Components of first-countability and various kinds of pseudoopen mappings

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Some new classes of pseudoopen continuous mappings are introduced. Using these, we provide some sufficient conditions for an image of a space under a pseudoopen continuous mapping to be first-countable, or for the mapping to be biquotient. In particular, we show that if a regular pseudocompact space \( Y \) is an image of a metric space \( X \) under a pseudoopen continuous almost \( \Sigma \)-mapping, then \( Y \) is first-countable. Among our main results are Theorems 2.5, 2.11, 2.12, 2.13, 2.14. See also Example 2.15, Corollary 2.7, and Theorem 2.18.

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1. General remarks, some concepts and techniques

In this article, a space is a topological \( T_1 \)-space. If \( X \) is a space, and \( A \) is a subset of \( X \), then \( \text{int}(A) \) is the interior of \( A \), that is, \( \text{int}(A) \) is the largest open set contained in \( A \). A base of a set \( A \subset X \) in a space \( X \) is a family of open neighbourhoods of \( A \) in \( X \) such that every open neighbourhood of \( A \) contains some element of this family. We say that the character of a set \( A \subset X \) in a space \( X \) is countable, if there exists a countable base of \( A \) in \( X \). Interchangeably, in most cases, we follow [8].

In many respects, this paper can be considered as a continuation of the paper [3] where, in particular, a new Tychonoff countable Fréchet–Urysohn space, which is not first-countable at any point, was constructed. Below, we introduce some special subclasses of the class of pseudoopen mappings. Recall that a mapping \( f \) of a space \( X \) onto a space \( Y \) is said to be pseudoopen if, for every \( y \in Y \) and every open neighbourhood \( U \) of \( f^{-1}(y) \) in \( X \), we have: \( y \in \text{int}(f(U)) \) [1,3].

To emphasize the importance of pseudoopen mappings, it is enough to mention the following facts. Every open mapping and every closed mapping is pseudoopen, and every pseudoopen continuous mapping is quotient. A Hausdorff space \( Y \) is Fréchet–Urysohn if and only if \( Y \) can be represented as an image of some metric space under a pseudoopen continuous mapping [1]. The \( \Sigma \)-product \( T \) of uncountably many closed intervals \( [0,1] \) is a standard example of a Tychonoff Fréchet–Urysohn space which is not first-countable at any point [8]. Besides, \( T \) is countably compact and normal. Thus, pseudoopen continuous mappings preserve the Fréchet–Urysohn property, but do not preserve the first-countability even when the image space is countably compact (“countably” in this statement can be removed). In this paper, we identify some situations in...
which an image of a metric space under a continuous pseudoopen mapping has to be first-countable. To do that, we distinguish and study some important components of first-countability, like countable fan-tightness and countable $\kappa$-fan-tightness, countable sensor and countable $\kappa$-sensor. Our main results in this direction are Theorems 2.5, 2.11, 2.12, 2.13, 2.14, and Example 2.15. In particular, we establish that a pseudocompact regular space has a point-countable base if and only if it can be represented as an image of some metric space under a pseudoopen continuous $S$-mapping.

Suppose that $A$ is a subset of a space $X$, and $B$ is a subset of a space $Y$. A mapping $f : X \to Y$ of $X$ onto $Y$ will be called pseudoopen at the pair $(A, B)$ if $f(A) = B$ and, for each open neighbourhood $U$ of $A$, the set $f(U)$ contains an open neighbourhood of $B$.

If nothing is explicitly stated to the contrary, then, whenever we consider below a pair $(A, B)$, we assume that $A$ and $B$ are subsets of the spaces $X$ and $Y$, respectively.

The next statement is easily established:

**Proposition 1.1.** A mapping $f : X \to Y$ of a space $X$ onto a space $Y$ is pseudoopen at a pair $(A, B)$ if and only if for each subset $M$ of $Y$ such that $B \cap M \neq \emptyset$, we have $A \cap f^{-1}(M) \neq \emptyset$.

We need a few more definitions. Given a subset $H$ of a space $X$, we will say that the $\kappa$-fan-tightness of $X$ at $H$ is countable if the following condition is satisfied:

(aft) For every sequence $\{ U_n : n \in \omega \}$ of open sets $U_n$ in $X$ such that $H \cap \overline{U_n} \neq \emptyset$ and $H \cap U_n = \emptyset$, for every $n \in \omega$, there exists a sequence $\{ G_n : n \in \omega \}$ of open sets such that $H$ does not intersect the closure of $G_n$, $G_n$ is a subset of $U_n$, and the intersection of $H$ with the closure of $\bigcup \{ G_n : n \in \omega \}$ is non-empty.

If this condition is satisfied for $H = \{ x \}$ whenever $x$ is an arbitrary point of $X$, then we say that $X$ is a space of countable $\kappa$-fan-tightness.

In connection with the above definition, recall that a space $X$ has countable fan-tightness at a point $x \in X$ if, for every sequence $\{ A_n : n \in \omega \}$ of subsets of $X$ such that $x \in \overline{A_n}$, for each $n \in \omega$, one can select a finite subset $B_n$ of $A_n$ so that $x \in \bigcup \{ B_n : n \in \omega \}$. If this condition is satisfied for every $x \in X$, then we say that $X$ is a space of countable fan-tightness.

