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A numerical method for nonlinear age-structured population models with finite maximum age

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ABSTRACT

We propose a new numerical method for the approximation of solutions to a non-autonomous form of the classical Gurtin–MacCamy population model with a mortality rate that is the sum of an intrinsic age-dependent rate that becomes unbounded as the age approaches its maximum value, plus a non-local, non-autonomous, bounded rate that depends on some weighted population size. We prove that our new quadrature based method converges to second-order and we show the results of several numerical simulations.

1. Introduction

The mathematical theory of age-structured populations is quite well developed [10,11]. They play a major role in ecological modeling, as well as in demography, epidemiology, cancer modeling, and other fields. Many models simplify the mathematical analysis by assuming that the mortality rate—as well as all other modeling parameters—is bounded, automatically leading to the possibility of immortality. Some more realistic models impose a maximum age that cannot be reached and must require that the mortality rate become unbounded at that age. Lotka [15,16] and McKendrick [18] are credited with the first age-structured model, a linear one that supports exponential solutions, just as the unstructured Malthus [17] model. A nonlinear form of that model was first proposed by Gurtin and MacCamy [9] by making the fertility and mortality rates dependent on the total population size, that is the integral of the age density.

Numerical methods to approximate the solution of such population models have been proposed and analyzed during the past twenty-five years. For an excellent review of these methods see [1]. Two of the main objectives driving the need for numerical methods are, first, the need to make projections about population growth for the future, usually for periods of 10–50 years. Secondly, there is the theoretical interest in long-term simulations for the purpose of analyzing trends under different scenarios. This is an important aspect of population models in theoretical biology. Some examples of numerical studies of structured population dynamics are found, for example, in [2–6,8,19].

For the use of real life data we have to limit ourselves to low regularity in the data, thus limiting the practicality of the numerical methods to second-order requiring just one continuous derivative for the coefficients. For long-term simulations higher-order methods would be more desirable and, therefore, second-order methods are a good compromise between performance and regularity demands. The first to point out that standard numerical methods based on uniform meshes degenerate and do not converge near the age of unbounded mortality were Iannelli and Milner [12].

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Kim and Kwon [13] introduced a numerical method that reaches its optimal order of convergence when the tail of the mortality function has some specific analytic form. Even though the partial differential equations in the model can be readily integrated along the characteristics as ordinary differential equations—thus providing natural numerical methods that consist of approximating that integral representation—very few numerical methods are based on that explicit analytic representation.

A linear model might be sufficient for short term population projections in which the use of finite maximum age can result in quantitatively better projections for advanced ages [7]. However, in order to provide more real-life applicability to the quantitative and qualitative study of population dynamics, we must consider a nonlinear population model with finite maximum age. Only these allow for the numerical study of the stability of steady states in the absence of theoretical results, as well as the possible appearance of bifurcations. Regularity is not a concern for such studies.

The new method we propose and analyze is based on quadratures of the integrals that appear in the explicit representation of solutions obtained by integration along characteristics. It converges at its optimal order—second—for more general tails that include some for which other methods in the literature fail. In particular, our method is applicable to families of mortality functions that include those most often used in population modeling.

The paper is organized as follows: Section 2 is devoted to a detailed description of the model. In Section 4 we present the convergence analysis of the numerical method introduced in Section 3. Finally, the fifth Section is devoted to numerical simulations and conclusions.

2. The model

We consider the problem of modeling the evolution of the age density of a population given by the following classical system:

$$u_t + u_a = -\big(m(a) + \mu\big(a, I_\mu(t), t\big)\big)u, \quad 0 < a < a_\dagger,\ t > 0, \tag{2.1}$$

$$u(0,t) = \int_0^{a_\dagger} \alpha\big(a, I_\alpha(t), t\big)u(a,t)\,da, \quad t > 0, \tag{2.2}$$

$$u(a,0) = u_0(a), \quad 0 \leqslant a \leqslant a_\dagger, \tag{2.3}$$

$$I_s(t) = \int_0^{a_\dagger} \gamma_s(a)u(a,t)\,da, \quad t > 0,\ s = \mu, \alpha, \tag{2.4}$$

where the independent variables a and t represent, respectively, age and time. The function $u(\cdot,t)$ is the age density of the population at time t and the vital functions are given by the age-specific mortality rate $(m(\cdot) + \mu(\cdot, I_\mu(t), t))$ and the age-specific fertility rate $(\alpha(\cdot, I_\alpha(t), t))$. For biological reasons, the latter will always be assumed to be bounded in all its arguments by a constant that we shall simply denote as $\|\alpha\|_\infty$. We consider the mortality rate given in separable form consisting of two terms, an intrinsic mortality $m(a)$—that is unbounded to take into account a maximum age $a_\dagger$—and a bounded mortality that, as the fertility rate, includes seasonality (through the dependence on the time) and resource competition (through the dependence on the non-local functionals $I_s(t)$). Finally, $u_0(\cdot)$ denotes the initial age distribution that, for biological reasons, will be assumed nonnegative and integrable. A derivation, analysis of well-posedness, and asymptotic behavior of solutions can be found, for example, in [10].

Note that the initial size of the population is $P_0 = \int_0^{a_\dagger} u_0(a)\,da$ and, at any time $t \geqslant 0$, the population size is certainly bounded by $\mathcal{P}(t) = P_0 e^{t\|\alpha\|_\infty}$. We shall use this bound $\mathcal{P}$ below in order to describe the regularity assumptions on the coefficients that we need for the solution to be smooth enough that our numerical method converge at second order.

In this paper we introduce a second order numerical method to approximate the solution of this nonlinear model and we also carry out its convergence analysis. Throughout the paper we assume the following regularity conditions on the data functions ($\bar{\varepsilon} > 0$ is a fixed positive constant):

(H1) $\gamma_\mu, \gamma_\alpha \in \mathcal{C}^2([0, a_\dagger])$;
(H2) $m \in \mathcal{C}^2([0, a_\dagger))$, is nonnegative and $\int_0^{a_\dagger} m(\sigma)\,d\sigma = +\infty$;
(H3) $\mu \in \mathcal{C}^2([0, a_\dagger] \times [-\bar{\varepsilon}, \mathcal{P}(T)\|\gamma_\mu\|_\infty + \bar{\varepsilon}] \times [0, T])$ is nonnegative;
(H4) $\alpha \in \mathcal{C}^2([0, a_\dagger] \times [-\bar{\varepsilon}, \mathcal{P}(T)\|\gamma_\alpha\|_\infty + \bar{\varepsilon}] \times [0, T])$ is nonnegative;
(H5) $u_0 \in \mathcal{C}^2([0, a_\dagger])$ satisfies the zero-oder compatibility condition $u_0(0) = \int_0^{a_\dagger} \alpha(a, I_\alpha(0), 0)u_0(a)\,da$, the corresponding first- and second-order relations to match to second order the boundary and initial conditions (2.2)–(2.3), as well as necessary decay conditions at $a_\dagger$, including

$$\lim_{a \to a_\dagger} u_0(a) \exp\left(\int_0^a m(s)\,ds\right) < +\infty.$$

Then the solution of problem (2.1)–(2.4) satisfies

$$u \in \mathcal{C}^2\big([0, a_\dagger) \times [0, T]\big), \qquad u(a,t) \geqslant 0, \quad \text{for } a \in [0, a_\dagger],\ t \geqslant 0. \tag{2.5}$$

The regularity part of (2.5) can be derived along the lines of the $\mathcal{C}^1$-results of Iannelli [10] but, as stated here, it is not in the literature and its proof escapes the scope of this article.

