A bivariate INAR(1) time series model with geometric marginals

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ARTICLE INFO	ABSTRACT
	In this paper we introduce a simple bivariate integer-valued time series model with positively correlated geometric marginals based on the negative binomial thinning mechanism. Some properties of the model are considered. The unknown parameters of the model are estimated using the modified conditional least squares method.

1. Introduction

The integer-valued autoregressive processes play an important role in time series analysis. They are useful in many areas, such as biology, medicine, business, etc. Detailed information about this kind of process can be found in [1,2]. In particular, bivariate integer-valued time series models maintain the pairing between two count variables that occur over specific times and play a major role in the analysis of the paired correlated count data. In the last two decades, special attention has been devoted to the multivariate integer-valued time series processes. Franke and Subba Rao [3] introduced the M-variate INAR(1) process given by $\mathbf{X}_t = \mathbf{A} \circ \mathbf{X}_{t-1} + \mathbf{Z}_t$, where $\{\mathbf{Z}_t\}$ is a sequence of independent and identically distributed (i.i.d.) random vectors and for a given matrix $\mathbf{A} = [a_{ij}]$, $a_{ij} \in [0, 1]$, the i-th component of the random vector $\mathbf{A} \circ \mathbf{X}$ is defined as $(\mathbf{A} \circ \mathbf{X})_i = \sum_{j=1}^{M} a_{ij} \circ X_j$, $i = 1, \ldots, M$. Here $a\circ$ represents the binomial thinning. Aly and Bouzar [4] introduced a multivariate process $\mathbf{X}_t = \mathbf{A}_{[\mathbf{a}],\theta} \bullet \mathbf{X}_{t-1} + \mathbf{Z}_t$, where $\{\mathbf{Z}_t\}$ is a sequence of i.i.d. random vectors, $\mathbf{a} = [\mathbf{a}_1, \ldots, \mathbf{a}_p]$, $\mathbf{a}_j = (a_{1j}, \ldots, a_{pj})'$, $\mathbf{A}_{[\mathbf{a}],\theta} \bullet \mathbf{X} = \sum_{i=1}^{p} A_{\mathbf{a}_i,\theta} \bullet X_i$, the operators $A_{\mathbf{a}_i,\theta}$ are independent and given as $A_{\mathbf{a}_j,\theta} \bullet X = \left(\sum_{i=1}^{N_1(X)} W_i^{(a_{j1})}, \ldots, \sum_{i=1}^{N_p(X)} W_i^{(a_{jp})} \right)'$, where $\left\{ W_i^{(a_{jk})} \right\}$, $k = 1, 2, \ldots, p$, are independent sequences of i.i.d. random variables with zero truncated Geom$((1 - a_{jk})\theta)$ distributions, $\theta \in [0, 1]$, and $(N_1(X), \ldots, N_p(X))$ has multinomial $(x, \mathbf{a}_j)$ distribution for given $X = x$. Latour [5] introduced a multivariate GINAR(p) model as $\mathbf{X}_t = \sum_{i=1}^{p} \mathbf{A}_i \star \mathbf{X}_{t-1} + \mathbf{Z}_t$, where $\mathbf{A}_i \star$ are generalized Steutel and van Harn operators. Based on generalized Steutel and van Harn thinning operators, Aly and Bouzar [4] discussed multiple INAR(1) processes via some multiple discrete distributions. Recently, there have been more attempts to derive BINAR type models. We refer the reader to Brännäs and Nordström [6], Quoreshi [7] and Pedeli and Karlis [8] for more details.

The purpose of this paper is to introduce a simple bivariate INAR(1) model with positively correlated geometric marginals. The model is constructed using the method proposed by Dewald et al. [9] for constructing a simple bivariate autoregressive model with exponential marginals and the negative binomial thinning introduced by Aly and Bouzar [4] and Ristić et al. [10].

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The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we consider the matrix representation of the model and derive some of its properties. In Section 4 we estimate the unknown parameters of the model using the modified conditional least squares method. Finally, some concluding remarks are given in Section 5.