A space $X$ is normal at a subset $H$ of $X$ if for every open neighbourhood $U$ of $H$ there exists an open neighbourhood $V$ of $H$ such that $\overline{V} \subset U$.

Recall that the tightness of a space $X$ at a point $x \in X$ is said to be countable if from $x \in \overline{A}$ it follows that there exists a countable subset $B$ of $A$ such that $x$ is in the closure of $B$.

**Theorem 1.2.** Suppose that $X$ is a pseudocompact regular space, and $H$ is a subset of $X$ such that $X$ is normal at $H$, and the tightness of $X$ at every point of $H$ is countable.

Then the $\kappa$-fan-tightness of $X$ at $H$ is countable.

**Proof.** Fix a sequence $\{ U_n : n \in \omega \}$ of open sets $U_n$ in $X$ such that $H \cap \overline{U_n} \neq \emptyset$ and $H \cap U_n = \emptyset$, for every $n \in \omega$.

Since $X$ is regular, and the tightness of $X$ at each point of $H$ is countable, and $X$ is normal at $H$, one can easily select open subsets $W_{n,i}$ of $U_n$ for $i \in \omega$ such that the closure of each $W_{n,i}$ does not intersect $H$, and the closure of $\bigcup \{ W_{n,i} : i \in \omega \}$ intersects $H$, for each $n \in \omega$.

Fix an arbitrary open neighbourhood $O(H)$ of $H$. For each $n \in \omega$ we can select $i(n) \in \omega$ so that $O(H) \cap W_{n,i(n)} \neq \emptyset$. Put $V_n = O(H) \cap W_{n,i(n)}$. Since $X$ is pseudocompact, the sequence $\xi = \{ V_n : n \in \omega \}$ of non-empty open sets has an accumulation point $x_\xi$ in $X$. Clearly, $x_\xi \in \overline{O(H)}$, since $V_n \subset O(H)$ for $n \in \omega$.

A point $z$ of $X$ will be called special if there exists a sequence $\eta$ of open sets $H_n$ in $X$ such that $H$ does not intersect the closure of $H_n$, $H_n \subset V_n$, for each $n \in \omega$, and $z$ is an accumulation point for the sequence $\eta = \{ H_n : n \in \omega \}$.

Thus, the point $x_\xi$ selected above is special. Since $X$ is normal at $H$, and $x_\xi$ is in the closure of $O(H)$, where $O(H)$ was fixed as an arbitrary open neighbourhood of $H$, the next claim holds:

**Claim 1.** Every open neighbourhood of $H$ intersects the set $S$ of all special points.

Claim 1 immediately implies the next statement:

**Claim 2.** $H \cap S \neq \emptyset$.

Fix a point $x \in H \cap S$. Since, by the assumption, the tightness of $X$ at $x$ is countable, we can find a sequence $\{ s_n : n \in \omega \}$ of points in $S$ such that $x$ is an accumulation point for this sequence. By the definition of a special point, we can also fix, for every $k \in \omega$, a sequence $\eta_k$ of open sets $H_{k,n}$ in $X$ such that $H$ does not intersect the closure of $H_{k,n}$, $H_{k,n} \subset U_n$, for each $n \in \omega$, and $s_k$ is an accumulation point for the sequence $\eta_k = \{ H_{k,n} : n \in \omega \}$.
Observe that, obviously, $s_k$ is also an accumulation point for the sequence $\{H_{k,n}: n \in \omega, k \leq n\}$. It follows that $x$ is in the closure of the set $E = \bigcup \{H_{k,n}: k, n \in \omega, k \leq n\}$.

Put $G_n = \bigcup \{H_{k,n}: k \in \omega, k \leq n\}$ for each $n \in \omega$. Clearly, $H$ does not intersect the closure of $G_n$. $G_n$ is a subset of $U_n$, and $x$ is in the closure of $\bigcup \{G_n: n \in \omega\}$, since $\bigcup \{G_n: n \in \omega\} = E$.

Thus, the $\kappa$-fan-tightness of $X$ at $H$ is countable. $\Box$

In particular, the conditions in the above statement are satisfied when $X$ is a pseudocompact Tychonoff space of countable tightness and $H$ is an arbitrary compact subspace of $X$.

**Proposition 1.3.** Suppose that $\{P_n: n \in \omega\}$ is an increasing sequence of closed sets in a space $X$, and $H$ is a subset of $P_0$ such that $X$ is normal at $H$, the $\kappa$-fan-tightness of $X$ at $H$ is countable, and $H$ is not contained in $\text{int}(P_n)$, for each $n \in \omega$.

Then there exists an open subset $W$ of $X$ such that $H$ intersects the closure of $W$, and, for each $n \in \omega$, $H \cap W \cap P_n = \emptyset$.

**Proof.** Put $U_n = X \setminus P_n$, for $n \in \omega$. Then $U_n$ is open, $H \cap U_n = \emptyset$, and $H \cap \overline{U_n}$ is non-empty, since $H$ is not contained in $\text{int}(P_n)$, by the assumption. Since $X$ is normal at $H$, and the $\kappa$-fan-tightness of $X$ at $H$ is countable, we can fix a countable family $\eta = \{V_n: n \in \omega\}$ of open sets such that $V_n \subset U_n$, $H$ does not intersect $V_n$, for each $n \in \omega$, and $H \cap \bigcup \{V_n: n \in \omega\} \neq \emptyset$.