3. The numerical method

The numerical method that we propose integrates the model along the characteristic lines $a - t = c$, c constant, where

$$\frac{d}{dt} u(t+c, t) = -\big(m(t+c) + \mu\big(t+c, I_\mu(t), t\big)\big) u(t+c, t). \tag{3.1}$$

Therefore, the integral representation of the solutions of (3.1) along the characteristics is given by the following relation: for each $\bar{a}$, with $0 < \bar{a} < a_\dagger$, and $h > 0$ such that $\bar{a} + h < a_\dagger$,

$$u(\bar{a}+h, t_0+h) = u(\bar{a}, t_0) \exp\left(-\int_0^h \big[m(\bar{a}+\tau) + \mu\big(\bar{a}+\tau, I_\mu(t_0+\tau), t_0+\tau\big)\big]\, d\tau\right). \tag{3.2}$$

The initial difficulty to produce a numerical method for a problem like (2.1)–(2.4) is that the intrinsic mortality is unbounded. Therefore, we consider an intermediate value $A^* \in (0, a_\dagger)$ such that m is bounded in $[0, A^*]$, and we know the function $f(a) = \int_{A^*}^a m(\sigma)\, d\sigma$, $A^* \leqslant a \leqslant a_\dagger$, and $f(a_\dagger) = +\infty$ [7,13]. Note that, in order to model the dynamics of a specific population, the parameters A^* and $a_\dagger$, and the function μ should be determined from the field data [7].

The numerical method we propose consists of the discretization of (3.2). First, we introduce an age grid on $[0, a_\dagger]$ but, taking into account the value A^* that we want to keep identified, we introduce a positive integer J^*, and we define the step size $h = A^*/J^*$. Then, the total number of grid points is given by $J = [a_\dagger/h]$, where $[\cdot]$ denotes the integer part and the nodes of the uniform partition of the interval $[0, a_\dagger]$ are $a_j = jh$, $0 \leqslant j \leqslant J$ (note that $a_{J^*} = A^*$ and $a_J \leqslant a_\dagger$). We shall also use the notation $a_{j+\frac{1}{2}} = a_j + \frac{h}{2} = (j + \frac{1}{2})h$ to denote the "midpoint nodes". We will integrate the problem in a fixed time interval $[0, T]$, so we define the discrete time levels, $t_n = nh$, $0 \leqslant n \leqslant N$, where $N = [T/h]$, as well as the intermediate time levels $t_{n+\frac{1}{2}} = t_n + \frac{h}{2} = (n + \frac{1}{2})h$.

The notation U_j^n will represent the numerical approximation to $u(a_j, t_n)$, $0 \leqslant j \leqslant J$, $0 \leqslant n \leqslant N$ (the subscript j refers to the age grid point a_j and the superscript n to the time level t_n). We also denote these approximations in vector form: $\mathbf{U}^n = [U_0^n, U_1^n, \ldots, U_J^n]$, $0 \leqslant n \leqslant N$.

We approximate now the directional derivatives (3.1) along characteristics using the explicit formula (3.2), where the evaluation of the right-hand side is performed by approximating the integral therein using the composite midpoint rule, thus providing second-order accuracy. The integrals that describe births are approximated by a second-order modified composite trapezoidal rule.

Given approximations of the initial age density on the age grid,

$$\mathbf{U}^0 = \big[U_0^0, U_1^0, \ldots, U_J^0\big]$$

(which could be taken as the grid restrictions of the initial density $U_j^0 = u_0(a_j)$, $0 \leqslant j \leqslant J$) the numerical method is defined by the following recursion that provides the numerical approximation at the time level $n+1$, ($\mathbf{U}^{n+1}$), from that at the time level n, ($\mathbf{U}^n$), $0 \leqslant n \leqslant N-1$:

$$U_{j+1}^{n+1} = U_j^n \exp\big(-h\big[m(a_{j+\frac{1}{2}}) + \mu\big(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\big(\boldsymbol{\gamma}_\mu \mathbf{U}^{n+\frac{1}{2}}\big), t_{n+\frac{1}{2}}\big)\big]\big), \quad 0 \leqslant j \leqslant J^* - 1, \tag{3.3}$$

$$U_{j+1}^{n+1} = U_j^n \exp\big(f(a_j) - f(a_{j+1})\big) \exp\big(-h\mu\big(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\big(\boldsymbol{\gamma}_\mu \mathbf{U}^{n+\frac{1}{2}}\big), t_{n+\frac{1}{2}}\big)\big), \quad J^* \leqslant j \leqslant J - 1, \tag{3.4}$$

$$U_0^{n+1} = \mathcal{Q}_h\big(\boldsymbol{\alpha}\big(\mathbf{U}^{n+1}\big)\mathbf{U}^{n+1}\big), \tag{3.5}$$

$0 \leqslant n \leqslant N-1$, where $\boldsymbol{\alpha}(\mathbf{U}^n)_j = \alpha(a_j, \mathcal{Q}_h(\boldsymbol{\gamma}_\alpha \mathbf{U}^n), t_n)$, $0 \leqslant j \leqslant J$, and the values at the half time-step, $\mathbf{U}^{n+\frac{1}{2}} = [U_{\frac{1}{2}}^{n+\frac{1}{2}}, U_{\frac{3}{2}}^{n+\frac{1}{2}}, \ldots, U_{J-\frac{1}{2}}^{n+\frac{1}{2}}]$, are computed by means of

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = U_j^n \exp\left(-\frac{h}{2}\big[m(a_j) + \mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{U}^n\big), t_n\big)\big]\right), \quad 0 \leqslant j \leqslant J^* - 1, \tag{3.6}$$