2. Construction of the model

Ristić et al. [10] introduced the negative binomial thinning operator $\alpha*$ as $\alpha * Z = \sum_{i=1}^{Z} G_i$, $\alpha \in (0, 1)$, with $\alpha * Z = 0$ for $Z = 0$. All the counting series $\{G_i\}$ are i.i.d. random variables with $\text{Geom}(\alpha/(1+\alpha))$ distribution, i.e. with probability mass function of the form $P(G_i = k) = \frac{\alpha^k}{(1+\alpha)^{k+1}}$, $k \geq 0$. Let us also consider the operator $\beta*$ given by $\beta * Z = \sum_{j=1}^{Z} W_j$, $\beta \in (0, 1)$, where $\{W_j\}$ are i.i.d. random variables with $\text{Geom}(\beta/(1+\beta))$ distribution. Based on these operators and the method proposed by Dewald [9], we introduce a bivariate time series model $\{(X_t, Y_t), \ t \geq 0\}$ given by

$$X_t = \begin{cases} \alpha * X_{t-1} + \varepsilon_t, & \text{w.p. } p, \\ \alpha * Y_{t-1} + \varepsilon_t, & \text{w.p. } 1-p, \end{cases} \tag{1}$$

$$Y_t = \begin{cases} \beta * X_{t-1} + \eta_t, & \text{w.p. } q, \\ \beta * Y_{t-1} + \eta_t, & \text{w.p. } 1-q, \end{cases} \tag{2}$$

where $p, q \in [0, 1]$, $\alpha, \beta \in (0, 1)$, $\{\varepsilon_t, \ t \geq 1\}$ and $\{\eta_t, \ t \geq 1\}$ are mutually independent sequences of i.i.d. random variables independent of (X_s, Y_s), for all $s < t$. We suppose that all thinnings $\alpha * X_{t-1}$, $\alpha * Y_{t-1}$, $\beta * X_{t-1}$ and $\beta * Y_{t-1}$ are mutually independent.

In the following theorem, we give a necessary and sufficient condition for a bivariate time series model $\{(X_t, Y_t)\}$ to be stationary.

Theorem 1. *Let $\mu > 0$, $\alpha, \beta \in (0, \mu/(1+\mu)]$ and $X_0 \stackrel{d}{=} Y_0 \stackrel{d}{=} \text{Geom}(\mu/(1+\mu))$. The bivariate time series $\{(X_t, Y_t), \ t \geq 0\}$ given by (1) and (2) is stationary with $\text{Geom}(\mu/(1+\mu))$ marginals if and only if the mutually independent sequences of i.i.d. random variables $\{\varepsilon_t, \ t \geq 1\}$ and $\{\eta_t, \ t \geq 1\}$ are distributed as*

$$\varepsilon_t \stackrel{d}{=} \begin{cases} \text{Geom}(\alpha/(1+\alpha)), & \text{w.p. } \alpha\mu/(\mu-\alpha), \\ \text{Geom}(\mu/(1+\mu)), & \text{w.p. } (\mu-\alpha-\alpha\mu)/(\mu-\alpha), \end{cases} \tag{3}$$

$$\eta_t \stackrel{d}{=} \begin{cases} \text{Geom}(\beta/(1+\beta)), & \text{w.p. } \beta\mu/(\mu-\beta), \\ \text{Geom}(\mu/(1+\mu)), & \text{w.p. } (\mu-\beta-\beta\mu)/(\mu-\beta). \end{cases} \tag{4}$$

Proof. Let us suppose that the random variables ε_t and η_t are distributed as (3) and (4). Ristić et al. [10] showed that these random variables have the probability generating functions (p.g.f.) given by $\Phi_\varepsilon(s) = (1+\alpha(1+\mu)(1-s))/((1+\alpha(1-s))(1+\mu(1-s)))$ and $\Phi_\eta(s) = (1 + \beta(1+\mu)(1-s))/((1 + \beta(1-s))(1 + \mu(1-s)))$, respectively. Let $\Phi_{X_t}(s)$ and $\Phi_{Y_t}(s)$ be the p.g.f. of the random variables X_t and Y_t, respectively. Since the p.g.f. of the random variables X_0 and Y_0 are $\Phi_{X_0}(s) = \Phi_{Y_0}(s) = 1/(1+\mu(1-s))$, we obtain from (1) and (2) that

$$\Phi_{X_1}(s) = \Phi_{X_0}\left(\frac{1}{1+\alpha(1-s)}\right) \cdot \frac{1+\alpha(1+\mu)(1-s)}{(1+\alpha(1-s))(1+\mu(1-s))} = \frac{1}{1+\mu(1-s)},$$

$$\Phi_{Y_1(s)} = \Phi_{Y_0}\left(\frac{1}{1+\beta(1-s)}\right) \cdot \frac{1+\beta(1+\mu)(1-s)}{(1+\beta(1-s))(1+\mu(1-s))} = \frac{1}{1+\mu(1-s)},$$

which implies that X_1 and Y_1 are distributed as $\text{Geom}(\mu/(1+\mu))$. Finally, by induction we can show that X_t and Y_t, $t \geq 0$, are distributed as $\text{Geom}(\mu/(1+\mu))$.