Put $W = \bigcup \{V_n: n \in \omega\}$. Then $W$ is open, and $H$ intersects the closure of $W$. Now take an arbitrary $k \in \omega$, and consider $P_k \cap W$.

**Claim.** $P_k \cap W \subset \bigcup \{P_k \cap V_i: i \leq k, i \in \omega\}$.

Indeed, if $i \geq k$, then $V_i \subset U_i \subset U_k = X \setminus P_k$, since the sets $P_n$ are increasing and, therefore, the sets $U_n$ are decreasing. Hence, $V_i \cap P_k = \emptyset$ whenever $i \geq k$. Now the claim follows from the definition of $W$.

Since $H \cap \overline{V_i} = \emptyset$, for every $i \in \omega$, it follows from the claim that $H$ does not intersect the closure of $P_k \cap W$. $\Box$

A family $S$ of subsets of a space $X$ is said to be a sensor (a $\kappa$-sensor) at a set $H \subset X$ if, for each open neighbourhood $O(H)$ of $H$ and each (open) set $U$ such that $H \cup U \neq \emptyset$, there exists $P \subset S$ satisfying the following conditions: $P \subset O(H)$ and $H \cup \overline{\bigcup P} \neq \emptyset$.

If there exists a countable $\kappa$-sensor at $H$, the space $X$ is said to be countably $\kappa$-sensitive at $H$.

Slightly different notions of a closure-sensor and of an $\text{FU}$-sensor of a space at a point in this space were introduced in [3]. We also used there the expression “$X$ is countably sensitive at $x \in X$” to mean that there exists a countable closure-sensor of $X$ at the set $H = \{x\}$. However, everywhere below this expression has a different meaning: it signifies that there exists a countable sensor of $X$ at $H = \{x\}$ in the sense of the definition of the notion of a sensor given in this article.

Clearly, every base of neighbourhoods of $H$ in $X$ is a sensor of $X$ at $H$. Observe also that if $S$ is a sensor (a $\kappa$-sensor) of $X$ at $H$, then the family $S_H = \{P \cup H: P \subset S\}$ is also a sensor (a $\kappa$-sensor) of $X$ at $H$.

The next result is a key piece of technique in our study of pseudoopen mappings.

**Theorem 1.4.** Suppose that $X$ is a space, and $H$ is a subset of $X$ such that the $\kappa$-fan-tightness of $X$ at $H$ is countable, and $X$ is normal at $H$. Suppose further that $X$ is countably $\kappa$-sensitive at $H$, and the character of $H$ in $X$ is countable.

**Proof.** Since $X$ is countably $\kappa$-sensitive at $H$, and $X$ is normal at $H$, we can fix a countable $\kappa$-sensor $S$ at $H$ all elements of which are closed sets. We may also assume that $H$ is contained in every element of $S$. Indeed, we can replace the family $S$ by the family $S_H$ which is also a countable $\kappa$-sensor of $X$ at $H$. Note that $H$ is closed in $X$, since $X$ is normal at $H$.

Let $O(H)$ be any open neighbourhood of $H$ in $X$, and $\gamma = \{P \subset S: P \subset O(H)\}$. Then $\gamma$ is countable, since $S$ is countable. Let $\gamma = \{B_n: n \in \omega\}$, and $P_n = \bigcup \{B_i: i \leq n\}$, for $n \in \omega$. Clearly, $\{P_n: n \in \omega\}$ is also a $\kappa$-sensor at $H$.

**Claim 1.** $H \subset \text{int}(P_n)$, for some $n \in \omega$.

Assume the contrary. Then all conditions in Proposition 1.3 are satisfied. Therefore, there exists an open set $W$ such that $H$ intersects the closure of $W$, and, for each $n \in \omega$, $H \cap \overline{W \cap P_n} = \emptyset$. However, this is impossible, since $\{P_n: n \in \omega\}$ is a $\kappa$-sensor at $H$. Claim 1 is established.

The next Claim 2 immediately follows from Claim 1.

**Claim 2.** Let $E$ be the family of all sets $G$ such that $H \subset G$ and $G = \text{int}(\bigcup \lambda)$, for some finite subfamily $\lambda$ of $S$. Then $E$ is countable, and $E$ is a base of the set $H$ in $X$. $\Box$

**Theorem 1.5.** Suppose that $X$ is a pseudocompact regular space of countable tightness, and that $H$ is a subset of $X$ such that $X$ is normal at $H$, and $X$ is countably $\kappa$-sensitive at $H$. Then the character of the set $H$ in the space $X$ is countable.
**Proof.** By Theorem 1.2, the $\kappa$-fan-tightness of $X$ is countable. Now it follows from Theorem 1.4 that the character of $H$ in $X$ is countable. □

It is well known, and very easy to verify, that a mapping $f$ of a space $X$ onto a space $Y$ is pseudoopen, if for every subset $V$ of $Y$ and every $y$ in the closure of $V$ there exists $x$ in the closure of $f^{-1}(V)$ such that $f(x) = y$.

