqslant \underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right)$$

$$= F_\ell\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right). \tag{3.13}$$

By Eq. (3.13), we have

$$F_\ell\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right) = \frac{p_{\ell-1}F_{\ell-1}\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right)}{1 - (1 - p_{\ell-1})F_{\ell-1}\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right)}. \tag{3.14}$$

For any $\ell \geqslant 1$, let $\varphi_\ell(\underline{x}) = \frac{1-F_\ell(\underline{x})}{F_\ell(\underline{x})}$, then $F_\ell(\underline{x}) = \frac{1}{1+\varphi_\ell(\underline{x})}$, substitute in Eq. (3.14), we have

$$\varphi_\ell\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right) = \frac{1}{p_{\ell-1}}\varphi_{\ell-1}\left(\underline{x} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right) \underline{\sigma}\right).$$

It follows by iteration that

$$\varphi_\ell(\underline{x} - (\ln p)\underline{\sigma}) = \frac{1}{p}\varphi_1(\underline{x} - (\ln p)\underline{\sigma}) \tag{3.15}$$

with $\ln p = \sum_{j=1}^{\ell-1} \ln p_j$. From Eq. (3.13), the joint cdf of $\underline{X}_{N_{\ell-1}}^{(\ell-1)} + (\ln p)\,\underline{\sigma}$ is

$$F_\ell(\underline{x} - (\ln p)\underline{\sigma}) = \frac{1}{1 + \varphi_\ell(\underline{x} - (\ln p)\underline{\sigma})} = \frac{1}{1 + \frac{1}{p}\varphi_1(\underline{x} - (\ln p)\underline{\sigma})}, \tag{3.16}$$

while the joint cdf of $\underline{X}^{(1)}$ is $F_1(\underline{x}) = \frac{1}{1+\varphi_1(\underline{x})} = \frac{1}{1+\varphi(\underline{x})}$, hence $\varphi_1(\underline{x}) \equiv \varphi(\underline{x})$ for all $\underline{x} \in R^k$, so the function $\varphi_1(\cdot)$ will also satisfy $\frac{1}{p}\,\varphi_1(\underline{x} - (\ln p)\,\underline{\sigma}) = \varphi_1(\underline{x})$.

Hence Eq. (3.16) becomes

$$F_\ell(\underline{x} - (\ln p)\,\underline{\sigma}) = \frac{1}{1 + \varphi_1(\underline{x})} = \frac{1}{1 + \varphi(\underline{x})} = F_1(\underline{x}).$$

Thus, $\underline{X}_{N_{\ell-1}}^{(\ell-1)} + (\ln p)\,\underline{\sigma} \stackrel{d}{=}; \underline{X}^{(1)}$, and hence (1) follows. ■

Analogously, Theorem 3.2 can also be extended to any finite steps of repeated geometric minimization procedure. It is stated as follows:

Theorem 3.4. *Let $\{\underline{Y}_i^{(1)}\}_1^\infty$ be a sequence of i.i.d. k-variate random vectors with common joint survival function $\overline{F}_1(\cdot)$. For each $\ell = 2, 3, \ldots$, define $\overline{F}_\ell(\cdot)$ sequentially in such manner that $\overline{F}_\ell(\cdot)$ is the joint survival function of a geometric $(p_{\ell-1})$ minima $(0 < p_{\ell-1} < 1)$, denoted by $\underline{Y}_{N_{\ell-1}}^{(\ell-1)} = \min_{1\leqslant j\leqslant N_{\ell-1}}\{\underline{Y}_j^{(\ell-1)}\}$ from a random sample of $\{\underline{Y}_i^{(\ell-1)}\}_1^\infty \overset{\text{i.i.d.}}{\sim} \overline{F}_{\ell-1}(\cdot)$, if there exists a parameter vector $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k) > \underline{0}$ as the scale parameter of $\overline{F}_1(\cdot)$, then the following two statements are equivalent:*

(1) *For each finite $\ell = 2, 3, \ldots$, we have*

$$\overline{F}_\ell\left(\underline{x} + \left(\sum_{j=1}^{\ell-1} \ln p_j\right)\underline{\sigma}\right) = \overline{F}_1(\underline{x}) \tag{3.17}$$

for any $\underline{x} \in R^k$, where $(\underline{x} + (\sum_{j=1}^{\ell-1} \ln p_j)\underline{\sigma}) \overset{\Delta}{=} (x_1 + (\sum_{j=1}^{\ell-1} \ln p_j)\sigma_1, \ldots, x_k + (\sum_{j=1}^{\ell-1} \ln p_j)\sigma_k)$, or equivalently,

$$\underline{Y}_{N_{\ell-1}}^{(\ell-1)} - \left(\sum_{j=1}^{\ell-1} \ln p_j\right)\underline{\sigma} \overset{d}{=} \underline{Y}^{(1)} \sim \overline{F}_1(\cdot). \tag{3.18}$$

(2) *The common joint survival function of $\{\underline{Y}_i^{(1)}\}$, $\overline{F}_1(\cdot)$ is a multivariate semi-logistic distribution of the second type, i.e., $\underline{Y}^{(1)} \sim MSL^{(2)}(p, \underline{\sigma})$ with $p = \prod_{j=1}^{\ell-1} p_j$.*

Note: The proof of Theorem 3.4 is analogous to Theorem 3.3, and hence is omitted.

Remark. The analogous results of Theorems 3.3 and 3.4 can be found in [16] Theorem 4.2.

