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Distributional and Analytic Solutions of Functional Differential Equations

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1. INTRODUCTION

Recently there has been considerable interest in problems concerning the existence of solutions to differential and functional differential equations (FDE) in various spaces of generalized functions. Research in this direction, still developed insufficiently, discovers new aspects and properties in the theory of FDE. Distributional solutions to homogeneous FDE may be originated either by singularities of their coefficients or by deviations of the argument. Thus, in [1] one of the authors proved that the system

$$x'(t) = Ax(t) + tBx(\lambda t), \qquad -1 < \lambda < 1$$

has a solution in the class of singular functionals—an impossible phenomenon for normal linear homogeneous systems of ordinary differential equations with infinitely smooth coefficients. A more general result was obtained in [2], where it was shown that, under certain conditions, the system

$$x'(t) = \sum_{j=0}^{\infty} A_j(t)\, x(\lambda_j t)$$

has a solution

$$x(t) = \sum_{n=0}^{\infty} x_n\, \delta^{(n)}(t) \qquad (1.1)$$

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in the generalized-function space $(S_o^\beta)'$ with arbitrary $\beta > 1$. In the sequel $\delta^{(n)}$ denotes the nth derivative of the Dirac δ measure and $\langle f, \phi \rangle$ is the value of the functional f applied to the testing function ϕ. The variable t is real in the case of distributional solutions and complex for analytic solutions. The norm of a matrix is defined to be

$$\|A\| = \max_i \sum_j |a_{ij}|$$

and E is the identity matrix. To ensure the convergence of series (1.1) it is sufficient to require that for $n \to \infty$ the vectors x_n satisfy the inequalities

$$\|x_n\| \leqslant ac^n n^{-n\rho}, \qquad \rho > 1. \tag{1.2}$$

In fact, since the testing functions $\phi(t) \in S_o^\beta$ are subject to the restriction [3]

$$|\phi^{(n)}(t)| \leqslant b\, d^n n^{n\beta},$$

then

$$\left\| \sum_{n=0}^{\infty} \langle x_n\, \delta^{(n)}(t), \phi(t) \rangle \right\| = \left\| \sum_{n=0}^{\infty} (-1)^n \phi^{(n)}(0) x_n \right\|$$
$$\leqslant \sum_{n=0}^{\infty} |\phi^{(n)}(0)|\, \|x_n\| \leqslant ab \sum_{n=0}^{\infty} (c\, dn^{\beta-\rho})^n < \infty,$$

for $\beta < \rho$. If series (1.1) converges, its sum represents the general form of a linear functional in $(S_o^\beta)'$ with the support $t = 0$. The particular importance of the system

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} (A_{ij} + tB_{ij})\, x^{(j)}(\lambda_i t) = tx(\lambda t)$$

which has been studied in [4] is that depending on the cofficients it combines either equations with a singular or regular point at $t = 0$ and in both cases there exists a solution of the form (1.1). The equation

$$tx'(t) = Ax(t) + tBx(\lambda t) \tag{1.3}$$

provides an interesting example of a system that may have two essentially different solutions in $(S_o^\beta)'$ concentrated on $t = 0$. If the matrix A assumes negative integer eigenvalues, (1.3) has a finite order solution

$$x = \sum_{j=0}^{m} x_j\, \delta^{(j)}.$$

At the same time there exists an infinite order solution (1.1), if $A \neq -nE$ for all $n \geqslant 1$.

In this article we extend the foregoing conclusions to comprehensive systems of any order with countable sets of variable argument deviations. Integral transformations establish close links between generalized and entire functions. Therefore the basic ideas in the method of proof are applied to investigate analytic solutions of linear FDE. In a number of works various authors have studied the solutions, especially their asymptotic behaviour as $t \to 0$ or $t \to \infty$, of equations with linear transformations of the argument. Particular attention is also given to such relations here, including systems with a singularity. The purpose of this paper is to present a unified treatment of some problems concerning generalized and analytic solutions of linear FDE.