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = U_j^n \exp\big(f(a_j) - f(a_{j+1/2})\big) \exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{U}^n\big), t_n\big)\right), \quad J^* \leqslant j \leqslant J - 1. \tag{3.7}$$

The notations $\mathcal{Q}_h(\mathbf{V})$ and $\mathcal{Q}_h^*(\mathbf{V})$ represent, respectively, the second-order composite half open quadrature and midpoint rules to approximate an integral over the interval $[0, a_\dagger]$ given by

$$\mathcal{Q}_h(\mathbf{V}) = hV_1 + \sum_{j=1}^{J-1} \frac{h}{2}(V_j + V_{j+1}), \qquad \mathbf{V} = (V_0, V_1, \dots, V_J), \tag{3.8}$$

$$\mathcal{Q}_h^*(\mathbf{V}) = \sum_{j=0}^{J-1} hV_{j+\frac{1}{2}}, \qquad \mathbf{V} = (V_{\frac{1}{2}}, \dots, V_{J-\frac{1}{2}}). \tag{3.9}$$

Furthermore, $\boldsymbol{\gamma}_s \mathbf{U}^n$, $0 \leqslant n \leqslant N$, $\boldsymbol{\gamma}_s \mathbf{U}^{n+\frac{1}{2}}$, $0 \leqslant n \leqslant N-1$, $s = \mu, \alpha$, and $\boldsymbol{\alpha}(\mathbf{U}^n)\mathbf{U}^n$, $0 \leqslant n \leqslant N$, denote the componentwise product of the corresponding vectors. Note that we use two different quadrature rules because, for each time step, we have two different nodes sets, one for levels t_n and other for intermediate levels $t_{n+\frac{1}{2}}$. It is important to notice that the numerical method in (3.3)–(3.5) is explicit which represents a great advantage in its implementation as compared to an implicit formulation, this is the reason because we use a modification of the composite trapezoidal quadrature rule.

4. Convergence analysis

We begin the analysis of the numerical method that we have described in Section 3. Convergence will be obtained by means of consistency and nonlinear stability. First, we rewrite the numerical method into the discretization framework developed by López-Marcos et al. [14]. From now on, C will denote a positive constant which is independent of h, n $(0 \leqslant n \leqslant N)$ and j $(0 \leqslant j \leqslant J)$; C has possibly different values in different places.

We assume that the age discretization parameter h takes values in the set $H = \{A^*/J^*,\ J^* \in \mathbb{N}\}$ and $J = [a_\dagger/h]$. In addition, we set $N = [T/h]$. For each $h \in H$, we define the space

$$X_h = \left(\mathbb{R}^{J+1}\right)^{N+1},$$

where $\mathbb{R}^{J+1}$ is used to consider the approximations to the theoretical solution on the grid nodes, $0 \leqslant n \leqslant N$. We also introduce the space

$$Y_h = \mathbb{R}^{J+1} \times \mathbb{R}^N \times \left(\mathbb{R}^J\right)^N,$$

where we consider the residuals which arise from the initial approximations (first term in the product), from the approximation to the solution at the boundary node (second term), and from the approximations to the solution at the other grid nodes, for every time step (except the first one). We note that the spaces, X_h and Y_h, have the same dimension.

In order to measure the size of the errors, we define

$$\|\mathbf{a}\|_{\infty,p} = \max_{1 \leqslant j \leqslant p} |a_j|, \qquad \mathbf{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p, \qquad \|\mathbf{V}\|_{1,J+1} = \sum_{j=0}^{J} h|V_j|, \quad \mathbf{V} \in \mathbb{R}^{J+1},$$

and $B_{\infty,p}(\mathbf{a}, r)$ the open ball with center $\mathbf{a}$ and radius r given by norm $\|\cdot\|_{\infty,p}$. Now, we endow the spaces X_h and Y_h with the following norms. If $(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) \in X_h$, then

$$\left\|\left(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N\right)\right\|_{X_h} = \max_{0 \leqslant n \leqslant N} \left\|\mathbf{V}^n\right\|_{\infty,J+1}.$$

On the other hand, if $(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \dots, \mathbf{P}^N) \in Y_h$, then

$$\left\|\left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \dots, \mathbf{P}^N\right)\right\|_{Y_h} = \left\|\mathbf{P}^0\right\|_{\infty,J+1} + \|\mathbf{P}_0\|_{\infty,N} + \sum_{n=1}^{N} h\left\|\mathbf{P}^n\right\|_{\infty,J}.$$

Let u represent the solution of (2.1)–(2.4). For each $h \in H$ we define

$$\mathbf{u}_h = \left(\mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^N\right), \qquad \mathbf{u}^n = \left(u_0^n, u_1^n, \dots, u_J^n\right) \in \mathbb{R}^{J+1},$$
$$u_j^n = u(a_j, t_n), \quad 0 \leqslant j \leqslant J,\ 0 \leqslant n \leqslant N.$$

For $\varepsilon > 0$ denote by $B_{X_h}(\mathbf{u}_h, \varepsilon) \subset X_h$ the open ball with center $\mathbf{u}_h$ and radius ε. Next, we introduce the mapping

$$\boldsymbol{\Phi}_h : B_{X_h}(\mathbf{u}_h, \varepsilon) \to Y_h,$$
$$\boldsymbol{\Phi}_h\left(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N\right) = \left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \dots, \mathbf{P}^N\right), \tag{4.1}$$

defined by the following equations:

$$\mathbf{P}^0 = \mathbf{V}^0 - \mathbf{U}^0 \in \mathbb{R}^{J+1}, \tag{4.2}$$

$$P_0^n = V_0^n - \mathcal{Q}_h\big(\boldsymbol{\alpha}\big(\mathbf{V}^n\big)\mathbf{V}^n\big), \quad 1 \leqslant n \leqslant N, \tag{4.3}$$

and for $0 \leqslant n \leqslant N-1$,

$$P_{j+1}^{n+1} = \frac{V_{j+1}^{n+1} - V_j^n \exp(-h[m(a_{j+\frac{1}{2}}) + \mu(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}})])}{h}, \quad 0 \leqslant j \leqslant J^*-1, \tag{4.4}$$

$$P_{j+1}^{n+1} = \frac{V_{j+1}^{n+1} - V_j^n \exp(f(a_j) - f(a_{j+1})) \exp(-h\mu(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}))}{h}, \quad J^* \leqslant j \leqslant J-1, \tag{4.5}$$

where

$$V_{j+\frac{1}{2}}^{n+\frac{1}{2}} = V_j^n \exp\left(-\frac{h}{2}\big[m(a_j) + \mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\big]\right), \quad 0 \leqslant j \leqslant J^*-1, \tag{4.6}$$

$$V_{j+\frac{1}{2}}^{n+\frac{1}{2}} = V_j^n \exp\big(f(a_j) - f(a_{j+1/2})\big) \exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\right), \quad J^* \leqslant j \leqslant J-1, \tag{4.7}$$

and the vector $\mathbf{U}^0$ represents an approximation of the analytical solution at $t = 0$, and we have used the same notation as in Section 3.