Let $A$ be a subset of a space $X$. A mixed base of $X$ at $A$ is a family $\mathcal{M}$ of open subsets of $X$ such that, for each $x \in A$ and each open neighbourhood $O(A)$ of $A$ in $X$, there exists $V \in \mathcal{M}$ such that $x \in V \subset O(A)$. We denote by $mw(A, X)$ the smallest infinite cardinal number $\tau$ such that there exists a mixed base of $X$ at $A$ of cardinality $\leq \tau$.

Recall that a point-wise base of $X$ at $A$ (called also an external base of $A$ in $X$) is a family $\mathcal{B}$ of open subsets of $X$ such that, for each $x \in A$ and each open neighbourhood $O(x)$ of $x$ in $X$, there exists $V \in \mathcal{B}$ such that $x \in V \subset O(x)$.

The point-wise weight $\omega(A, X)$, is defined as the smallest infinite cardinal number $\tau$ such that there exists a point-wise base of $X$ at $A$ of cardinality $\leq \tau$.

Obviously, every base of open neighbourhoods of a set $A \subset X$ in a space $X$ is a mixed base of $A$ in $X$, and therefore, $mw(A, X)$ does not exceed the character of $A$ in $X$. On the other hand, every point-wise base of $X$ at $A$ is a mixed base of $X$ at $A$. Hence, $mw(A, X) \leq \omega(A, X)$. Observe that if $A$ is an open subset of a space $X$, then the mixed weight of $A$ in $X$ is countable.

2. Results on pseudoopen mappings satisfying some additional restrictions

First, we introduce some new classes of pseudoopen continuous mappings.

A mapping $f : X \to Y$ of a space $X$ onto a space $Y$ is called $\omega$-pseudoopen (strictly $\omega$-pseudoopen) at a point $y \in Y$ if there exists a subset $P$ of $X$ such that $f$ is pseudoopen at the pair $(P, \{y\})$, and the mixed weight (the point-wise weight, respectively) of $P$ in $X$ is countable.

Notice that $f$ is $\omega$-pseudoopen at every isolated point of $Y$.

If $f$ is $\omega$-pseudoopen (strictly $\omega$-pseudoopen) at each $y \in Y$, then we say that $f$ is $\omega$-pseudoopen (strictly $\omega$-pseudoopen, respectively).

Now we are going to define a slightly different version of this concept. A mapping $f : X \to Y$ of a space $X$ onto a space $Y$ is almost $\omega$-pseudoopen at a point $y \in Y$ if either $y$ is isolated in $Y$, or there exist a subset $P$ of $X$ and a subset $H$ of $Y$ such that $y \in H$, $Y$ is normal at $H$, the mapping $f$ is pseudoopen at the pair $(P, H)$, and the point-wise weight of $P$ in $X$ is countable. Again, we say that $f$ is almost $\omega$-pseudoopen if $f$ is almost $\omega$-pseudoopen at every $y \in Y$.

Obviously, every $\omega$-pseudoopen mapping is almost $\omega$-pseudoopen.

We will need below the next statement:

**Lemma 2.1.** Suppose that $f$ is a continuous mapping of a space $X$ onto a space $Y$, and that $P$ is a subset of $X$ and $y$ is a point of $Y$ satisfying the following conditions:

1. $f$ is pseudoopen at the pair $(P, \{y\})$; and
2. the set $P$ has a countable mixed base $\mathcal{B}$ in $X$.

Then $Y$ is a Fréchet–Urysohn space at $y$, and hence, the tightness of $Y$ at $y$ is countable.

**Proof.** Take any subset $M$ of $Y$ such that $y \notin M$, and put $A = f^{-1}(M)$. Then $x \in A$ for some $x \in P$, since $f$ is pseudoopen at the pair $(P, \{y\})$ (see Proposition 1.1). Put $\eta = \{W \in \mathcal{B} : x \in W\}$. The family $\eta$ is countable, since $\mathcal{B}$ is countable. Hence, we can write $\eta$ as a sequence: $\eta = \{W_n : n \in \omega\}$. Let $\mathcal{V}_k = \bigcap\{W_n : n \leq k\}$ for $k \in \omega$, and let $y = \{V_n : n \in \omega\}$.

For each $n \in \omega$ the set $V_n \cap A$ is non-empty, and we fix a point $x_n \in V_n \cap A$. The sequence $\xi = \{f(x_n) : n \in \omega\}$, clearly, is contained in $M$.

Let us show that $\xi$ converges to $y$. Take any open neighbourhood $O(y)$ of $y$. Since $f$ is continuous and $f(P) = y$, there exists an open neighbourhood $O(P)$ of $P$ such that $f(O(P)) \subset O(y)$. Since $x \in P$ and $\mathcal{B}$ is a mixed base of $X$ at $P$, there exists $W \in \mathcal{B}$ such that $x \in W \subset O(P)$. Then $\mathcal{W}_k = \{W_n : n \in \omega\}$ for some $k \in \omega$, and therefore, $x_n \in W_n$ for every $n \geq k$. It follows that $f(x_n) \in f(O(P)) \subset O(y)$ for $n \geq k$. Hence, the sequence $\xi$ converges to $y$. □

The next statement immediately follows from Lemma 2.1.

**Proposition 2.2.** If $f$ is a continuous $\omega$-pseudoopen mapping of a space $X$ onto a space $Y$, then $Y$ is a Fréchet–Urysohn space.