2. Existence of Distributional Solutions

We prove existence theorems for FDE in the space $(S_o^\beta)'$. The study of the system

$$\sum_i \sum_{j=0}^{m} A_{ij}(t)\, x^{(j)}(\lambda_{ij}(t)) = 0 \tag{2.1}$$

generalizes the corresponding results of [4] and [20]. The choice of the coefficients enables us to consider both equations with a singular or regular point and to show that distributional solutions of FDE may be originated by deviations of the argument. We also investigate the system

$$t^p x'(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t)\, x^{(j)}(\lambda_{ij}(t)), \tag{2.2}$$

the particular cases of which

$$t^p x'(t) = A(t)\, x(t)$$

and

$$t^p x'(t) = \sum_{i=0}^{\infty} A_i(t)\, x(\lambda_i t)$$

have been studied in [5] and [6], respectively.

Lemma. *If the function $\lambda(t) \in C^1$, $\lambda(0) = 0$ and $\lambda'(0) = \alpha \neq 0$, then for the distribution* (1.1)

$$x^{(j)}(\lambda(t)) = \sum_n |\alpha|^{-1} \alpha^{-n-j} x_n \delta^{(n+j)}(t) \tag{2.3}$$

in some neighborhood of the origin.

Proof. From $\lambda(t) = t\phi(t)$ it follows that $\phi(0) = \alpha$, and there exists a neighborhood T of $t = 0$ in all points of which $\phi(t) \neq 0$, since assuming the opposite we find a sequence $t_n \to 0$ such that $\phi(t_n) \to 0$; hence, $\lambda'(0) = 0$. In T, $\delta^{(n)}(t\phi) = \delta^{(n)}(\alpha t) = 0$, for $t \neq 0$. But $t\phi = \alpha t$, for $t = 0$, and the relation

$$\delta^{(n)}(\lambda(t)) = \delta^{(n)}(\alpha t)$$

holds for all $t \in T$. The functional δ is the derivative of the function $H(t)$ which equals 1 for $t > 0$ and 0 for $t < 0$. Differentiating $n + 1$ times $H(\alpha t) = H(t)$, $\alpha > 0$ and $H(\alpha t) = 1 - H(t)$, $\alpha < 0$ we obtain $\delta^{(n)}(\lambda(t)) = |\alpha|^{-1}\alpha^{-n}\,\delta^{(n)}(t)$.

THEOREM 2.1. *Let* (2.1) *with a finite number of argument deviations, in which x is an r-vector and A_{ij} are $r \times r$-matrices, satisfy the following hypotheses.*

(i) *The coefficients $A_{ij}(t)$ are polynomials in t of degree not exceeding p:*

$$A_{ij}(t) = \sum_{k=0}^{p} A_{ijk}t^k, \qquad A_{oo}(t) = At^p, \qquad p \geqslant 1.$$

(ii) *The real-valued functions $\lambda_{ij}(t) \in C^1$ in a neighborhood of the origin, $\lambda_{ij}(0) = 0$ and*

$$0 < |\alpha_{oo}| < 1, \qquad |\alpha_{ij}| \geqslant 1, \qquad i + j \geqslant 1, \qquad \alpha_{ij} = \lambda'_{ij}(0).$$

(iii) *The matrix A is nonsingular and*

$$c = |\alpha_{oo}|^{-p-1}\,\|A\| - \sum_{i \geqslant 1} |\alpha_{io}|^{-p-1}\,\|A_{iop}\| > 0.$$

Then in the space of generalized functions $(S_o^\beta)'$ with arbitrary $\beta > 1$ there exists a solution $x(t)$ supported on $t = 0$.

Proof. By virtue of (2.3) and the formula

$$t^k\,\delta^{(n)} = \begin{cases} (-1)^k n!\,\delta^{(n-k)}(t)/(n-k)!, & n \geqslant k \\ 0, \quad n < k \end{cases}$$

we obtain the equation

$$\sum_{i,j,k} (-1)^k A_{ijk} \sum_{n+j \geqslant k} (n+j)!\,|\alpha_{ij}|^{-1}\alpha_{ij}^{-n-j} x_n\,\delta^{(n+j-k)}(t)/(n+j-k)! = 0$$

for the unknowns x_n of the solution (1.1). The replacement of $n+j-k$ by n gives the relations

$$\sum (-1)^k (n+k)!\, |\alpha_{ij}|^{-1} \alpha_{ij}^{-n-k} A_{ijk} x_{n+k-j} = 0, \qquad n \geqslant 0$$

which can be written as

$$\sum_{k-j<p} (-1)^{p-k} (n+k)!\, |\alpha_{ij}|^{-1} \alpha_{ij}^{-n-k} A_{ijk} x_{n+k-j}/(n+p)!$$
$$+ \left(\sum_{i \geqslant 0} |\alpha_{io}|^{-1} \alpha_{io}^{-n-p} A_{iop} \right) x_{n+p} = 0.$$