Conversely, let us suppose that the bivariate time series $\{(X_t, Y_t), \ t \geq 0\}$ is stationary with $\text{Geom}(\mu/(1+\mu))$ marginals. Then from the facts that $\Phi_\varepsilon(s) = \Phi_X(s)/\Phi_X(1/(1+\alpha(1-s)))$ and $\Phi_\eta(s) = \Phi_X(s)/\Phi_X(1/(1+\beta(1-s)))$, it is obtained that $\{\varepsilon_t, \ t \geq 1\}$ and $\{\eta_t, \ t \geq 1\}$ are distributed as (3) and (4), respectively. $\square$

Let us consider the case when X_0 and Y_0 have the same arbitrary distribution. From (1) it follows that

$$\begin{aligned} \Phi_{X_t}(s) &= \Phi_{X_{t-1}}\left(\frac{1}{1+\alpha-\alpha s}\right) \Phi_\varepsilon(s) \\ &= \Phi_{X_0}\left(\frac{1-\alpha^t-\alpha(1-\alpha^{t-1})s}{1-\alpha^{t+1}-\alpha(1-\alpha^t)s}\right) \prod_{j=0}^{t-1} \Phi_\varepsilon\left(\frac{1-\alpha^j-\alpha(1-\alpha^{j-1})s}{1-\alpha^{j+1}-\alpha(1-\alpha^j)s}\right) \\ &= \Phi_{X_0}\left(\frac{1-\alpha^t-\alpha(1-\alpha^{t-1})s}{1-\alpha^{t+1}-\alpha(1-\alpha^t)s}\right) \frac{1-\alpha^{t+1}+(1-\alpha)\alpha^t\mu-\alpha(1-\alpha^t+(1-\alpha)\alpha^{t-1}\mu)s}{(1-\alpha^{t+1}-\alpha(1-\alpha^t)s)(1+\mu-\mu s)}. \end{aligned}$$

Taking the limit as $t \to \infty$, we obtain that X_t converges to $\text{Geom}(\mu/(1+\mu))$ distribution.

Now, we will derive the joint conditional distribution of X_t and Y_t for given X_{t-1} and Y_{t-1}. This property can be used for conditional maximum likelihood estimation and prediction, for example. Let $p(x, y|u, v) = P(X_t = x, Y_t = y|X_{t-1} = u, Y_{t-1} = v)$. To simplify the derivation, we define the functions $\psi_\varepsilon(x, u, \alpha)$ and $p_{A_{t-1}, B_{t-1}}(x, y|u, v)$ as

$$\psi_\varepsilon(x, u, \alpha) = (1 + \alpha)^{-u} \left\{ \pi_\varepsilon(x) + I(u \geq 1) \sum_{g=1}^{x} \pi_\varepsilon(x - g) \binom{u + g - 1}{g} \alpha^g (1 + \alpha)^{-g} \right\},$$

$$p_{A_{t-1}, B_{t-1}}(x, y|u, v) = P(\alpha * A_{t-1} + \varepsilon_t = x, \beta * B_{t-1} + \eta_t = y|X_{t-1} = u, Y_{t-1} = v),$$