4. Characterization of the GMSL distribution

The GMSL defined in Definition 2.3 has the $SL^{(1)}$ and $SL^{(2)}$ as marginal distributions. The characterization of GMSL is based on both random geometric maximization and geometric minimization procedures which is the extension work of Arnold et al. [18].

Let $\{\underline{X}_{ij}\}$ be a double array of i.i.d. k-variate random vectors with common joint cdf $F(\cdot)$, and let $N_1 \sim$ geometric (p_1) and $N_2^{(i)}$ $(i = 1, 2, \ldots)$ be i.i.d. geometric (p_2) random variables independent of the $\underline{X}_{ij}$'s.

Define two k-variate random vectors as

$$\underline{W} = \min_{i\leqslant N_1} \max_{j\leqslant N_2^{(i)}} \underline{X}_{ij} = (W_1, \ldots, W_k), \tag{4.1}$$

and

$$\underline{\tilde{W}} = \max_{i\leqslant N_1} \min_{j\leqslant N_2^{(i)}} \underline{X}_{ij} = (\tilde{W}_1, \ldots, \tilde{W}_k). \tag{4.2}$$

Such two random vectors might arise in competition for employment models, and W_ℓ and $\tilde{W}_\ell$ are the ℓth $(1 \leqslant \ell \leqslant k)$ marginal of the k-variate random vectors $\underline{W}$ and $\underline{\tilde{W}}$ respectively.

For $j = 1, 2, \ldots, k$, let $\overline{F}_j(\cdot)$ be the jth marginal survival function of $F(\cdot)$, and the jth marginal survival function of $\underline{W}$, $\underline{\tilde{W}}$ are calculated straightforward as

$$\overline{F}_{W_j}(x) = \frac{p_1\overline{F}_j(x)}{p_2 + (p_1 - p_2)\overline{F}_j(x)}, \tag{4.3}$$

and

$$\overline{F}_{\tilde{W}_j}(x) = \frac{p_2\overline{F}_j(x)}{p_1 + (p_2 - p_1)\overline{F}_j(x)}. \tag{4.4}$$

The following theorem is the characterization of the GMSL distribution.

Theorem 4.1. *Suppose $\{\underline{X}_{ij}\}$ is a sequence double array of i.i.d. k-variate random vectors and $\{\underline{X}_{ij}\} \overset{\sim}{\text{i.i.d.}} F(\underline{x})$, and the two random vectors $\underline{W}$ and $\underline{\tilde{W}}$ are defined as in Eqs. (4.1) and (4.2), then the following two statements are equivalent.*

(1) *The equality in marginal distributions of any two of the three random vectors $\{\underline{X}_{11}, \underline{W} + \underline{\sigma} \ln(\frac{p_2}{p_1}), \underline{\tilde{W}} + \underline{\sigma} \ln(\frac{p_1}{p_2})\}$ for some p_1, p_2 with $p_1 \neq p_2$, $0 < p_1, p_2 < 1$.*
(2) *The common joint cdf $F(\cdot)$ is GMSL.*

Proof. (1)⇒(2):

(i) If $\{\underline{X}_{11}$ and $\underline{W} + \underline{\sigma}\ \ln(p_2/p_1)\}$ are identical in marginal distributions, i.e.,

$$X_{11j} \overset{d}{=} W_j + \sigma_j \ln\left(\frac{p_2}{p_1}\right), \quad j = 1, 2, \ldots, k, \tag{4.5}$$

where X_{11j}, W_j, σ_j are the jth component in $\underline{X}_{11}$, $\underline{W}$, and $\underline{\sigma}$ separately. Let $\overline{F}_j(\cdot)$, $F_j(\cdot)$ be the survival function and cdf of X_{11j} respectively. If by usual notation, express $\psi_j(x) = \frac{F_j(x)}{\overline{F}_j(x)}$, then $\overline{F}_j(x) = \frac{1}{1+\psi_j(x)}$ and denote $\varphi_j(x) = \frac{1-F_j(x)}{F_j(x)}$, then $F_j(x) = \frac{1}{1+\varphi_j(x)} = 1 - \overline{F}_j(x) = \frac{1}{1+\frac{1}{\psi_j(x)}}$. Hence the relation between $\psi_j(\cdot)$ and $\varphi_j(\cdot)$ is

$$\psi_j(x) = \frac{1}{\varphi_j(x)}, \quad x \in \mathbf{R}. \tag{4.6}$$

By the assumption (4.5) and Eq. (4.3), we have

$$P(W_j + \sigma_j \ln(p_2/p_1) \geqslant x) = \frac{1}{1 + \psi_j(x)} \quad \text{for } x \in \mathbf{R}. \tag{4.7}$$

On the other hand, the LHS of Eq. (4.7) is

$$\begin{aligned} p\,(W_j \geqslant x - \sigma_j \ln(p_2/p_1)) &= \frac{p_1\{\frac{1}{1+\psi_j(x-\sigma_j \ln(p_2/p_1))}\}}{p_2 + (p_1 - p_2)\{\frac{1}{1+\psi_j(x-\sigma_j \ln(p_2/p_1))}\}} \\ &= \frac{1}{1 + p_2/p_1 \psi_j(x - \sigma_j \ln(p_2/p_1))}. \end{aligned} \tag{4.8}$$