Since $A_{ook} = 0$ $(k < p)$ the first sum does not include terms with α_{oo}. According to (iii) the coefficients B_n of x_{n+p} are nonsingular matrices and $\|B_n^{-1}\| \leqslant c\,\|A\|\,|\alpha_{oo}|^n$. Consequently,

$$\|x_{n+p}\| \leqslant \mu q^{n+p} \sum_{k=0}^{m+p-1} \|x_{n+k-m}\|, \qquad 0 < q < 1 \tag{2.4}$$

where μ is some positive constant. Using the notation

$$M_n = \max_{0 \leqslant i \leqslant n} \|x_i\| \tag{2.5}$$

we conclude from (2.4) that

$$\|x_{n+p}\| \leqslant \mu(m+p)\, q^{n+p} M_{n+p-1}.$$

For large n there is $\mu(m+p)\, q^{n+p} \leqslant 1$. Hence, $\|x_{n+p}\| \leqslant M_{n+p-1}$ and $M_{n+p} = M_{n+p-1}$. Thus, starting with some N,

$$M_n = M_N, \qquad n \geqslant N. \tag{2.6}$$

The application of (2.6) to (2.4) successively yields:

$$\|x_{N+p+i}\| \leqslant \mu(m+p)\, q^{N+p} M_N,$$
$$\|x_{N+p+(m+p)+i}\| \leqslant \mu^2 (m+p)^2 q^{N+p} q^{N+p+(m+p)} M_N,$$
$$\|x_{N+p+2(m+p)+i}\| \leqslant \mu^3 (m+p)^3 q^{N+p} q^{N+p+(m+p)} q^{N+p+2(m+p)} M_N$$
$$(0 \leqslant i \leqslant m+p-1).$$

The conjecture

$$\|x_{N+p+n(m+p)+i}\| \leqslant \mu^{n+1} (m+p)^{n+1} q^{n(N+p)+n(n+1)(m+p)/2} M_N \tag{2.7}$$

may readily be ascertained by induction, for all n and the mentioned values of i, and proves the theorem since the condition $0 < q < 1$ makes it more restrictive than (1.2).

THEOREM 2.2. *System* (2.1) *with a countable set of argument deviations has a solution* (1.1) *if, in addition to the conditions of Theorem* 2.1, *there exists a neighborhood of the origin in which each function* $\lambda_{ij}(t)$ *has the only zero* $t=0$ *and the series* $\sum_{i=1}^{\infty} \alpha_i^{-1} A_i$ *converges where*

$$A_i = \max_{j,k} \|A_{ijk}\|, \qquad \alpha_i = \inf_j |\alpha_{ij}|, \qquad i+j \geqslant 1.$$

THEOREM 2.3. *Suppose that system* (2.2), *in which* x *is an* r*-vector and* A_{ij} *are* $r \times r$*-matrices, satisfies the following conditions.*

(i) *The* $A_{ij}(t)$ *are polynomials in* t *of degree not exceeding* $p+j-2$:

$$A_{ij}(t) = \sum_{k=0}^{p+j-2} A_{ijk} t^k, \qquad p \geqslant 2.$$

(ii) *There exists a neighborhood* T *of the origin in which the real-valued functions* $\lambda_{ij} \in C^1$ *have the only zero* $t=0$ *and* $|\alpha_{io}| \geqslant 1$, $\inf |\alpha_{ij}| > 1$, *for* $i \geqslant 0$, $j \geqslant 1$, $\alpha_{ij} = \lambda'_{ij}(0)$.

(iii) *The series* $A = \sum_{i=0}^{\infty} \alpha_i^{-1} A_i$ *converges where*

$$\alpha_i = \inf_j |\alpha_{ij}|, \qquad A_i = \max_{j,k} \|A_{ijk}\|.$$

Then there is a solution of (2.2) *in* $(S_o^\beta)'$ *with some* $\beta > 1$ *supported on* $t=0$.