Recall that a mapping $f : X \to Y$ is an $S$-mapping, if the subspace $f^{-1}(y)$ is separable for every $y \in Y$. We will say that $f : X \to Y$ is an almost $S$-mapping, if for every non-isolated point $y \in Y$, the subspace $f^{-1}(y)$ is separable. We have:

**Proposition 2.3.** Every pseudoopen almost $S$-mapping of a metric space $X$ onto a space $Y$ is strictly $\omega$-pseudoopen.
**Proof.** The condition to verify is obviously satisfied at any isolated point of \( Y \). Every metric space has a point-countable base [8]. It remains to refer to the following obvious statement: every separable subset of a space with a point-countable base has a countable point-wise base in this space. Therefore, \( f \) is pseudoopen at any pair \((f^{-1}(y), y)\) where \( y \) is an arbitrary non-isolated point of \( Y \). \( \square \)

Below we will establish a theorem on \( \omega \)-pseudoopen continuous mappings with a pseudocompact range (Theorem 2.5). This is one of the main results of this article. Its proof is based on some results obtained above and on the next general statement:

**Proposition 2.4.** Suppose that \( f : X \to Y \) is a continuous mapping of a space \( X \) onto a regular space \( Y \), and that \( f \) is pseudoopen at a pair \((A, B)\), where the mixed weight of \( A \) in \( X \) is countable. Then \( Y \) has a countable sensor at \( B \).

**Proof.** Fix a countable mixed base \( B \) of \( X \) at \( A \), and put \( S = \{ f(U) : U \in B \} \). Clearly, \( S \) is countable.

Claim. \( S \) is a sensor of \( Y \) at \( B \).

Indeed, take any open neighbourhood \( O(B) \) of \( B \) and any \( M \subseteq Y \) such that \( B \cap M \neq \emptyset \). Then, by Proposition 1.1, \( A \cap f^{-1}(M) \neq \emptyset \), so that we can fix \( x \in A \cap f^{-1}(M) \). Put \( O(A) = f^{-1}(O(B)) \). Since \( f \) is continuous and \( f(A) = B \), it follows that \( O(A) \) is an open neighbourhood of \( A \) in \( X \).

Since \( B \) is a mixed base of \( X \) at \( A \), we can find \( U \in B \) such that \( x \in U \subseteq O(A) \). Put \( y = f(x) \). Then \( y \in B \), and \( y \in f(U) \subseteq O(B) \). Since \( x \in f^{-1}(M) \) and \( U \) is an open neighbourhood of \( x \), it follows that \( x \in f^{-1}(M) \cap U \). Therefore, by continuity of \( f \), \( y = f(x) \) belongs to the interior of the set \( f(U) \subseteq \bigcup \{ f(U) : U \in \gamma \} \).

**Theorem 2.5.** If \( f \) is a continuous \( \omega \)-pseudoopen mapping of a space \( X \) onto a pseudocompact regular space \( Y \), then the space \( Y \) is first-countable.

**Proof.** Take any point \( y \in Y \). Since \( f \) is \( \omega \)-pseudoopen, Proposition 2.4 implies that \( Y \) has a countable sensor at \( y \).

It follows from Lemma 2.1 that the tightness of \( Y \) at the point \( y \) is countable. Now Theorem 1.5 is applicable, and we conclude that \( Y \) is first-countable at \( y \). \( \square \)

The proof of the next statement is quite similar to the proof of an analogous statement in [10] about pseudoopen mappings onto strongly Fréchet–Urysohn spaces, but for the sake of completeness we present its proof here. For a discussion of biquotient mappings, see [9,10]. Here we just recall the definition of this important notion.

A continuous mapping \( f \) of a space \( X \) onto a space \( Y \) is called **biquotient** if, for every \( y \in Y \) and every family \( \eta \) of open subsets of \( X \) such that \( f^{-1}(y) \subseteq \bigcup \eta \), there exists a finite subfamily \( \gamma \) of \( \eta \) such that \( y \) belongs to the interior of the set \( \bigcup \{ f(U) : U \in \gamma \} \).

**Proposition 2.6.** Every continuous strictly \( \omega \)-pseudoopen mapping \( f \) of a space \( X \) onto a space \( Y \) of countable fan-tightness is biquotient.

**Proof.** Fix a point \( y \in Y \) and a subset \( P \) of \( X \) such that the point-wise weight of \( X \) at \( P \) is countable, and \( f \) is pseudoopen at the pair \((P, \{y\})\). Fix also a countable point-wise base \( B \) of \( X \) at \( P \). Take any family \( \gamma \) of open subsets of \( X \) such that the set \( F = f^{-1}(y) \) is covered by \( \gamma \). Clearly, \( P \subseteq F \), and \( P \) is Lindelöf. Therefore, we can find a countable subfamily \( \eta \) of \( \gamma \) such that \( P \subseteq \bigcup \eta \) and every element of \( \eta \) intersects \( P \). Obviously, there exists an increasing sequence \( \{W_n : n \in \omega \} \) of open sets in \( X \) such that every \( W_n \) is the union of some finite subcollection of \( \eta \), and \( \bigcup \{ W_n : n \in \omega \} = \bigcup \eta \). Clearly, \( y \in f(W_n) \) for every \( n \in \omega \). It is enough to show that \( y \in \text{int}(f(W_n)) \), for some \( n \in \omega \).