Comparing Eqs. (4.7) and (4.8), we have $\psi_j(x) = \frac{p_2}{p_1}\psi_j(x - \sigma_j \ln(p_2/p_1))$, or equivalently,

$$\psi_j(x) = \frac{p_1}{p_2}\psi_j(x + \sigma_j \ln(p_2/p_1)). \tag{4.9}$$

Without loss of generality, assuming $p_1 < p_2$ and let $p = p_1/p_2 (< 1)$, Eq. (4.9) is equivalent to that there exists $0 < p < 1$, such that $\psi_j(\cdot)$ satisfies

$$\psi_j(x) = \frac{1}{p}\psi_j(x + \sigma_j \ln\ p) \tag{4.10}$$

which is Eq. (2.10). Also, because $\varphi_j(x) = 1/\psi_j(x)$, so the reciprocal of Eq. (4.10) is

$$\varphi_j(x) = p\varphi_j(x + \sigma_j \ln\ p). \tag{4.11}$$

Eq. (4.11) is equivalent to

$$\varphi_j(x) = \frac{1}{p}\varphi_j(x - \sigma_j \ln\ p), \tag{4.12}$$

which is Eq. (2.8). Thus $\underline{X} \sim$ GMSL follows.

(ii) Analogously, if $\{\underline{X}_{11}$ and $\underline{\tilde{W}} + \underline{\sigma}\ \ln(p_1/p_2)\}$ are marginal identical distributed, then follow the similar discussions as in (i) and by Eq. (4.4), we obtain

$$\psi_j(x) = \frac{p_2}{p_1}\psi_j(x + \sigma_j \ln(p_1/p_2)). \tag{4.13}$$

Eq. (4.13) is also equivalent to Eq. (4.10), hence $\underline{X} \sim \mathit{GMSL}$ holds.

(iii) If $\{\underline{W} + \underline{\sigma}\ \ln(p_2/p_1)$ and $\underline{\tilde{W}} + \underline{\sigma}\ \ln(p_1/p_2)\}$ are identical in marginal distributions, i.e., for $j = 1, 2, \ldots, k$, $W_j + \sigma_j \ln(p_2/p_1) =^d \tilde{W}_j + \sigma_j \ln(p_1/p_2)$.

To apply Eqs. (4.3) and (4.4) and after some algebraic calculations, we have

$$\frac{p_2}{p_1}\psi_j(x - \sigma_j \ln(p_2/p_1)) = \frac{p_1}{p_2}\psi_j(x - \sigma_j \ln(p_1/p_2)). \tag{4.14}$$

Eq. (4.14) is equivalent to

$$\psi_j(x) = \left(\frac{p_1}{p_2}\right)^2 \psi_j\left(x + \sigma_j \ln\left(\frac{p_2}{p_1}\right)^2\right). \tag{4.15}$$

Without loss of generality, assuming $p_1 > p_2$ and let $p = (p_2/p_1)^2$, then Eq. (4.15) is

$$\psi_j(x) = \frac{1}{p}\psi_j(x + \sigma_j \ln\, p) \tag{4.16}$$

which is Eq. (2.10). Thus $\underline{X} \sim GMSL$, therefore, (1) implies (2) follows.

To prove (2)⇒(1):

If $\underline{X}_{11} \sim GMSL$, then according to the Definition 2.3, the marginal distribution of $\underline{X}_{11}$ is both $SL^{(1)}(p, \sigma_i)$ and $SL^{(2)}(p, \sigma_i)$ distributed. We have to check that

$$X_{11j} \stackrel{d}{=} W_j + \sigma_j \ln\,(p_2/p_1) \stackrel{d}{=} \tilde{W}_j + \sigma_j \ln\,(p_1/p_2) \tag{4.17}$$

for $j = 1, 2, \ldots, k$.

(i) To check

$$X_{11j} \stackrel{d}{=} W_j + \sigma_j \ln\,(p_2/p_1). \tag{4.18}$$

From Eqs. (4.3) and (4.8), and by the assumption that the survival function of X_{11j} is $\overline{F}_j(x) = \frac{1}{1+\psi_j(x)}$, and $\psi_j(x)$ satisfies Eq. (4.10), thus Eq. (4.18) holds.

(ii) The derivations of $X_{11j} \stackrel{d}{=} \tilde{W}_j + \sigma_j \ln\,(p_1/p_2)$ and $W_j + \sigma_j \ln\,(p_2/p_1) \stackrel{d}{=} \tilde{W}_j + \sigma_j \ln\,(p_2/p_1)$ are analogous to the procedure in (i). Thus (2) implies (1) follows. ■

Remarks. Some interesting and remarkable observations inherent in the expressions of Eqs. (4.3) and (4.4) and the content in Theorem 4.1 include:

(i) According to Eqs. (4.3) and (4.4), it is discerned that the marginal distributions of $\underline{W}$ and $\underline{\tilde{W}}$ are of the same form.
(ii) If $\underline{\tilde{W}}$ were defined in Eq. (4.2) with parameters p_2 and p_1, instead of p_1 and p_2, it would have exactly the same marginal distributions as $\underline{W}$ (defined by parameters p_1 and p_2).
(iii) The most remarkably, if $p_1 = p_2$, then the three random vectors $\underline{X}_{11}$, $\underline{W}$, and $\underline{\tilde{W}}$ have identical marginal distributions.
(iv) The statement (1) in Theorem 4.1 with $p_1 \neq p_2$ can be used to characterize GMSL distribution. However if $p_1 = p_2$, we do not have a characterization i.e., any distribution will provide a solution.
(v) Alternatively, if we have equality in distribution of any two of $\{X_{11j}, W_j + \sigma_j \ln\,(p_2/p_1), \tilde{W}_j + \sigma_j \ln\,(p_1/p_2)\}$ for every $p_1 \neq p_2$ or for two suitably chosen pairs (p_1, p_2) and $(\tilde{p}_1, \tilde{p}_2)$, we may conclude that the common distribution of $\underline{X}_{11}$ is the general multivariate logistic distribution with the usual traditional univariate logistic distribution as marginal distribution.