Proof. According to (2.3), in T Eq. (2.2) has the same distributional solutions as the similar system with constants α_{ij} instead of $\lambda_{ij}(t)$, and it is easy to obtain the relations

$$(n+p)!\, x_{n+p-1} = \sum_{i,j,k} (-1)^{p-k} (n+k)!\, |\alpha_{ij}|^{-1} \alpha_{ij}^{-n-k} A_{ijk} x_{n+k-j}$$

for the coefficients x_n of (1.1). Assumptions (ii) and (iii) imply that

$$(n+p)\, F_{n+p-1} \leqslant A \alpha^{-n} \sum_{j=0}^{m} \sum_{k=0}^{p+j-2} F_{n+k-1}, \qquad F_n = \|x_n\|\, n!$$

where $\alpha = \inf |\alpha_{ij}|$, $i, j \geqslant 0$, and the procedure of Theorem 2.1 yields the inequalities

$$F_{N+n(m+p-1)+k} \leqslant (m+1)^n (m+p-1)^n \alpha^{-n(n-1)(m+p-1)/2}$$
$$\times A^n F_N \Big/ \prod_{i=1}^{n} (N+1+(m+p-1)i) \qquad (0 \leqslant k \leqslant m+p-1)$$

more stringent than (1.2), if $\alpha > 1$. Hence, the space $(S_o^\beta)'$ with arbitrary $\beta > 1$ contains a solution of (2.2) concentrated on $t = 0$. For $\alpha = 1$,

$$\|x_{N+n(m+p-1)+k}\| \leqslant (m+1)^n A^n M_N/(N+n(m+p-1)+k)!\, n!,$$

and applying Stirling's formula we get

$$\|x_\nu\| \leqslant ac^\nu \nu^{-\nu\rho}, \qquad \nu = N + n(m+p-1) + k, \quad \rho = 1 + (m+p-1)^{-1}.$$

Therefore, if $\inf|\alpha_{io}| = 1$, (2.2) has a solution (1.1) in $(S_o^\beta)'$ with $1 < \beta < \rho$.

3. Equations with Rotation of the Argument

In [7–10] a method has been discovered for the study of differential equations with periodic transformations of the argument. This basically algebraic approach was developed also in a number of other works and culminated in the monograph [11]. Though numerous papers continue to appear in this field [12], some aspects of the theory still require further investigation. In connection with the purposes of our article we mention only such topics as higher-order equations (especially with singularities in the coefficients) with reflection and rotation of the argument, influence of the method on the study of systems with deviations of more general nature, and solutions in spaces of distributions. Linear first-order equations with constant coefficients and with reflection have been examined in detail in [11]. There is also an indication (p. 169) that "the problem is much more difficult in the case of a differential equation with reflection of order greater than one." Meanwhile, general results for systems of any order appeared in [7, 9, 10]. Here we consider the scalar equation

$$\sum_{k=0}^{n} a_k x^{(k)}(t) = \sum_{k=0}^{n} b_k x^{(k)}(\varepsilon t) + \psi(t), \qquad \varepsilon^m = 1 \tag{3.1}$$

$$x^{(k)}(0) = x_k, \qquad k = 0,\ldots, n-1$$

with complex constants a_k, b_k, ε, then the method is extended to some systems with variable coefficients. We are interested mainly in equations with a regular singular point and establish the structure of solutions in the case of deviations proportional to t.

Turning to (3.1) and assuming that ψ is smooth enough we introduce the operators

$$A_j = \sum_{k=0}^{n} a_k \varepsilon^{-jk} A_{j-1}\, d^k/dt^k,$$

$$B_j = \sum_{k=0}^{n} b_k \varepsilon^{jk} B_{j-1}\, d^k/dt^k \qquad (A_{-1} = B_{-1} = E;\ j = 0,\ldots, m-1)$$

and apply A_1 to the given equation

$$A_0 x = (B_0 x)(\varepsilon t) + \psi.$$

Since

$$A_1 [(B_0 x)(\varepsilon t)] = (B_1 B_0 x)(\varepsilon^2 t) + (B_0 \psi)(\varepsilon t)$$