where $\pi_\varepsilon(x) = P(\varepsilon_t = x)$, I denotes the indicator function and $(A_{t-1}, B_{t-1}) \in \{(X_{t-1}, X_{t-1}), (X_{t-1}, Y_{t-1}), (Y_{t-1}, X_{t-1}), (Y_{t-1}, Y_{t-1})\}$. Using the independency of the thinnings and the random variables ε_t and η_t, and the fact that $\alpha * X_{t-1}$ and $\beta * X_{t-1}$ for given $X_{t-1} = 0$ have zero values, we obtain for $v \geq 0$ that $p_{X_{t-1}, X_{t-1}}(x, y|0, v) = \pi_\varepsilon(x)\pi_\eta(y) = \psi_\varepsilon(x, 0, \alpha)\psi_\eta(y, 0, \beta)$. Similarly, using the independency of the thinnings and the random variables ε_t and η_t, and the fact that random variables $\alpha * X_{t-1}$ and $\beta * X_{t-1}$ for given $X_{t-1} = u$, $u \geq 1$, have negative binomial distributions with parameters u and $\alpha/(1 + \alpha)$, and u and $\beta/(1 + \beta)$, respectively, we obtain that

$$p_{X_{t-1}, X_{t-1}}(x, y|u, v) = P\left(\sum_{i=1}^{u} G_i + \varepsilon_t = x\right) P\left(\sum_{j=1}^{u} W_j + \eta_t = y\right) = \psi_\varepsilon(x, u, \alpha)\psi_\eta(y, u, \beta).$$

Thus, we obtain for $u \geq 0$ and $v \geq 0$ that $p_{X_{t-1}, X_{t-1}}(x, y|u, v) = \psi_\varepsilon(x, u, \alpha)\psi_\eta(y, u, \beta)$. The same derivation can be applied for $p_{X_{t-1}, Y_{t-1}}(x, y|u, v)$, $p_{Y_{t-1}, X_{t-1}}(x, y|u, v)$ and $p_{Y_{t-1}, Y_{t-1}}(x, y|u, v)$, which implies after some calculations that the joint conditional distribution of X_t and Y_t for given X_{t-1} and Y_{t-1} is given by $p(x, y|u, v) = [p\psi_\varepsilon(x, u, \alpha) + (1 - p)\psi_\varepsilon(x, v, \alpha)][q\psi_\eta(y, u, \beta) + (1 - q)\psi_\eta(y, v, \beta)]$.

3. Matrix representation and properties

The bivariate time series model $\{(X_t, Y_t), t \geq 0\}$ given by (1) and (2) can be represented as

$$X_t = U_{t1} * X_{t-1} + U_{t2} * Y_{t-1} + \varepsilon_t, \tag{5}$$

$$Y_t = V_{t1} * X_{t-1} + V_{t2} * Y_{t-1} + \eta_t, \tag{6}$$

where the random vectors $\{(U_{t1}, U_{t2})\}$ and $\{(V_{t1}, V_{t2})\}$ are mutually independent and identically distributed as $P(U_{t1} = \alpha, U_{t2} = 0) = 1 - P(U_{t1} = 0, U_{t2} = \alpha) = p$ and $P(V_{t1} = \beta, V_{t2} = 0) = 1 - P(V_{t1} = 0, V_{t2} = \beta) = q$. If we define $\mathbf{A}_t * \mathbf{X}$ as

$$\mathbf{A}_t * \mathbf{X} = \begin{bmatrix} U_{t1} * X + U_{t2} * Y \\ V_{t1} * X + V_{t2} * Y \end{bmatrix}, \quad \text{where } \mathbf{A}_t = \begin{bmatrix} U_{t1} & U_{t2} \\ V_{t1} & V_{t2} \end{bmatrix},$$

and $\mathbf{X} = (X, Y)'$, then (5) and (6) can be represented in matrix form $\mathbf{X}_t = \mathbf{A}_t * \mathbf{X}_{t-1} + \mathbf{Z}_t$, where $\mathbf{X}_t = (X_t, Y_t)'$ and $\mathbf{Z}_t = (\varepsilon_t, \eta_t)'$. Using this representation, we can easily derive the autocovariance matrix. The autocovariance matrix is given by

$$\text{Cov}(\mathbf{X}_h, \mathbf{X}_0) = \begin{bmatrix} \text{Cov}(X_h, X_0) & \text{Cov}(X_h, Y_0) \\ \text{Cov}(Y_h, X_0) & \text{Cov}(Y_h, Y_0) \end{bmatrix}.$$