5. The multivariate logistic distribution in multiple logistic regression

The most well-known application of multivariate logistic distribution is the multiple logistic regression model [26], which is defined as follows:

Definition 5.1. Suppose $\{Y_i\}$ are independent Bernoulli random variables with expected values $E(Y_i) = \pi_i$, the multiple logistic response function is $E(Y) = \frac{1}{1+e^{-\sum_{i=1}^k \beta_i x_i}}$, the logit transformation $\pi_i' = \ln(\frac{\pi_i}{1-\pi_i}) = \sum_{i=1}^k \beta_i x_i$, is a multiple linear regression model with no intercept where X_i, $(1 \leqslant i \leqslant k)$ observations are considered to be known constants. Alternatively, if the X variables are random, then E(Y) is viewed as a conditional mean $E(Y \mid \underline{X} = \underline{x})$.

Let $\underline{X} = (X_1, X_2, \ldots, X_k)$, if the cdf of $\underline{X}$ is of the form

$$F(\underline{x}) = \frac{1}{1 + e^{-\sum_{i=1}^{k} \beta_i x_i}}, \quad \underline{x} \in R^k, \tag{5.1}$$

then $F(\cdot)$ is referred to as the logit multivariate logistic distribution (denoted by LML). It is clear to see that LML is radial symmetric about $\underline{0}$, i.e., $F(\cdot)$ satisfies $F(-\underline{x}) = 1 - F(\underline{x})$ for all $\underline{x} \geqslant \underline{0}$.

Galambos and Kotz [22] provided an interesting joint characterization of the univariate logistic and exponential distribution. A parallel joint characterization for multivariate logistic distribution of Eq. (5.1) and independence of k univariate exponential distributions is as follows:

Theorem 5.1. *Suppose that $\underline{X}$ is a continuous k-variate random vector and radial symmetric about the origin $\underline{0}$. Then $\underline{X}$ is a k-variate logistic distribution LML with cdf as Eq. (5.1) if and only if*

$$P(\underline{X} < -\underline{x} | \underline{X} < \underline{x}) = \prod_{i=1}^{k} e^{-\beta_i x_i} \quad \textit{for all } \underline{x} \geqslant \underline{0}, \tag{5.2}$$

or, if and only if

$$\frac{1 - F(\underline{x} + \underline{y})}{(1 - F(\underline{x}))\,(1 - F(\underline{y}))} = \frac{F(\underline{x} + \underline{y})}{F(\underline{x})F(\underline{y})} \quad \text{for all } \underline{x}, \underline{y} \geqslant \underline{0}. \tag{5.3}$$

Proof. The multivariate logistic distribution Eq. (5.1) is evidently continuous, and radial symmetric about the origin. It satisfies both assumptions Eqs. (5.2) and (5.3) of the theorem, and hence we need only prove the converse of this theorem. To prove Eq. (5.2) implies Eq. (5.1):

Since the cdf of $\underline{X}$ is radial symmetric about the origin $\underline{0}$, and the conditional cdf is assumed to be the product of k independent univariate exponential variables, hence Eq. (5.2) is reduced to

$$\frac{F(-\underline{x})}{F(\underline{x})} = \frac{1 - F(\underline{x})}{F(\underline{x})} = e^{-\sum_{i=1}^{k} \beta_i x_i} \quad \text{for all } \underline{x} \geqslant \underline{0}. \tag{5.4}$$

The solution of Eq. (5.4) is the multivariate logistic distribution with cdf as Eq. (5.1), hence Eq. (5.2) implies Eq. (5.1) follows. Now, to prove Eq. (5.3) implies Eq. (5.1):

Let $G(\underline{x})$ be the conditional cdf of $(\underline{X} < -\underline{x})$ given $\underline{X} < \underline{x}$ for all $\underline{x} \geqslant \underline{0}$, and by the radial symmetric about $\underline{0}$, then

$$G(\underline{x}) = P(\underline{X} < -\underline{x} | \underline{X} < \underline{x}) = \frac{1 - F(\underline{x})}{F(\underline{x})},$$

and the assumption of Eq. (5.3) reduces to

$$G(\underline{x} + \underline{y}) = G(\underline{x})G(\underline{y}) \quad \text{for all } \underline{x}, \underline{y} \geqslant 0. \tag{5.5}$$

In view of Theorem 5.2.1 on Galambos and Kotz [22] p. 105, the only continuous solution of Eq. (5.5) is $G(\underline{x}) = e^{-\sum_{i=1}^{k} \beta_i x_i}$ with some $\beta_i > 0$, $1 \leqslant i \leqslant k$, for all $\underline{x} \geqslant \underline{0}$. Hence $F(\underline{x})$ is a multivariate logistic distribution with cdf as Eq. (5.1) and thus the proof of this theorem is complete. ■

The more general logit multivariate logistic distribution (denoted by GLML) than the LML in Eq. (5.1) is defined as follows:

Definition 5.2. A k-variate random vector $\underline{X}$ is said to have a GLML with location vector parameter $\underline{\mu} = (\mu_1, \ldots, \mu_k) \in R^k$ and scale parameter $\underline{\sigma} = (\sigma_1, \ldots, \sigma_k) \in R^k_+$, if its joint cdf is of the form

$$F(\underline{x}) = \frac{1}{1 + e^{-\sum_{i=1}^{k} (\frac{x_i - \mu_i}{\sigma_i})}}, \quad \underline{x} \in R^k \tag{5.6}$$

and $\underline{X}$ is denoted by $\underline{X} \sim GLML(\underline{\mu}, \underline{\sigma})$.