we obtain

$$A_1 A_0 x = (B_1 B_0 x)(\varepsilon^2 t) + A_1 \psi + (B_0 \psi)(\varepsilon t)$$

and act on this relation by A_2. From

$$A_2 [(B_1 B_0 x)(\varepsilon^2 t)] = (B_2 B_1 B_0 x)(\varepsilon^3 t) + (B_1 B_0 \psi)(\varepsilon^2 t),$$
$$A_2 [(B_0 \psi)(\varepsilon t)] = (A_1 B_0 \psi)(\varepsilon t)$$

it follows that

$$A_2 A_1 A_0 x = (B_2 B_1 B_0 x)(\varepsilon^3 t) + A_2 A_1 \psi + (A_1 B_0 \psi)(\varepsilon t) + (B_1 B_0 \psi)(\varepsilon^2 t).$$

Finally, this process leads to the ordinary differential equation

$$(A_0^{(m-1)} - B_0^{(m-1)})x = \sum_{j=0}^{m-1} (A_1^{(m-1-j)} B_0^{(j-1)} \psi)(\varepsilon^j t) \tag{3.2}$$

where

$$A_i^{(j)} = A_j A_{j-1} \cdots A_i, \qquad B_i^{(j)} = B_j B_{j-1} \cdots B_i, \qquad i \leqslant j$$
$$A_1^{(0)} = B_0^{(-1)} = E.$$

Thus, (3.1) is reduced to the ODE (3.2) of order mn. To agree the initial conditions for (3.2) with the original problem it is necessary to attach to conditions (3.1) the additional relations

$$(A_0^{(j)} - \varepsilon^{k(j+1)} B_0^{(j)})\, x^{(k)}(t)|_{t=0} = \sum_{i=0}^{j} \varepsilon^{ik} A_1^{(j-i)} B_0^{(i-1)} \psi^{(k)}(t)|_{t=0}$$
$$(j = 0,..., m-2;\ k = 0,..., n-1). \tag{3.3}$$

System (3.3) has a unique solution for $x^{(k)}(0)$ $(n \leqslant k \leqslant mn-1)$, iff

$$a_n^j \neq (\varepsilon^i b_n)^j \qquad (0 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant m-1). \tag{3.4}$$

These considerations enable us to formulate

THEOREM 3.1. *If* $\psi \in C^{(m-1)n}$ *and inequalities* (3.4) *are fulfilled, the solution of ordinary differential equation* (3.2) *with initial conditions* (3.1)–(3.3) *satisfies problem* (3.1).

THEOREM 3.2. *If* $\varepsilon \neq 1$, *the substitution*

$$y = x \exp(\alpha t/1 - \varepsilon)$$

transforms the equation

$$Ay = \exp(\alpha t)(By)(\varepsilon t) + \psi \tag{3.5}$$

with operators A and B defined by (3.1) *to*

$$Px = (Qx)(\varepsilon t) + \psi \exp(-\alpha t/1 - \varepsilon)$$

with constant coefficients p_k, q_k *and* $p_n = a_n$, $q_n = b_n$.

COROLLARY. *Under assumptions* (3.4) *and* $\varepsilon^m = 1$, (3.5) *is reducible to a linear ordinary differential equation with constant coefficients.*

Remark. Conditions (3.4) hold if, in particular, $|a_n| \neq |b_n|$. Theorems 3.1 and 3.2 sharpen the corresponding results of [13] and [14] established for homogeneous equations (3.1) and (3.5) by operational methods under the restriction $|a_n| > |b_n|$.

EXAMPLE. The substitution $y = x \exp t$ reduces the equation

$$y'(t) = (5y(-t) + 2y'(-t)) \exp 2t, \qquad y(0) = y_0$$

to the form

$$x'(t) + x(t) = 7x(-t) + 2x'(-t), \qquad x(0) = y_0.$$

Therefore (3.2) gives for $x(t)$ the ODE $x'' - 16x = 0$ with the initial conditions $x(0) = y_0$, $x'(0) = -6y_0$. The unknown solution is

$$y(t) = y_0(5 \exp(-3t) - \exp 5t)/4.$$

THEOREM 3.3. *Suppose that the coefficients of the equation*

$$\sum_{k=0}^{n} a_k(t)\, x^{(k)}(t) = x(\varepsilon t) + \psi(t), \qquad x^{(k)}(0) = x_k,\ k = 0, \dots, n-1 \tag{3.6}$$

beong to $C^{(m-1)n}$, $\varepsilon^m = 1$, $a_n(0) \neq 0$ *and*

$$L_j = \sum_{k=0}^{n} \varepsilon^{-jk} a_k(\varepsilon^j t)\, d^k/dt^k, \qquad j = 0, \dots, m-1. \tag{3.7}$$