Since $E(\mathbf{A}_h * \mathbf{X}_{h-1}) = \mathbf{A}E(\mathbf{X}_{h-1})$ and $E((\mathbf{A}_h * \mathbf{X}_{h-1})\mathbf{X}_0') = \mathbf{A}E(\mathbf{X}_{h-1}\mathbf{X}_0')$, where $\mathbf{A} = E(\mathbf{A}_h)$, we obtain for $h \geq 0$ that the autocovariance matrix is $\text{Cov}(\mathbf{X}_h, \mathbf{X}_0) = \mathbf{A}^h \text{Var}(\mathbf{X}_0)$. Let us consider the matrix $\text{Var}(\mathbf{X}_0)$. First, since the random variables X_0 and Y_0 have $\text{Geom}(\mu/(1 + \mu))$ distribution, it follows that $\text{Var}(X_0) = \text{Var}(Y_0) = \mu(1 + \mu)$. Second, the covariance $\text{Cov}(X_0, Y_0)$ can be easily derived from (1) and (2) and the fact that the bivariate time series model $\{(X_t, Y_t)\}$ is stationary. Thus we obtain that

$$\text{Cov}(X_0, Y_0) = \frac{\alpha\beta(pq + (1-p)(1-q))}{1 - \alpha\beta(p(1-q) + (1-p)q)} \cdot \mu(1 + \mu) \equiv \rho \cdot \mu(1 + \mu).$$

Obviously, ρ is positive. On the other hand, since $\rho - 1 = (\alpha\beta - 1)/(1 - \alpha\beta(p(1-q) + (1-p)q)) < 0$, we obtain that $\rho < 1$. Using the obtained results, we derive the matrix $\text{Var}(\mathbf{X}_0)$ as

$$\text{Var}(\mathbf{X}_0) = \mu(1 + \mu) \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

Since the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \alpha p & \alpha(1 - p) \\ \beta q & \beta(1 - q) \end{bmatrix},$$

we can conclude that all the elements of Cov($\mathbf{X}_h, \mathbf{X}_0$) are positive. Thus, we have shown that the two time series are positively correlated. Now, we will determine how much they are positively correlated. First, we can conclude that $\rho = \alpha\beta$ for $p = q = 0$ or $p = q = 1$. Second, we have for any $p, q \in [0, 1]$ that

$$\rho - \alpha\beta = \frac{-\alpha\beta(1 - \alpha\beta)(p(1 - q) + q(1 - p))}{1 - \alpha\beta(p(1 - q) + q(1 - p))} \leq 0,$$

which implies that $\rho \leq \alpha\beta$. Since $0 < \alpha, \beta \leq \mu/(1 + \mu)$, we finally obtain that $0 \leq \rho \leq \mu^2(1 + \mu)^{-2}$. Therefore, for large values of μ the two time series can be highly positively correlated, while for a small μ the two series are weakly positively correlated.

Now, we will show that Cov($\mathbf{X}_h, \mathbf{X}_0$) $\to$ $\mathbf{0}$, as $h \to \infty$. The eigenvalues $\lambda_1 < \lambda_2$ of the matrix $\mathbf{A}$ satisfy: (i) $\lambda_1 + \lambda_2 = \alpha p + \beta(1 - q) > 0$, which implies that $\lambda_2 > 0$, and (ii) $\lambda_1\lambda_2 = \alpha\beta(p - q)$. Using these results, we have that $(1 - \lambda_1)(1 - \lambda_2) = (1 - \alpha)(1 - \beta(1 - q)) + \alpha(1 - p)(1 - \beta) > 0$ and $(1 + \lambda_1)(1 + \lambda_2) = 1 - \alpha\beta(1 - p) + \alpha p + \beta(1 + \alpha)(1 - q) > 0$. Since $\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < \alpha + \beta < 2$, we obtain that $|\lambda_1| < 1$ and $0 < \lambda_2 < 1$. This implies that $\mathbf{A}^h \to \mathbf{0}$ and Cov($\mathbf{X}_h, \mathbf{X}_0$) $\to$ $\mathbf{0}$, as $h \to \infty$. Thus, this well known property of autoregressive processes is confirmed for our bivariate process. While the model is autoregressive with respect to the vector, this is not the case for the marginals. Namely, it is clear from the definition of the process (1) and (2) that these series $\{X_t\}$ and $\{Y_t\}$ are not autoregressive, which may also be verified from the diagonal elements of the matrix $\mathbf{A}^h\text{Var}(\mathbf{X}_0)$.