Arnold et al. [18] and Yeh [15] respectively study the characterizations of the univariate and multivariate Pareto (III) distribution via the scale transformation of the geometric minima. The related characterizations have extensions to the univariate logistic distribution which is discussed by Arnold and Laguna [27], as for the characterizations of the GLML is parallelly studied in the following.

Theorem 5.2. *Suppose* $\{\underline{X}^i = (X_1^i, X_2^i, \ldots, X_k^i)\}$ *are i.i.d. k-variate random vectors with common joint cdf* $F(\cdot)$ *satisfying*

$$\lim_{\substack{\min x_i \to -\infty \\ 1 \leqslant i \leqslant k}} F(\underline{x}) e^{-\sum_{i=1}^{k} (x_i / \sigma_i)} = \delta \tag{5.7}$$

for some $\underline{\sigma} = (\sigma_1, \ldots, \sigma_k) > \underline{0}$ *and* $\delta > 0$*. Let N* $\sim$ *geometric (p) be independent of all the* $\underline{X}^i$*'s and define the k-variate geometric maxima* $\underline{M}$ *as in Section 3. If* $\underline{M} + \underline{s} \stackrel{d}{=} \underline{X}^1$*, then we must have* $\underline{X}^1 \sim GLML(\underline{\mu}, \underline{\sigma})$ *(note that* $\underline{s} < \underline{0}$*), where* $\underline{\mu} = -(\ln \delta)\,\underline{\sigma}$ *and* $\underline{s} = (\ln p)\,\underline{\sigma}$*.*

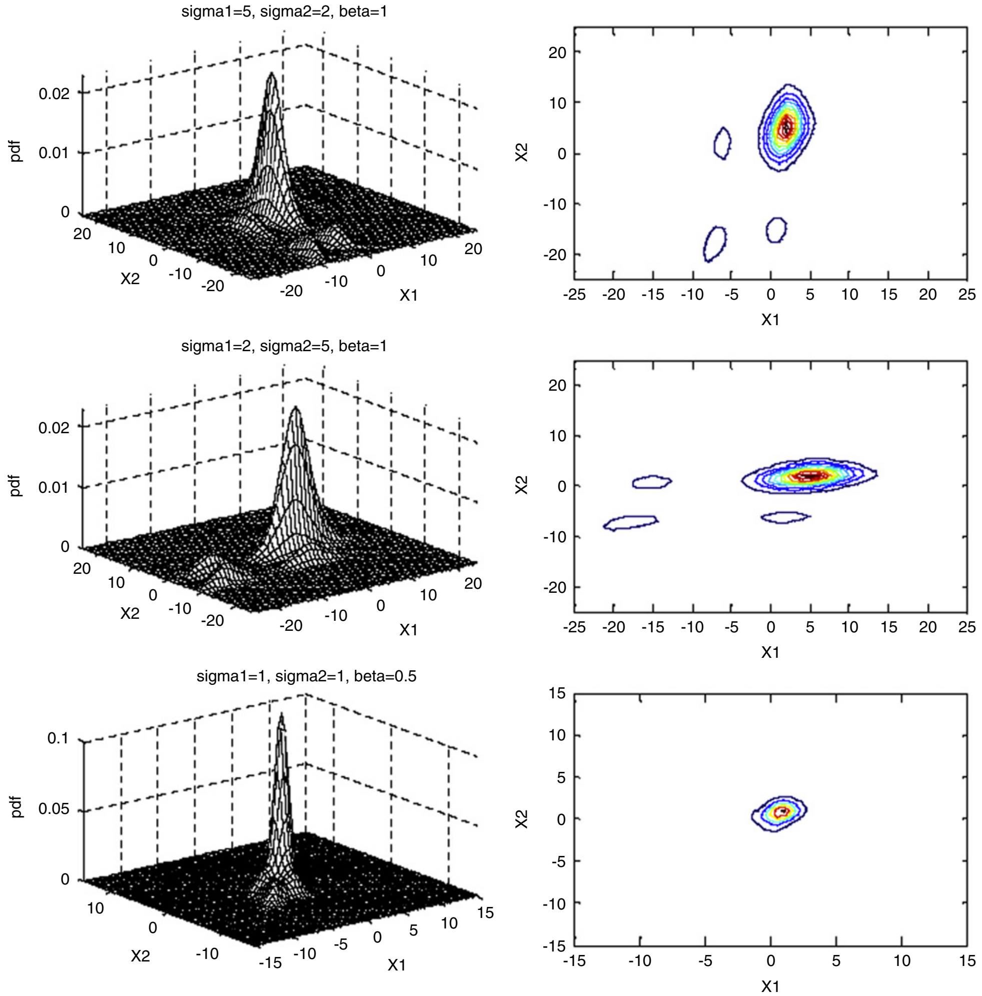

Fig. 2.1. PDF and contours of the $\text{BSL}^{(1)}(p = e^{-2\pi}, \underline{\sigma})$ where $\underline{\sigma} = (\sigma_1, \sigma_2)$.