Then the solution of the linear ordinary differential equation

$$L_0^{(m-1)}x(t) = x(t) + \sum_{k=1}^{m-1} (L_k^{(m-1)}\psi)(\varepsilon^{k-1}t) + \psi(\varepsilon^{m-1}t)$$
$$(L_k^{(m-1)} = L_{m-1}L_{m-2} \cdots L_k, 0 \leqslant k \leqslant m-1) \tag{3.8}$$

with the initial conditions

$$x^{(k)}(0) = x_k(k = 0,..., n-1),$$
$$L_0 x^{(k)}(t)|_{t=0} = \varepsilon^k x^{(k)}(0) + \psi^{(k)}(0), \qquad k = 0,..., n(m-1)-1$$

satisfies problem (3.6).

Proof. Applying the operator L_1 to (3.6) and taking into account that

$$(L_0 x)(\varepsilon t) = x(\varepsilon^2 t) + \psi(\varepsilon t)$$

we get

$$L_1 L_0 x(t) = x(\varepsilon^2 t) + L_1 \psi(t) + \psi(\varepsilon t)$$

and act on this equation by L_2 to obtain

$$L_2 L_1 L_0 x(t) = x(\varepsilon^3 t) + L_2 L_1 \psi(t) + (L_2 \psi)(\varepsilon t) + \psi(\varepsilon^2 t).$$

It is easy to verify the relations

$$(L_j x)(\varepsilon^j t) = x(\varepsilon^{j+1} t) + \psi(\varepsilon^j t), \qquad j = 0,..., m-1.$$

In particular,

$$(L_{m-1} x)(\varepsilon^{m-1} t) = x(t) + \psi(\varepsilon^{m-1} t).$$

Thus, the use of the operator L_{m-1} at the conclusive stage yields (3.8).

THEOREM 3.4. *The system*

$$tAX'(t) + BX(t) = X(\varepsilon t) \tag{3.9}$$

with constant matrices A and B is integrable in the closed form if $\varepsilon^m = 1$, $\det A \neq 0$.

Proof. For (3.9) the operators L_j defined by formula (3.7) are

$$L_j = tA\, d/dt + B.$$

Hence, on the basis of the previous theorem, (3.9) is reducible to the ordinary system

$$(tA\, d/dt + B)^m X(t) = X(t). \tag{3.10}$$

This is Euler's equation with matrix coefficients. Since its order is higher than that of (3.9) we substitute the general solution of (3.10) into (3.9) and equate the coefficients of the like terms in the corresponding logarithmic sums to find the additional unknown constants.

EXAMPLE. We associate with the equation

$$tx'(t) - 2x(t) = x(\varepsilon t), \qquad \varepsilon^3 = 1 \tag{3.11}$$

the relation

$$(td/dt - 2)^3 x(t) = x(t).$$

The substitution of its general solution

$$x(t) = C_1 t^3 + t^{3/2}(C_2 \sin(\sqrt{3}/2 \ln t) + C_3 \cos(\sqrt{3}/2 \ln t))$$

into (3.11) gives $C_2 = C_3 = 0$. A solution of (3.11) is

$$x = Ct^3.$$

THEOREM 3.5. *The system*

$$tAX'(t) = BX(t) + tX(\varepsilon t) \tag{3.12}$$

with constant coefficients A and B, $\det A \neq 0$ and $\varepsilon^m = 1$ is integrable in the closed form and has a solution

$$X(t) = P(t)\, t^{A^{-1}B} \tag{3.13}$$

where the matrix $P(t)$ is a finite linear combination of exponential functions.