Assume the contrary, and put \( A_n = Y \setminus f(W_n) \), for \( n \in \omega \). Then \( y = \bigcap A_n \). Since the fan-tightness of \( Y \) is countable, we can select finite subsets \( B_n \) of \( A_n \) such that \( y \in B_n \), where \( B = \bigcup \{ B_n : n \in \omega \} \). Since \( f \) is pseudoopen at the pair \((P, \{y\})\), it follows that \( x \in f^{-1}(B) \), for some \( x \in P \). There exists \( k \in \omega \) such that \( x \in W_k \). Then, obviously, \( W_k \cap B \) is infinite. However, \( A_n \cap f(W_k) \) is empty for \( n \geq k \), since \( A_k \cap f(W_k) = \emptyset \), and the family \( \{ A_n : n \in \omega \} \) is decreasing. Therefore, \( f(W_k) \cap B \) is contained in the finite set \( \bigcup \{ B_i : i < k \} \). We have arrived at a contradiction. \( \square \)

We obtain from Theorem 2.5 the following conclusion:

**Corollary 2.7.** Every continuous strictly \( \omega \)-pseudoopen mapping of a space \( X \) onto a pseudocompact regular space \( Y \) is biquotient.

**Proof.** By Theorem 2.5, the space \( Y \) is first-countable. Since, obviously, every first-countable space has countable fan-tightness, it follows from Proposition 2.6 that the mapping \( f \) is biquotient. \( \square \)
Obviously, the last statement implies the next two results:

**Corollary 2.8.** If \( f \) is a continuous strictly \( \omega \)-pseudoopen mapping of a locally compact space \( X \) onto a pseudocompact regular space \( Y \), then \( Y \) is locally compact and \( \omega \)-first-countable.

**Corollary 2.9.** If \( f \) is a continuous strictly \( \omega \)-pseudoopen mapping of a locally metrizable and locally separable space \( X \) onto a pseudocompact regular space \( Y \), then \( Y \) is locally compact and \( \omega \)-metrizable.

To derive Corollary 2.9 from Corollary 2.7, observe that the closure of an open subset of a pseudocompact space is pseudocompact, and that every pseudocompact regular space with a countable network is compact and metrizable (see [8]).

Theorem 2.5 can be expanded in a non-trivial way to continuous almost \( \omega \)-pseudoopen mappings. To do this, we need the next technical result:

**Proposition 2.10.** Suppose that \( f : X \to Y \) is a continuous almost \( \omega \)-pseudoopen mapping of a space \( X \) of countable tightness onto a regular space \( Y \). Then:

1. The tightness of \( Y \) is countable; and
2. for each \( y \in Y \), there exists a closed subspace \( H \) of \( Y \) such that \( y \in H \), the subspace \( H \) has a countable network, \( Y \) is normal at \( H \), and \( Y \) is countably sensitive at \( H \).

**Proof.** Take any non-isolated point \( y \in Y \). Since \( f \) is almost \( \omega \)-pseudoopen, we can select a subset \( P \) of \( X \) and a subset \( H \) of \( Y \) such that \( y \in H \), \( Y \) is normal at \( H \), the mapping \( f \) is pseudoopen at the pair \((P, H)\), and the point-wise weight of \( P \) in \( X \) is countable. Clearly, \( P \) is a space with a countable base. Since \( f \) is continuous and \( f(P) = H \), it follows that the space \( H \) has a countable network.

Take any non-closed subset \( B \) of \( Y \). We have to find a point in \( Y \setminus B \) which belongs to the closure of some countable subset of \( B \). Since \( y \) is an arbitrary non-isolated point of \( Y \), we may assume that \( y \in \overline{B} \setminus B \).

If \( y \) also belongs to the closure of \( B \cap H \) then, using the fact that \( H \) has a countable network, we can take \( C \) to be a countable dense subset of \( B \cap H \). Then \( C \) is a countable dense subset of \( B \), \( y \) is in the closure of \( C \), and \( y \) is not in \( B \).

It remains to consider the case when \( y \) is not in the closure of \( B \cap H \). Since \( Y \) is regular, we can fix an open neighbourhood \( O(y) \) of \( y \) such that the closure of \( O(y) \) does not intersect the closure of the set \( B \cap H \). Put \( B_1 = O(y) \cap B \).

Clearly, \( y \) is in the closure of \( B_1 \). Since \( f \) is pseudoopen at the pair \((P, H)\), it follows that \( P \cap f^{-1}(B_1) \) is not empty. Fix \( x \in P \cap f^{-1}(B_1) \), and put \( z = f(x) \). Then \( z \in H \), and, by continuity of \( f \), \( z \in \overline{B} \). Therefore, \( z \) belongs to the closure of \( O(y) \) and hence, \( z \) is not in \( B \cap H \). Since \( z \in H \), it follows that \( z \notin B \). Since the tightness of \( X \) is countable, and \( x \) is in the closure of \( f^{-1}(B_1) \), we can find a countable subset \( M \) of \( f^{-1}(B_1) \) such that \( x \) is in the closure of \( M \). Then \( C = f(M) \) is a countable subset of \( B_1 \cap B \) such that \( z \in C \). Since \( z \) is not in \( B \), we can conclude that the tightness of \( Y \) is countable. Thus, we have established (1).