It is clear that $(\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N) \in X_h$ is a solution of (3.3)–(3.5) if and only if

$$\boldsymbol{\Phi}_h\big(\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N\big) = \mathbf{0}. \tag{4.8}$$

We begin with the following auxiliary result.

Proposition 1. *Assume hypotheses* (H1)–(H4)*. Let be* $\mathbf{V}^n, \mathbf{W}^n \in B_{\infty, J+1}(\mathbf{u}^n, \varepsilon)$, $1 \leqslant n \leqslant N-1$*. Then, for h sufficiently small, the following hold*:

$$\big|\mathcal{Q}_h\big(\boldsymbol{\gamma}_\phi \mathbf{V}^n\big) - \mathcal{Q}_h\big(\boldsymbol{\gamma}_\phi \mathbf{W}^n\big)\big| \leqslant C\big\|\mathbf{V}^n - \mathbf{W}^n\big\|_1, \quad 1 \leqslant n \leqslant N, \tag{4.9}$$

$$\big|\mathcal{Q}_h\big(\boldsymbol{\alpha}\big(\mathbf{V}^n\big)\mathbf{V}^n\big) - \mathcal{Q}_h\big(\boldsymbol{\alpha}\big(\mathbf{W}^n\big)\mathbf{W}^n\big)\big| \leqslant C\big\|\mathbf{V}^n - \mathbf{W}^n\big\|_1, \quad 1 \leqslant n \leqslant N, \tag{4.10}$$

$$\big|V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}}\big| \leqslant \big|V_j^n - W_j^n\big| + Ch\big\|\mathbf{V}^n - \mathbf{W}^n\big\|_{1,J+1}, \quad 0 \leqslant j \leqslant J-1, \tag{4.11}$$

$$\big|\mathcal{Q}_h^*\big(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\big) - \mathcal{Q}_h^*\big(\boldsymbol{\gamma}_\phi \mathbf{W}^{n+\frac{1}{2}}\big)\big| \leqslant C\big\|\mathbf{V}^n - \mathbf{W}^n\big\|_{1,J+1}, \tag{4.12}$$

for $\phi = \mu, \alpha$.

Proof. Inequalities (4.12)–(4.10) follow from (3.8) and hypothesis (H1). Next, from (4.6) we have that, for $0 \leqslant j \leqslant J^*-1$,

$$\begin{aligned} V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \big(V_j^n - W_j^n\big) \exp\left(-\frac{h}{2}\big[m(a_j) + \mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\big]\right) \\ &\quad + W_j^n \exp\left(-\frac{h}{2}m(a_j)\right)\left[\exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\right) - \exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{W}^n\big), t_n\big)\right)\right]. \end{aligned}$$

Also, using (4.7), we have for $J^* \leqslant j \leqslant J-1$,

$$\begin{aligned} V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \big(V_j^n - W_j^n\big) \exp\big(f(a_j) - f(a_{j+1/2})\big) \exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\right) \\ &\quad + W_j^n \exp\big(f(a_j) - f(a_{j+1/2})\big)\left[\exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big), t_n\big)\right)\right. \\ &\quad \left. - \exp\left(-\frac{h}{2}\mu\big(a_j, \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{W}^n\big), t_n\big)\right)\right]. \end{aligned}$$

Now, the regularity hypotheses (H1)–(H4), inequality (4.9) and the relation $\|\mathbf{W}^n\|_{\infty, J+1} \leqslant C$ yield, for $0 \leqslant j \leqslant J-1$, $0 \leqslant n \leqslant N-1$,

$$\big|V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}}\big| \leqslant \big|V_j^n - W_j^n\big| + Ch\big|\mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{V}^n\big) - \mathcal{Q}_h\big(\boldsymbol{\gamma}_\mu \mathbf{W}^n\big)\big| \leqslant \big|V_j^n - W_j^n\big| + Ch\big\|\mathbf{V}^n - \mathbf{W}^n\big\|_{1,J+1}.$$

Finally, using (3.9), the inequality (4.11) and the hypothesis (H1), we see that, for $1 \leqslant n \leqslant N-1$,

$$\begin{aligned}\left|\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right)-\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{W}^{n+\frac{1}{2}}\right)\right| &= \left|\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi\left[\mathbf{V}^{n+\frac{1}{2}}-\mathbf{W}^{n+\frac{1}{2}}\right]\right)\right| \\ &\leqslant C\sum_{i=0}^{J-1} h\left|V_{i+\frac{1}{2}}^{n+\frac{1}{2}}-W_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right| \leqslant C\sum_{i=0}^{J-1} h\left(\left|V_i^n-W_i^n\right|+Ch\left\|\mathbf{V}^n-\mathbf{W}^n\right\|_{1,J+1}\right) \\ &\leqslant C\left\|\mathbf{V}^n-\mathbf{W}^n\right\|_{1,J+1},\end{aligned}$$

as desired. □

The next result shows that operator defined by (4.1) is well defined.

Proposition 2. *Assume that hypotheses* (H1)–(H4) *hold. If*

$$\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) \in B_{X_h}(\mathbf{u}_h, \varepsilon),$$

where $\varepsilon > 0$ is a fixed positive constant. Then, for h sufficiently small,

$$\mathcal{Q}_h\left(\boldsymbol{\gamma}_\phi \mathbf{V}^n\right) \in \left[-\bar{\varepsilon}, \mathcal{P}(T)\|\gamma_\phi\|_\infty+\bar{\varepsilon}\right], \quad 0 \leqslant n \leqslant N,\ \phi=\mu, \alpha, \tag{4.13}$$

and

$$\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right) \in \left[-\bar{\varepsilon}, \mathcal{P}(T)\|\gamma_\phi\|_\infty+\bar{\varepsilon}\right], \quad 0 \leqslant n \leqslant N-1,\ \phi=\mu, \alpha. \tag{4.14}$$