4. Modified conditional least squares estimation

In this section we briefly consider the estimation of the unknown parameters of the model by the modified conditional least squares method. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a bivariate sample from the bivariate INAR(1) model with geometric marginals. First we consider the estimation of the parameter μ. Using the fact that $\mu = E(X_t) = E(Y_t)$, an estimator obtained by the Yule–Walker method has the following form: $\hat{\mu} = (2N)^{-1}\sum_{t=1}^{N}(X_t + Y_t)$. In order to obtain the conditional least squares estimators of the parameters α, β, p and q, we introduce the new parametrization $\theta_1 = \alpha p, \theta_2 = \alpha(1 - p), \theta_3 = \beta q$ and $\theta_4 = \beta(1 - q)$. Using the matrix representation of the model, we obtain that the conditional expectation is

$$E(\mathbf{X}_t|\mathbf{X}_{t-1}) = E(\mathbf{A}_t * \mathbf{X}_{t-1} + \mathbf{Z}_t|\mathbf{X}_{t-1}) = \begin{bmatrix} \alpha p X_{t-1} + \alpha(1 - p)Y_{t-1} \\ \beta q X_{t-1} + \beta(1 - q)Y_{t-1} \end{bmatrix} + \begin{bmatrix} E(\varepsilon_t) \\ E(\eta_t) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha p & \alpha(1 - p) \\ \beta q & \beta(1 - q) \end{bmatrix} \cdot \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \mu \begin{bmatrix} 1 - \alpha \\ 1 - \beta \end{bmatrix} = \begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{bmatrix} \cdot \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \mu \begin{bmatrix} 1 - \theta_1 - \theta_2 \\ 1 - \theta_3 - \theta_4 \end{bmatrix}.$$

Let $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. Then the conditional least squares estimators of the parameter vector Θ are obtained by minimizing the sum of squares:

$$Q_N(\Theta) = \sum_{t=2}^{N}(\mathbf{X}_t - \mathbf{A}\mathbf{X}_{t-1} - E(\mathbf{Z}_t))'(\mathbf{X}_t - \mathbf{A}\mathbf{X}_{t-1} - E(\mathbf{Z}_t))$$

$$= \sum_{t=2}^{N}(X_t - \theta_1 X_{t-1} - \theta_2 Y_{t-1} - (1 - \theta_1 - \theta_2)\mu)^2 + \sum_{t=2}^{N}(Y_t - \theta_3 X_{t-1} - \theta_4 Y_{t-1} - (1 - \theta_3 - \theta_4)\mu)^2.$$

Thus the estimators are the solutions of the following system:

$$\sum_{t=2}^{N}(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})' \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \sum_{t=2}^{N}(X_t - \hat{\mu})(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})',$$

$$\sum_{t=2}^{N}(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})' \begin{bmatrix} \hat{\theta}_3 \\ \hat{\theta}_4 \end{bmatrix} = \sum_{t=2}^{N}(Y_t - \hat{\mu})(\mathbf{X}_{t-1} - \hat{\boldsymbol{\mu}})',$$

where $\hat{\boldsymbol{\mu}} = (\hat{\mu}, \hat{\mu})'$. Following the definition of the new introduced parametrization, the conditional least squares estimators of the parameters α, β, p and q are derived as $\hat{\alpha} = \hat{\theta}_1 + \hat{\theta}_2, \hat{\beta} = \hat{\theta}_3 + \hat{\theta}_4, \hat{p} = \hat{\theta}_1/(\hat{\theta}_1 + \hat{\theta}_2)$, and $\hat{q} = \hat{\theta}_3/(\hat{\theta}_3 + \hat{\theta}_4)$. It is interesting to derive the properties of these estimators, which we leave for future work.

5. Concluding remarks

In this paper we introduced a new approach to a bivariate INAR(1) time series analysis. For this purpose, we based our investigation on a time series of geometrically distributed marginals, generated by negative binomial thinning. Nevertheless, further research in this field might be performed in some new directions. First, the marginal distribution could be generalized by using a negative binomial distribution, which is more applicable and thus can be used in far more situations. Hence, this assumption would make the model much more realistic. Second, a binomial thinning operator might be applied together

with the Poisson or negative binomial marginal distribution which might produce more model real-life applications. Third, one may consider a bivariate distribution for the innovations so that the correlation comes from two sources. Finally, a very important subject of our interest is to define and analyze an m-variate case of the INAR process, for $m > 2$.

Acknowledgments

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