We also know that \( y \in H \), \( H \) has a countable network, and \( Y \) is normal at \( H \). The last fact implies that \( H \) is closed in \( Y \). Therefore, to establish (2), it is enough to show that \( Y \) is countably sensitive at \( H \). This follows directly from Proposition 2.4.

**Theorem 2.11.** If \( f \) is a continuous almost \( \omega \)-pseudoopen mapping of a space \( X \) onto a pseudocompact regular space \( Y \), then \( Y \) is \( \omega \)-first-countable.

**Proof.** Take any \( y \in Y \). By Proposition 2.10, there exists a subspace \( H \) of \( Y \) such that \( H \) has a countable network, \( Y \) is normal at \( H \), \( Y \) has a countable sensor at \( H \), and \( y \in H \). Proposition 2.10 also implies that the tightness of \( Y \) is countable.

Thus, we can apply Theorem 1.5. It follows that \( H \) is a \( G_\delta \)-subset of \( Y \). However, \( y \) is a \( G_\delta \)-point in \( H \), since \( H \) is regular and has a countable network. Therefore, \( y \) is a \( G_\delta \)-point in \( Y \). Since \( Y \) is pseudocompact and regular, it follows that \( Y \) is \( \omega \)-first-countable at \( y \).

The following two theorems are among main results on pseudoopen \( S \)-mappings of metric spaces in this article.

**Theorem 2.12.** Suppose that \( f \) is a continuous pseudoopen almost \( S \)-mapping of a metric space \( X \) onto a pseudocompact regular space \( Y \). Then \( Y \) is \( \omega \)-first-countable.

**Proof.** The space \( X \) has a point-countable base. Therefore, the mapping \( f \) is almost \( \omega \)-pseudoopen (and \( \omega \)-pseudoopen).

Now it follows from Theorem 2.11 (or from Theorem 2.5) that the space \( Y \) is \( \omega \)-first-countable.

**Theorem 2.13.** If \( f \) is a continuous pseudoopen \( S \)-mapping of a metric space \( X \) onto a pseudocompact regular space \( Y \), then \( Y \) has a point-countable base.
Proof. By Theorem 2.12, $Y$ is first-countable. Since $f$ is a pseudoopen $S$-mapping, it follows that $f$ is biquotient (see Problem 21 in Chapter 6, Section 1 in [5], or apply Proposition 2.6). Observe that $X$ has a point-countable base, since $X$ is a metric space. Now we can apply a remarkable theorem of V.V. Filippov saying that if $Y$ is an image of a metric space under a biquotient $S$-mapping, then $Y$ also has a point-countable base [9].

It is clear from the above argument that the last theorem also holds for any space $X$ with a point-countable base. The conclusion in Theorem 2.13 cannot be strengthened to the statement that $Y$ is metrizable, since every space with a point-countable base can be represented as an image of a metric space under an open continuous $S$-mapping (see [5]). Observe also that Theorem 2.13 cannot be extended to pseudoopen almost $S$-mappings. We will see it below, Example 2.15.

**Theorem 2.14.** If $f$ is a continuous $\omega$-pseudoopen mapping of a space $X$ onto a topological group $G$, then $G$ is metrizable.

Proof. It follows from Proposition 2.2 that the space $G$ is Fréchet–Urysohn. Since $G$ is a topological group, a theorem of P. Nyikos from [11] implies that the space $G$ is strongly Fréchet–Urysohn. Hence, the fan-tightness of $G$ is countable. Note also that $G$ is regular, since $G$ is a topological group (see [6]). Applying Proposition 2.4, we conclude that the space $G$ has a countable sensor at every point. Fix $e \in G$ and $H = \{e\}$. Theorem 1.4 implies that the space $G$ has a countable base at $e$. It follows that $G$ is metrizable, since $G$ is a topological group (see [6]).

In connection with Theorem 2.12 we should mention that if a compact Hausdorff space $Y$ is an image of a metric space under a pseudoopen continuous $S$-mapping, then $Y$ is metrizable. This is a special case of a remarkable result of V.V. Filippov [9] on quotient $S$-images of metric spaces.

**Example 2.15.** There exists a pseudocompact Tychonoff separable non-metrizable space $M$ with an open covering $\gamma$ such that the following conditions are satisfied:

(a) Every element of $\gamma$ is a metrizable subspace of $M$;
(b) Each non-isolated in $M$ point belongs to at most one element of $\gamma$.

For example, the famous Mrowka space (often denoted by $\Psi$) satisfies the above restrictions on $M$.

Let $X$ be the free topological sum of the spaces in the family $\gamma$, and $f$ be the natural mapping of $X$ onto $M$ (thus, the restrictions of $f$ to elements of $\gamma$ are the identity mappings). Obviously, $X$ is a metric space, and $f$ is a continuous mapping of $X$ onto $M$. It is also well known, and easily seen, that $f$ is an open mapping. Hence, $f$ is pseudoopen as well. It follows from condition (b) that the inverse image under $f$ of any non-isolated point of $M$ consists of exactly one point. Therefore, $f$ is an almost $S$-mapping. Thus, we have constructed an open (hence, pseudoopen) continuous almost $S$-mapping of a metric space $X$ onto a Tychonoff pseudocompact separable non-metrizable space $M$. Notice, that the space $M$ cannot have a point-countable base but is locally metrizable and hence, is first-countable. Observe also that both $X$ and $M$ can be selected to be, in addition, locally compact and locally countable. However, $M$ cannot be made, in this situation, countably compact. This follows from the next statement.