Proof. Definition (3.8), hypotheses (H1)–(H4), the regularity result (2.5), the inequality (4.9), and the fact that $\mathbf{V}^n$ is bounded, allow us to obtain, for $0 \leqslant n \leqslant N$ and h sufficiently small,

$$\left|\mathcal{Q}_h\left(\boldsymbol{\gamma}_\phi \mathbf{V}^n\right)-I_\phi(t_n)\right| \leqslant \left|\mathcal{Q}_h\left(\boldsymbol{\gamma}_\phi \mathbf{V}^n\right)-\mathcal{Q}_h\left(\boldsymbol{\gamma}_\phi \mathbf{u}^n\right)\right|+\left|\mathcal{Q}_h\left(\boldsymbol{\gamma}_\phi \mathbf{u}^n\right)-I_\phi(t_n)\right| \leqslant C\left\|\mathbf{V}^n-\mathbf{u}^n\right\|_{1,J+1}+O(h). \tag{4.15}$$

Therefore, (4.13) holds. On the other hand, we can see from (3.9), hypotheses (H1)–(H4), the regularity result (2.5), inequality (4.12), the property

$$\left|u(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}})-u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right| = O(h), \quad 0 \leqslant j \leqslant J-1, \tag{4.16}$$

and the boundedness of $\mathbf{V}^n$, that for $0 \leqslant n \leqslant N-1$,

$$\begin{aligned}\left|\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right)-I_\phi(t_{n+\frac{1}{2}})\right| &\leqslant \left|\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right)-\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{u}^{n+\frac{1}{2}}\right)\right|+\left|\mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{u}^{n+\frac{1}{2}}\right)-I_\phi(t_{n+\frac{1}{2}})\right| \\ &\leqslant C\left\|\mathbf{V}^n-\mathbf{u}^n\right\|_{1,J+1}+O(h),\end{aligned}$$

for h sufficiently small, where $\mathbf{u}^{n+\frac{1}{2}}$ is defined in (4.6)–(4.7). Therefore, (4.14) holds. □

Now we define *the local discretization error* as

$$\mathbf{l}_h = \boldsymbol{\Phi}_h(\mathbf{u}_h) \in Y_h,$$

and we say that the discretization (4.1) is *consistent* if

$$\lim_{h\to 0}\left\|\boldsymbol{\Phi}_h(\mathbf{u}_h)\right\|_{Y_h} = \lim_{h\to 0}\|\mathbf{l}_h\|_{Y_h} = 0.$$

The next theorem establishes the consistency of the numerical method defined by (3.3)–(3.5).

Theorem 3. *Assume that hypotheses* (H1)–(H4) *hold. Then, for h sufficiently small, the local discretization error satisfies,*

$$\left\|\boldsymbol{\Phi}_h(\mathbf{u}_h)\right\|_{Y_h} = \left\|\mathbf{u}^0-\mathbf{U}^0\right\|_{\infty,J+1}+O\left(h^2\right). \tag{4.17}$$

Proof. Let us denote $\boldsymbol{\Phi}_h(\mathbf{u}_h) = (\mathbf{L}^0, \mathbf{L}_0, \mathbf{L}^1, \mathbf{L}^2, \ldots, \mathbf{L}^N)$. First we set the bounds for $\mathbf{L}^{n+1}$, $0 \leqslant n \leqslant N-1$. Using (3.2) and (4.4), the regularity hypotheses (H1)–(H4) and result (2.5), and the standard error bound of the mid-point quadrature rule, we have for $0 \leqslant j \leqslant J^*-1$,

$$\begin{aligned}
\left|L_{j+1}^{n+1}\right| \leqslant \frac{|u_j^n|}{h}\Bigg\{\Bigg|&\exp\left(-\int_0^h \left[m(a_j+\sigma)+\mu\left(a_j+\sigma, I_\mu(t_n+\sigma), t_n+\sigma\right)\right]d\sigma\right)\\
&-\exp\left(-h\left[m(a_{j+\frac12})+\mu\left(a_{j+\frac12}, I_\mu(t_{n+\frac12}), t_{n+\frac12}\right)\right]\right)\Bigg| + \left|\exp\left(-h\left[m(a_{j+\frac12})+\mu\left(a_{j+\frac12}, I_\mu(t_{n+\frac12}), t_{n+\frac12}\right)\right]\right)\right.\\
&\left.-\exp\left(-h\left[m(a_{j+\frac12})+\mu\left(a_{j+\frac12}, \mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\mu \mathbf{u}^{n+\frac12}\right), t_{n+\frac12}\right)\right]\right)\right|\Bigg\}\\
\leqslant C\Big\{h^2 + \Big|&\mu\left(a_{j+\frac12}, I_\mu(t_{n+\frac12}), t_{n+\frac12}\right) - \mu\left(a_{j+\frac12}, \mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\mu \mathbf{u}^{n+\frac12}\right), t_{n+\frac12}\right)\Big|\Big\}\\
\leqslant C\Big\{h^2 + \Big|&I_\mu(t_{n+\frac12}) - \mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\mu \mathbf{u}^{n+\frac12}\right)\Big|\Big\},
\end{aligned} \tag{4.18}$$

where $\mathbf{u}^{n+\frac12}$ are computed by means of (4.6)–(4.7) and, therefore, they are not the values of the solution at $t_{n+\frac12}$. Following an analogous argument we can derive, for $J^* \leqslant j \leqslant J-1$,

$$\left|L_{j+1}^{n+1}\right| \leqslant C\left\{h^2 + \left|I_\mu(t_{n+\frac12}) - \mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\mu \mathbf{u}^{n+\frac12}\right)\right|\right\}. \tag{4.19}$$

Also, from the convergence properties of the quadrature rules employed, we obtain the following bounds:

$$\left|u(a_{j+\frac12}, t_{n+\frac12}) - u_j^n \exp\left(-\frac{h}{2}\left[m(a_j)+\mu\left(a_j, \mathcal{Q}_h\left(\boldsymbol{\gamma}_\mu \mathbf{u}^n\right), t_n\right)\right]\right)\right| \leqslant Ch^2, \quad 0 \leqslant j \leqslant J^*-1, \tag{4.20}$$

$$\left|u(a_{j+\frac12}, t_{n+\frac12}) - u_j^n e^{[f(a_j)-f(a_{j+1/2})]} \exp\left(-\frac{h}{2}\mu\left(a_j, \mathcal{Q}_h\left(\boldsymbol{\gamma}_\mu \mathbf{u}^n\right), t_n\right)\right)\right| \leqslant Ch^2, \quad J^* \leqslant j \leqslant J-1, \tag{4.21}$$

that lead to the relation

$$\left|I_\mu(t_{n+\frac12}) - \mathcal{Q}_h^*\left(\boldsymbol{\gamma}_\mu \mathbf{u}^{n+\frac12}\right)\right| \leqslant Ch^2, \tag{4.22}$$

which can be substituted in (4.18) and (4.19) to see that

$$\left|L_{j+1}^{n+1}\right| \leqslant Ch^2, \quad 0 \leqslant j \leqslant J-1. \tag{4.23}$$