Proof. Let $\varphi(\underline{x}) = \frac{1-F(\underline{x})}{F(\underline{x})}$, and the joint cdf of $\underline{M}$ is $P(\underline{M} \leqslant \underline{x}) = P(\underline{X}^1 - \underline{s} \leqslant \underline{x}) = F(\underline{x} + \underline{s})$ and by conditioning on N, we have

$$P(\underline{M} \leqslant \underline{x}) = \sum_{n=1}^{\infty} (F(\underline{x}))^n p\,(1-p)^{n-1} = \frac{pF(\underline{x})}{1-(1-p)F(\underline{x})}. \tag{5.8}$$

Alternatively Eq. (5.8) can be expressed as

$$\varphi\,(\underline{x}) = p\varphi\,(\underline{x} + \underline{s}). \tag{5.9}$$

To iterate Eq. (5.9) and use Eq. (5.7), we obtain

$$\varphi(\underline{x}) = \lim_{\ell\to\infty} p^{\ell}\varphi(\underline{x} + \ell\underline{s}) = \lim_{\ell\to\infty} p^{\ell} e^{-\sum_{i=1}^{k}(\frac{x_i+\ell s_i}{\sigma_i})} e^{\sum_{i=1}^{k}(\frac{x_i+\ell s_i}{\sigma_i})} \varphi(\underline{x} + \ell\underline{s}). \tag{5.10}$$

Note $\underline{s} = (s_1, \ldots, s_k) < \underline{0}$ and $\ell \to +\infty$, so Eq. (5.10) reduces to

$$\varphi(\underline{x}) = \delta^{-1} e^{-\sum_{i=1}^{k}(x_i/\sigma_i)} \lim_{\ell\to\infty} p^{\ell} e^{-\ell\sum_{i=1}^{k}(s_i/\sigma_i)}. \tag{5.11}$$

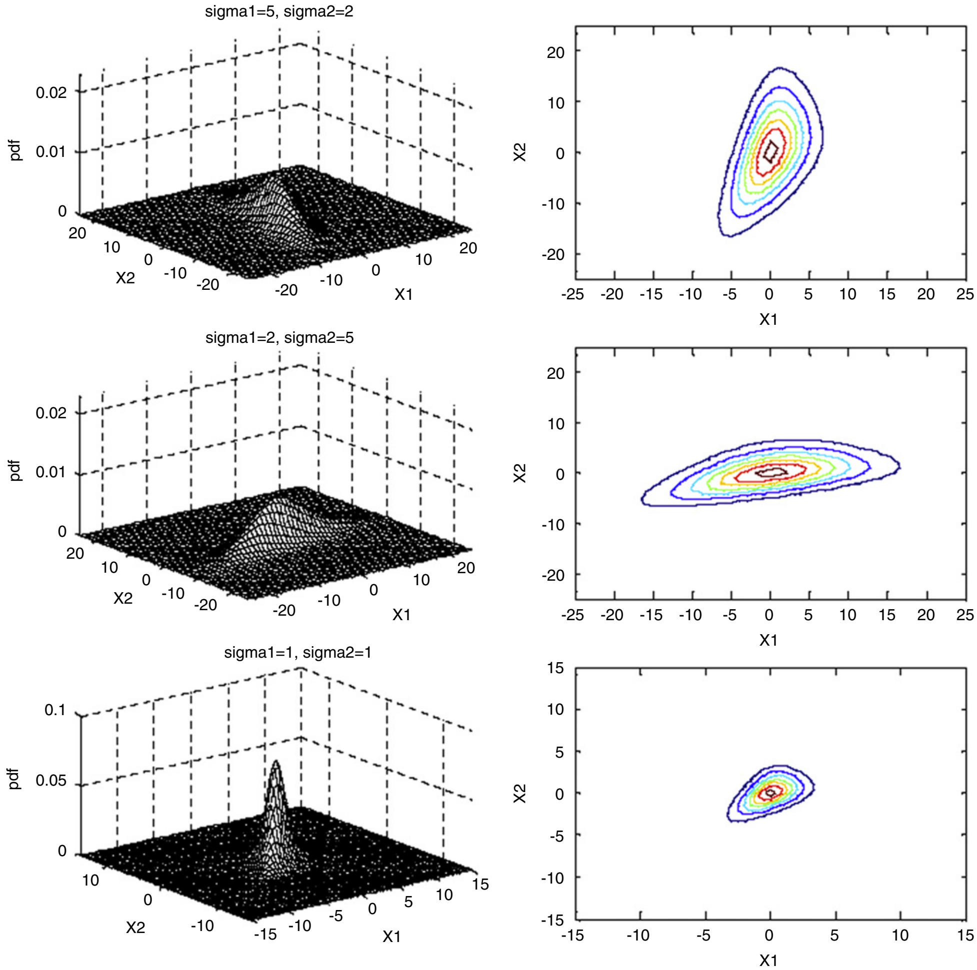

Fig. 2.2. PDF and contours of Gumbel's BL($\underline{\sigma}$) where $\underline{\sigma} = (\sigma_1, \sigma_2)$.