**Proposition 2.16.** If $f$ is a continuous open almost $S$-mapping of a metric space $X$ onto a regular countably compact space $Y$, then $Y$ is separable, metrizable, and compact.

Proof. Let $Z$ be the subspace of $Y$ consisting of all non-isolated points of $Y$. Clearly, $Z$ is closed in $Y$. Therefore, $Z$ is countably compact. Let $g$ be the restriction of $f$ to the subspace $P = f^{-1}(Z)$. Then $g : P \to Z$ is an open continuous $S$-mapping of the metric space $P$ onto the regular countably compact space $Z$. It follows that the space $Z$ has a point-countable base, since $P$ has a point-countable base [8]. Therefore, by A.S. Mischenko’s Theorem [8], the space $Z$ has a countable base. Hence, $Z$ is separable, metrizable, and compact.

**Claim 1.** $Z$ is a $G_\delta$-subset of $Y$.

Since $Z$ is compact, Claim 1 obviously follows from the next statement:

**Claim 2.** The space $X$ has a countable point-wise base at the set $Z$.

Indeed, there exists a point-countable base $B$ for $X$. Put $\eta = \{U \in B: U \cap P \neq \emptyset\}$ and $C = \{f(U): U \in \eta\}$. Then, clearly, $C$ is a point-wise base of $Y$ at $Z$, every element of $Z$ is contained in at most countably many elements of $C$, and $V \cap Z \neq \emptyset$, for every $V \in C$. Since $Z$ is separable, it follows that the family $C$ is countable. Thus, $C$ is a countable point-wise base of $Y$ at $Z$. Claims 2 and 1 are established.

It follows easily from Claim 1 that $Y \setminus Z$ is the union of a countable family $\xi$ of closed subsets of $Y$. 


Claim 3. Each \( F \in \xi \) is finite.

Indeed, every point of \( Y \setminus Z \) is isolated in \( Y \). Thus, each \( F \in \xi \) is a countably compact discrete subspace of \( Y \). It follows that \( F \) is finite.

Using Claim 3, we conclude that \( Y \setminus Z \) is a countable discrete subspace of \( Y \). Thus, \( Y \) is the union of two separable metrizable subspaces. Since \( Y \) is also countably compact and regular, it follows from the Addition Theorem for the weight that \( Y \) is a separable metrizable compact space [8]. □

The next example of a mapping, considered on many other occasions, shows that the class of pseudoopen continuous mappings is much wider than the class of \( \omega \)-pseudoopen continuous mappings.

Example 2.17. Let \( G \) be the \( \Sigma \)-product of uncountably many copies of the discrete space \( D = \{0, 1\} \). It is well known that \( G \) is a countably compact Fréchet–Urysohn Tychonoff space [8]. The space \( G \) has many other nice properties as well. In particular, \( G \) is a topological group, and therefore, \( G \) is homogeneous. It is also clear that at no point \( G \) is first-countable. Hence, \( G \) is not metrizable.

Since the space \( G \) is Fréchet–Urysohn and Hausdorff, one can represent \( G \), in a standard way, as an image of a locally compact locally countable metric space \( X \) under a (natural) pseudoopen continuous mapping \( f \) [1]. This mapping \( f \) is not \( \omega \)-pseudoopen, since the space \( G \) is pseudocompact (even countably compact) and is not first-countable. Observe also that the mapping \( f \) is not biquotient.

We should also refer the reader to an example of a Tychonoff countable Fréchet–Urysohn space described in [3]. This space is not first-countable at any point. However, it is shown in [3] that the space can be represented as an image of a countable metrizable space under a continuous pseudoopen mapping (which is, clearly, an \( S \)-mapping). On the other hand, it was shown in [3] that if a topological group \( G \) is an image of a separable metrizable space under a pseudoopen continuous mapping, then \( G \) is metrizable.

Finally, we give an application of our results on pseudoopen \( S \)-mappings to closed \( S \)-mappings. Recall that a pseudo-compact regular space with a uniform base is compact and metrizable [12]. We generalize this result as follows:

Theorem 2.18. Suppose that \( f \) is a continuous closed almost \( S \)-mapping of a regular space \( X \) with a uniform base onto a pseudocompact regular space \( Y \). Then \( Y \) is compact and metrizable.

Proof. Every uniform base, called also a point-regular base in [8], is point-countable (see [8]). Obviously, Theorem 2.12 extends from metrizable spaces to spaces with a point-countable base. Since every closed mapping is pseudoopen, it follows from the extended version of Theorem 2.12 that \( Y \) is first-countable. Now, applying the well-known Vainshtein–Morita–Stone Theorem [8,5], we conclude that the boundary of \( f^{-1}(y) \) is compact, for each \( y \in Y \). Hence, the restriction of \( f \) to some closed subspace \( Z \) of \( X \) is a perfect mapping of \( Z \) onto \( Y \) [8,5]. Since \( Z \) also has a uniform base, it follows that \( Y \) has a uniform base as well (see [7]). It remains to refer to the fact that every pseudocompact regular space with a uniform base is metrizable and compact (see [12]). □

References