Now we consider bounding $\mathbf{L}_0$. Using (4.3), the regularity hypotheses (H1)–(H4) and result (2.5), the standard error bounds for the quadrature rule, and similar arguments to those employed above, we have for $1 \leqslant n \leqslant N$,

$$\left|L_0^n\right| \leqslant \left|\int_0^{a_\dagger} \alpha\left(a, I_\alpha(t_n), t_n\right)u(a,t_n)\,da - \mathcal{Q}_h\left(\boldsymbol{\alpha}\left(\mathbf{u}^n\right)\mathbf{u}^n\right)\right| \leqslant Ch^2 + \left|I_\alpha(t_n) - \mathcal{Q}_h\left(\boldsymbol{\gamma}_\alpha \mathbf{u}^n\right)\right| \leqslant Ch^2. \tag{4.24}$$

This completes the proof of (4.17). □

Another notion that plays an important role in the analysis of the numerical method is that of *stability with h-dependent thresholds*. For $h \in H$, let $M_h > 0$ be a real number (*the stability threshold*); we say that the discretization (4.1) is *stable* for $\mathbf{u}$ restricted to the thresholds M_h, if there exist two positive constants h_0 and S (*the stability constant*) such that, for any $h \in H$ with $h \leqslant h_0$, the open ball $B_{X_h}(\mathbf{u}_h, M_h)$ is contained in the domain of $\boldsymbol{\Phi}_h$ and, for all $\mathbf{V}$, $\mathbf{W}_h$ in that ball,

$$\|\mathbf{V}_h - \mathbf{W}_h\| \leqslant S\left\|\boldsymbol{\Phi}_h(\mathbf{V}_h) - \boldsymbol{\Phi}_h(\mathbf{W}_h)\right\|.$$

Next, we introduce a theorem that establishes the *stability* of the discretization defined by Eqs. (3.3)–(3.5).

Theorem 4. *Assume that hypotheses* (H1)–(H4) *hold and let* $\varepsilon > 0$ *be a fixed constant. Then, the discretization* (3.3)–(3.5) *is stable for* $\mathbf{u}_h$ *with thresholds* $M_h = \varepsilon$.

Proof. Let $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N)$, $(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N) \in B_{X_h}(\mathbf{u}_h, M_h)$ and set

$$\begin{aligned}
&\mathbf{E}^n = \mathbf{V}^n - \mathbf{W}^n \in \mathbb{R}^{J+1}, \quad 0 \leqslant n \leqslant N,\\
&\boldsymbol{\Phi}_h\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) = \left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^N\right),\\
&\boldsymbol{\Phi}_h\left(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N\right) = \left(\mathbf{R}^0, \mathbf{R}_0, \mathbf{R}^1, \mathbf{R}^2, \ldots, \mathbf{R}^N\right).
\end{aligned}$$

From (4.4) we have, for $0 \leqslant j \leqslant J^* - 1$,

$$\begin{aligned} E_{j+1}^{n+1} &= E_j^n \exp\bigl(-h\bigl[m(a_{j+\frac{1}{2}}) + \mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr]\bigr) \\ &\quad + W_j^n \exp\bigl(-hm(a_{j+\frac{1}{2}})\bigr)\bigl(\exp\bigl(-h\mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr) \\ &\quad - \exp\bigl(-h\mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{W}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr)\bigr) + h\bigl(P_{j+1}^{n+1} - R_{j+1}^{n+1}\bigr), \end{aligned} \tag{4.25}$$

and from (4.5) we have, for $J^* \leqslant j \leqslant J - 1$,

$$\begin{aligned} E_{j+1}^{n+1} &= E_j^n e^{[f(a_j) - f(a_{j+1})]} \exp\bigl(-h\mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr) \\ &\quad + W_j^n e^{[f(a_j) - f(a_{j+1})]}\bigl[\exp\bigl(-h\mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr) \\ &\quad - \exp\bigl(-h\mu\bigl(a_{j+\frac{1}{2}}, \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{W}^{n+\frac{1}{2}}\bigr), t_{n+\frac{1}{2}}\bigr)\bigr)\bigr] + h\bigl(P_{j+1}^{n+1} - R_{j+1}^{n+1}\bigr). \end{aligned} \tag{4.26}$$

The regularity result (2.5) and hypotheses (H1)–(H4), formulae (4.25)–(4.26), inequality (4.12), and $\|\mathbf{W}\|_{\infty, J+1} \leqslant C$ imply that, for $0 \leqslant j \leqslant J - 1$,

$$\begin{aligned} \bigl|E_{j+1}^{n+1}\bigr| &\leqslant \bigl|E_j^n\bigr| + Ch\bigl|\mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{V}^{n+\frac{1}{2}}\bigr) - \mathcal{Q}_h^*\bigl(\boldsymbol{\gamma}_\mu \mathbf{W}^{n+\frac{1}{2}}\bigr)\bigr| + h\bigl|P_{j+1}^{n+1} - R_{j+1}^{n+1}\bigr| \\ &\leqslant \bigl|E_j^n\bigr| + Ch\bigl\|\mathbf{E}^n\bigr\|_{1, J+1} + h\bigl|P_{j+1}^{n+1} - R_{j+1}^{n+1}\bigr|. \end{aligned} \tag{4.27}$$

Thus, when $N \geqslant n > j \geqslant 1$, from (4.27) we have

$$\bigl|E_j^n\bigr| \leqslant \bigl|E_0^{n-j}\bigr| + Ch\sum_{l=1}^{j}\bigl\|\mathbf{E}^{n-l}\bigr\|_{1, J+1} + h\sum_{l=0}^{j-1}\bigl|P_{j-l}^{n-l} - R_{j-l}^{n-l}\bigr| \leqslant \bigl|E_0^{n-j}\bigr| + Ch\sum_{l=0}^{n-1}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + h\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J}. \tag{4.28}$$

On the other hand, when $j > n \geqslant 1$, from (4.27) we obtain

$$\bigl|E_j^n\bigr| \leqslant \bigl|E_{j-n}^0\bigr| + Ch\sum_{l=1}^{n}\bigl\|\mathbf{E}^{n-l}\bigr\|_{1, J+1} + h\sum_{l=0}^{n-1}\bigl|P_{j-l}^{n-l} - R_{j-l}^{n-l}\bigr| \leqslant \bigl|E_{n-j}^0\bigr| + Ch\sum_{l=0}^{n-1}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + h\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J}. \tag{4.29}$$