The only choice that Eq. (5.11) will yield a value of $\varphi(\underline{x})$ in $(0, \infty)$ as it must for Eq. (5.11) hold is when taking $s_i = (\ln p)\sigma_i$ and in such case $\varphi(\underline{x}) = \delta^{-1} e^{-\sum_{i=1}^{k}(x_i/\sigma_i)}$. It follows that the cdf of $\underline{X}^1$ is

$$F(\underline{x}) = \frac{1}{1 + \delta^{-1} e^{-\sum\limits_{i=1}^{k}(x_i/\sigma_i)}} = \frac{1}{1 + e^{-\sum\limits_{i=1}^{k}\left(\frac{x_i + \sigma_i \ln \delta}{\sigma_i}\right)}}. \tag{5.12}$$

Thus $\underline{X}^1$ is distributed as a k-variate GLML($\underline{\mu}, \underline{\sigma}$) distribution with location parameter $\underline{\mu} = (-\ln \delta)\underline{\sigma}$, scale parameter $\underline{\sigma} = (\sigma_1, \ldots, \sigma_k)$ and the translation parameter $\underline{s} = (\ln p)\underline{\sigma}$, therefore the proof is complete. ■

As a parallel to Theorem 5.2, we have the following characterization theorem for a particular Arnold's [2] multivariate logistic distribution via the translation transformation of the k-variate geometric minima $\underline{m}$ in Section 3.

Theorem 5.3. *Let $\{\underline{X}^i\}$ be i.i.d. k-variate random vector with common joint survival function $\overline{F}(\cdot)$ satisfying*

$$\lim_{\substack{\max x_i \to \infty \\ 1 \leqslant i \leqslant k}} \overline{F}(\underline{x}) e^{\sum\limits_{i=1}^{k}(x_i/\sigma_i)} = \theta \tag{5.13}$$

for some $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k) > \underline{0}$ and $\theta > 0$. Let $N \sim$ geometric(p) be independent of all the $\underline{X}^i$'s. If $\underline{m} - \underline{\tau} \overset{d}{=} \underline{X}^1$, then $\underline{X}^1$ is distributed as Arnold [2] ML($\underline{\mu}, \underline{\sigma}$) with survival function

$$\overline{F}(\underline{x}) = \frac{1}{1 + e^{\sum\limits_{i=1}^{k}\left(\frac{x_i - \mu_i}{\sigma_i}\right)}}, \quad \underline{x} \in R^k, \tag{5.14}$$

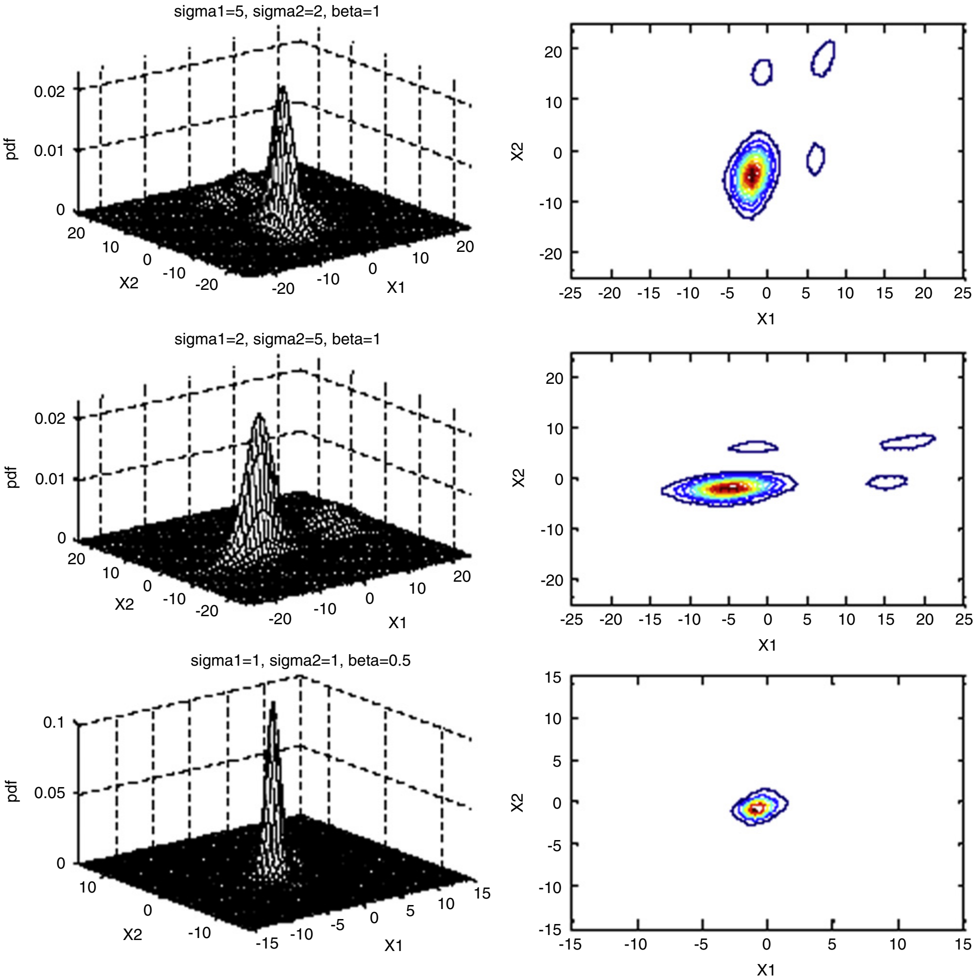

Fig. 2.3. PDF and contours of $BSL^{(2)}(p = e^{-2\pi}, \underline{\sigma})$ where $\underline{\sigma} = (\sigma_1, \sigma_2)$.

where the location parameter $\underline{\mu} = (-\ln\theta)\underline{\sigma}$*, scale parameter* $\underline{\sigma}$*, and the translation parameter* $\underline{\tau} = (\ln p)\underline{\sigma}$*.*

Note: The proof of Theorem 5.3 is analogous to Theorem 5.2, and thus is omitted.