Now, by (4.3)

$$\begin{aligned} E_0^n &= \mathcal{Q}_h\bigl(\boldsymbol{\alpha}\bigl(\mathbf{V}^n\bigr)\mathbf{V}^n\bigr) - \mathcal{Q}_h\bigl(\boldsymbol{\alpha}\bigl(\mathbf{W}^n\bigr)\mathbf{W}^n\bigr) + \bigl(P_0^n - R_0^n\bigr) \\ &= \mathcal{Q}_h\bigl(\boldsymbol{\alpha}\bigl(\mathbf{V}^n\bigr)\mathbf{E}^n\bigr) + \mathcal{Q}_h\bigl(\bigl[\boldsymbol{\alpha}\bigl(\mathbf{V}^n\bigr) - \boldsymbol{\alpha}\bigl(\mathbf{W}^n\bigr)\bigr]\mathbf{W}^n\bigr) + \bigl(P_0^n - R_0^n\bigr). \end{aligned} \tag{4.30}$$

Also, using relation (2.5), hypotheses (H1)–(H4), inequalities (4.9) and (4.10), and $\|\mathbf{W}^n\|_{\infty, J+1} \leqslant C$, we have

$$\bigl|E_0^n\bigr| \leqslant \bigl\|\mathbf{E}^n\bigr\|_{1, J+1} + C\bigl|\mathcal{Q}_h\bigl(\boldsymbol{\gamma}_\alpha \mathbf{V}^n\bigr) - \mathcal{Q}_h\bigl(\boldsymbol{\gamma}_\alpha \mathbf{W}^n\bigr)\bigr| + \bigl|P_0^n - R_0^n\bigr| \leqslant C\bigl\|\mathbf{E}^n\bigr\|_{1, J+1} + \bigl|P_0^n - R_0^n\bigr|. \tag{4.31}$$

Next, multiplying $|E_j^n|$ by h and summing on j, $0 \leqslant j \leqslant J$, from (4.28), (4.29), and (4.31) we have, for $1 \leqslant n \leqslant N$,

$$\begin{aligned} \bigl\|\mathbf{E}^n\bigr\|_{1, J+1} &= h\bigl|E_0^n\bigr| + \sum_{j=1}^{n-1} h\bigl|E_j^n\bigr| + \sum_{j=n}^{J} h\bigl|E_j^n\bigr| \\ &\leqslant h\bigl(C\bigl\|\mathbf{E}^n\bigr\|_{1, J+1} + \bigl|P_0^n - R_0^n\bigr|\bigr) + \sum_{j=1}^{n-1} h\Biggl(\bigl|E_0^{n-j}\bigr| + Ch\sum_{l=0}^{n-1}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + h\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J}\Biggr) \\ &\quad + \sum_{j=n}^{J} h\Biggl(\bigl|E_{n-j}^0\bigr| + Ch\sum_{l=0}^{n-1}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + h\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J}\Biggr) \\ &\leqslant C\bigl\|\mathbf{E}^0\bigr\|_{1, J+1} + h\bigl(C\bigl\|\mathbf{E}^n\bigr\|_{1, J+1} + \bigl|P_0^n - R_0^n\bigr|\bigr) + \sum_{j=1}^{n-1} h\bigl(C\bigl\|\mathbf{E}^{n-j}\bigr\|_{1, J+1} + \bigl|P_0^{n-j} - R_0^{n-j}\bigr|\bigr) \\ &\quad + Ch\sum_{l=0}^{n-1}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + Ch\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J} \\ &\leqslant C\bigl\|\mathbf{E}^0\bigr\|_{1, J+1} + Ch\sum_{l=0}^{n}\bigl\|\mathbf{E}^l\bigr\|_{1, J+1} + C\sum_{l=1}^{n} h\bigl|P_0^l - R_0^l\bigr| + Ch\sum_{l=1}^{n}\bigl\|\mathbf{P}^l - \mathbf{R}^l\bigr\|_{\infty, J}. \end{aligned}$$

Using the discrete Gronwall lemma, it follows that

$$\left\|\mathbf{E}^n\right\|_{1,J+1} \leqslant C\left(\left\|\mathbf{E}^0\right\|_{1,J+1} + \|\mathbf{P}_0 - \mathbf{R}_0\|_{\infty,N} + h\sum_{l=1}^{n}\left\|\mathbf{P}^l - \mathbf{R}^l\right\|_{\infty,J}\right). \tag{4.32}$$

Then, we substitute (4.32) in (4.28) and (4.31) to conclude the proof. □

Now, we define the *global discretization error* as

$$\mathbf{e}_h = \mathbf{u}_h - \mathbf{U}_h \in X_h.$$

We say that the discretization (4.1) is *convergent* if there exists $h_0 > 0$ such that, for each $h \in H$ with $h \leqslant h_0$, (4.8) has a solution $\mathbf{U}_h$ for which

$$\lim_{h\to 0}\|\mathbf{u}_h - \mathbf{U}_h\|_{X_h} = \lim_{h\to 0}\|\mathbf{e}_h\|_{X_h} = 0.$$

To conclude our convergence analysis we shall use the following result from the general discretization framework introduced by López-Marcos et al. [14].

Theorem 5. *Assume that* (4.1) *is consistent and stable with thresholds M_h. If Φ_h is continuous in $B(\mathbf{u}_h, M_h)$ and $\|\mathbf{l}_h\|_{Y_h} = o(M_h)$ as $h \to 0$, then for h sufficiently small,*

(i) *the discrete equations* (4.8) *possess a unique solution in $B(\mathbf{u}_h, M_h)$;*
(ii) *the solutions converge and $\|\mathbf{e}_h\|_{X_h} = O(\|\mathbf{l}_h\|_{Y_h})$.*

Finally, we state the theorem that establishes the *convergence* of our numerical method defined by Eqs. (3.3)–(3.5).

Theorem 6. *Assume that hypotheses* (H1)–(H4) *hold. Then, for h sufficiently small, the numerical method defined by* (3.3)–(3.5) *has a unique solution $\mathbf{U}_h \in B_{X_h}(\mathbf{u}_h, M)$ and*

$$\|\mathbf{U}_h - \mathbf{u}_h\|_{X_h} \leqslant C\left[\left\|\mathbf{u}^0 - \mathbf{U}^0\right\|_{\infty} + O\left(h^2\right)\right].$$

The proof of Theorem 6 is immediate from the consistency—Theorem 3, the stability—Theorem 4, and Theorem 5.