Application: Many practical and recent categorical data analysis by logistic regression are given in Agresti [5]. The usefulness of Theorems 5.1–5.3 is the characterization of the multivariate logistic distributions in multiple logit regression.

Note: Both Gumbel's ML and Arnold's ML do not satisfy the logit transformations $\pi_i' = \ln(\frac{\pi_i}{1-\pi_i}) = \sum_{i=1}^{k} \beta_i x_i$, where β_i plays the role of $1/\sigma_i$ in Eqs. (2.3) and (2.6); so their multivariate logistic distributions cannot be used in multiple logistic regression.

6. Conclusion and discussion

Three general multivariate semi-logistic distributions, $MSL^{(1)}$, $MSL^{(2)}$, and GMSL are introduced in this paper. They may serve as competitors to Gumbel's [1] and Arnold's [2] multivariate logistic distributions. Two more particular multivariate logistic distributions of (5.6) and (5.14) used in the multiple logistic regression model are introduced in Section 5. Some characterization properties of the various multivariate semi-logistic distributions are studied in Sections 3–5 respectively. As for the moment problems such as the moment generating functions, moments, the covariance structures and the statistical inferences of the various multivariate semi-logistic distributions constitute the ongoing research work. Another interesting topic for further research is the application of the proportional hazard models by using the multivariate semi-logistic distributions $MSL^{(1)}$, $MSL^{(2)}$, and GMSL to the multivariate survival data, and the comparison of the powers of the goodness of fit test for these three multivariate semi-logistic distributions with Gumbel's [1] and Arnold's [2] multivariate logistic distributions.

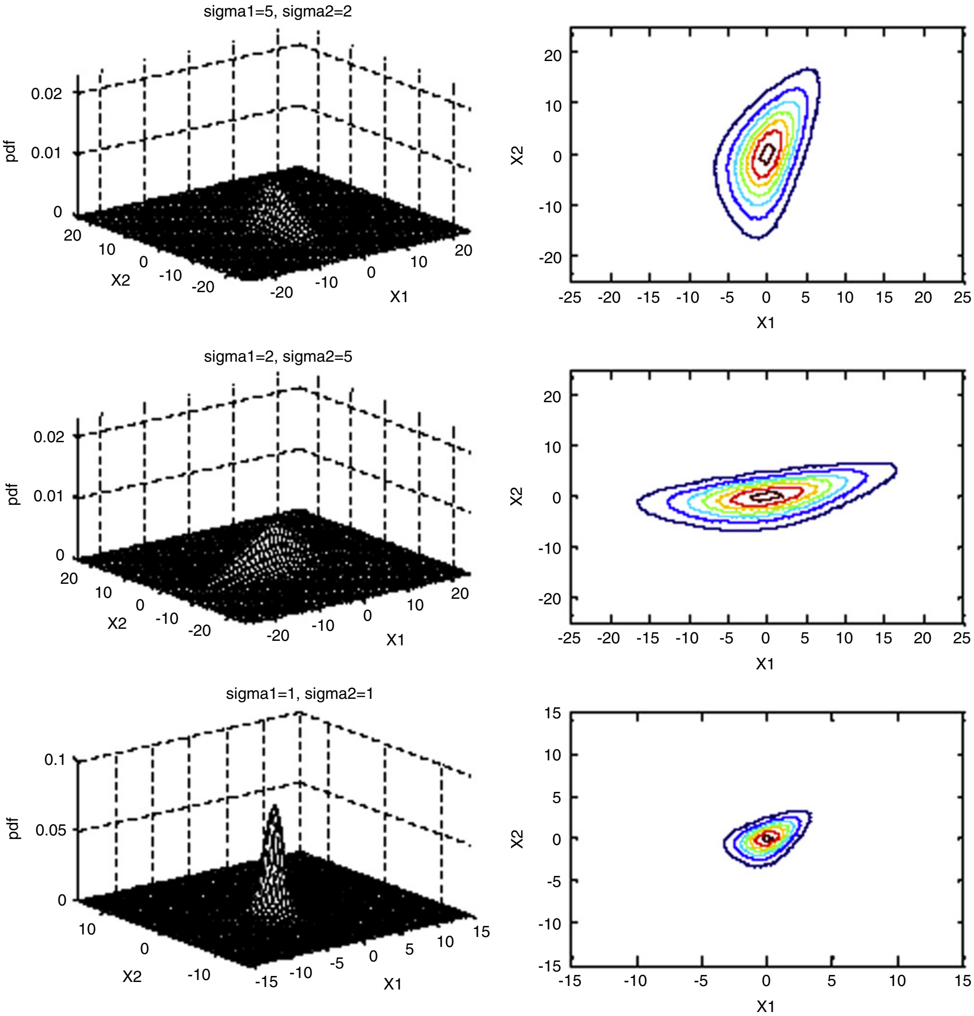

Fig. 2.4. PDF and contours of Arnold's BL($\underline{\sigma}$) where $\underline{\sigma} = (\sigma_1, \sigma_2)$.

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Appendix

See Figs. 2.1–2.4.

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