5. Numerical results and conclusions

We have carried out several numerical experiments using the algorithm defined in Section 3. We considered different test problems presenting meaningful nonlinearities that appear in the literature [13]. The numerical integration for each numerical experiment was carried out over the time interval $[0, 1]$. In all the simulations we used the parameter values $a_\dagger = 1$, $A^* = 0.9$.

Problem 1. This is one of the examples present in [13], using the fertility and mortality rates $\alpha(a,z,t) = 4$, $m(a) = \frac{1}{1-a}$ and $\mu(a,z,t) = z$. The weight functions are taken as $\gamma_\mu \equiv \gamma_\alpha \equiv 1$, and we consider as initial age density the function $u_0(a) = 4(1-a)e^{-\lambda a}$, where $\lambda = 2.5569290855$. Problem (2.1)–(2.4) then has the following solution,

$$u(a,t) = 4(1-a)e^{-\lambda a}\frac{\lambda}{(\lambda-1)e^{-\lambda t}+1}.$$

Problem 2. We take now $m(a) = \frac{0.5}{1-a}$ and the other functions as in Problem 1. Problem (2.1)–(2.4) now has the following solution,

$$u(a,t) = 4\sqrt{1-a}\,e^{-\lambda a}\frac{\lambda}{(\lambda-1)e^{-\lambda t}+1},$$

where $\lambda = 3.22540174092$.

Problem 3. In this case, we chose $\mu(a,z,t) = z^2$, the other functions as in Problem 2. The solution to problem (2.1)–(2.4) is then given by

$$u(a,t) = 4\sqrt{1-a}\,e^{-\lambda a}\sqrt{\frac{\lambda}{(\lambda-1)e^{-2\lambda t}+1}}.$$

Table 1
Errors and convergence order for Problem 1.

k	$\max_{0\leqslant n\leqslant N}\lvert \mathcal{Q}_h(\mathbf{U}^n)-P(t_n)\rvert$	Order	$\max_{0\leqslant n\leqslant N,0\leqslant j\leqslant J}\lvert u(a_j,t_n)-U_j^n\rvert$	Order
1.25000E−02	6.164114E−03		2.465645E−02	
6.25000E−03	1.561714E−03	1.98	6.246857E−03	1.98
3.12500E−03	3.928375E−04	1.99	1.571350E−03	1.99
1.56250E−03	9.849879E−05	2.00	3.939952E−04	2.00
7.81250E−04	2.466012E−05	2.00	9.864050E−05	2.00
3.90625E−04	6.169412E−06	2.00	2.467765E−05	2.00

Table 2
Errors and convergence order for Problem 2.

k	$\max_{0\leqslant n\leqslant N}\lvert \mathcal{Q}_h(\mathbf{U}^n)-P(t_n)\rvert$	Order	$\max_{0\leqslant n\leqslant N,0\leqslant j\leqslant J}\lvert u(a_j,t_n)-U_j^n\rvert$	Order
1.25000E−02	7.745942E−03		3.098377E−02	
6.25000E−03	1.998691E−03	1.95	7.994764E−03	1.95
3.12500E−03	5.156830E−04	1.95	2.062732E−03	1.95
1.56250E−03	1.334500E−04	1.95	5.337998E−04	1.95
7.81250E−04	3.438362E−05	1.96	1.375345E−04	1.96
3.90625E−04	8.441683E−06	2.03	3.376673E−05	2.03

Table 3
Errors and convergence order for Problem 3.

k	$\max_{0\leqslant n\leqslant N}\lvert \mathcal{Q}_h(\mathbf{U}^n)-P(t_n)\rvert$	Order	$\max_{0\leqslant n\leqslant N,0\leqslant j\leqslant J}\lvert u(a_j,t_n)-U_j^n\rvert$	Order
1.25000E−02	3.417214E−03		1.366886E−02	
6.25000E−03	8.738994E−04	1.97	3.495597E−03	1.97
3.12500E−03	2.243750E−04	1.96	8.974998E−04	1.96
1.56250E−03	5.794316E−05	1.95	2.317727E−04	1.95
7.81250E−04	1.497745E−05	1.95	5.990980E−05	1.95
3.90625E−04	3.764917E−06	1.99	1.505967E−05	1.99

Since we know the exact solution for each of these problems, we can show numerically that our method is second order accurate by means of error tables. In each table the second column shows the global error for the total population computed as follows,

$$e_h=\max_{0\leqslant n\leqslant N}\left|\mathcal{Q}_h\left(\mathbf{U}^n\right)-P(t_n)\right|.$$

The fourth column represents the global error for the age density of the population computed using the formula

$$e_h=\max_{0\leqslant n\leqslant N,0\leqslant j\leqslant J}\left|u(a_j,t_n)-U_j^n\right|.$$

The third and fifth columns display the experimental order of convergence of the method, s, computed as

$$s=\frac{\log(e_{2h}/e_h)}{\log(2)},$$

using the values of the second and forth column, respectively. Each row of Tables 1–3 corresponds to different value of the discretization parameter.

We have described a new numerical method to approximate solutions of the initial-boundary value problem for a nonlinear, age-structured population model with finite maximum age. The method is based on quadratures of the integrals resulting from the explicit integration of the differential equation in the model along characteristics. It is biologically relevant because of its wide applicability to both short- and long-term projections.

We have proved the second-order convergence of the method provided the analytical solution is sufficiently regular. This order of convergence seems like a good compromise between efficiency for long time simulations and regularity constraints on the coefficient functions. When the required compatibility condition between initial age density and births is not satisfied, two types of singularities may arise: a jump discontinuity in the first derivative, or a jump discontinuity in the density function itself along the characteristic $0\leqslant t=a\leqslant a_+$. The former leads to no loss in the order of convergence since the composite quadratures used are local and always utilize the point of singularity. In the latter case the numerical method can be modified by considering an open quadrature rule over the two affected subintervals [6]. Alternatively, a double value for the solution can be considered on the nodes $0\leqslant t_n=a_n\leqslant a_+$ and still use the trapezoidal rule as in [8], though the convergence analysis becomes more tedious in such case.

The implementation of the method is very straightforward since it is explicit and uses fractional time steps in order to avoid iterations on the nonlinearities. The results in Tables 1–3 above clearly confirm the theoretical second order of convergence. The decrease of this order that would result from lack of regularity actually provides information about the regularity of the density being approximated, which could contribute to questioning whether the model chosen and/or the measured initial data are realistic.

We point out that the results for Problems 2 and 3 are novel because, as indicated in [12,13], the numerical methods that have been previously proposed to approximate the solution of this model do not converge at their optimal order for mortality functions $m(a) = \frac{c}{1-a}$, $c < 1$, such as the one we chose for these problems.

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