Proof. The transition from (3.12) to an ordinary equation is realized by means of the operators

$$L_j = \varepsilon^{-j}(Ad/dt - t^{-1}B), \qquad j = 0, \ldots, m-1$$

in consequence of which we obtain the relation

$$(Ad/dt - t^{-1}B)^m X(t) = \varepsilon^{m(m-1)/2} X(t). \tag{3.14}$$

Since $\varepsilon^{m(m-1)/2} = \pm 1$, it takes the form

$$\prod_{k=1}^{m} [Ad/dt - (\varepsilon_k E + t^{-1}B)]\, X(t) = 0$$

where ε_k are the m-order roots of 1 or -1. The solutions of the equations

$$AX'(t) = (\varepsilon_k E + t^{-1} B)\, X(t)$$

are matrices

$$X_k(t) = \exp(\varepsilon_k t A^{-1})\, t^{A^{-1}B}, \qquad k = 1,..., m.$$

Their linear combination represents the general solution of (3.14).

EXAMPLE. In accordance with (3.14), to the equation

$$tx'(t) = 3x(t) + tx(-t) \tag{3.15}$$

there correspond two ordinary relations

$$x'(t) = (3t^{-1} + i)\, x(t), \qquad x'(t) = (3t^{-1} - i)\, x(t).$$

We substitute into (3.15) the linear combination of their solutions

$$x(t) = t^3 (C_1 \exp(it) + C_2 \exp(-it))$$

and find $C_2 = iC_1$. A solution of (3.15) is

$$x(t) = Ct^3 (\sin t + \cos t).$$

4. GLOBAL SOLUTIONS OF LINEAR SYSTEMS

The results of the earlier chapter provide information on the structure of solutions to FDE with rather special transformations of the argument. However, the techniques developed for the study of distributional solutions may be used to explore entire solutions of equations with variable coefficients and deviations proportional to the argument. There is abundant material and references on the asymptotic behavior of solutions to linear FDE in [15, 16] and also in numerous articles. At first we extend to more general systems the well-known Izumi's theorem [17] that if in the equation

$$f^{(n)}(t) + a_1(t) f^{(n-1)}(\omega_1(t)) + \cdots + a_n(t)\, f(\omega_n(t)) = b(t)$$

$a_i(t), b(t), \omega_i(t)$ are regular in the disk $|t| \leqslant 1$ and $\omega_i(0) = 0$, $|\omega_i(t)| < 1$, for $|t| \leqslant 1$, there exists a unique solution with the given $f^{(i)}(0)$ regular in $|t| \leqslant 1$.

THEOREM 4.1. *Suppose the system*

$$X'(t) = \sum_{i=0}^{\infty} A_i(t)\, X(\lambda_i(t)) + \sum_{i=0}^{\infty} B_i(t)\, X'(\mu_i(t)), \qquad X(0) = X_0 \tag{4.1}$$

in which A_i, B_i, X are $r \times r$-matrices satisfies the following hypotheses.

(i) *The coefficients and the deviations are regular in the disk $|t| \leqslant 1$.*

(ii) $\lambda_i(0)=0$, $\mu_i(0)=0$, $|\lambda_i(t)| \leqslant q < 1$, $|\mu_i(t)| \leqslant q$ *for* $|t| \leqslant 1$.

(iii) *The series* $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|A_{ij}\|$ *and* $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|B_{ij}\|$,

where A_{ij} and B_{ij} are the Taylor coefficients of $A_i(t)$ and $B_i(t)$, respectively, converge and $\sum_{i=0}^{\infty} \|B_i(0)\| < 1$.

Then the problem has a unique solution regular in $|t| < 1$.

Proof. Let

$$X(t)=\sum_{n=0}^{\infty} X_n t^n, \quad (\lambda_i(t))^n=\sum_{k=0}^{\infty} \lambda_{ink} t^k, \quad (\mu_i(t))^n=\sum_{k=0}^{\infty} \mu_{ink} t^k.$$

Then

$$X(\lambda_i(t))=\sum_{n=0}^{\infty} X_n(\lambda_i(t))^n=\sum_{n=0}^{\infty} X_n \sum_{k=0}^{\infty} \lambda_{ink} t^k,$$

$$X'(\mu_i(t))=\sum_{n=0}^{\infty} (n+1) X_{n+1}(\mu_i(t))^n=\sum_{n=0}^{\infty} (n+1) X_{n+1} \sum_{k=0}^{\infty} \mu_{ink} t^k.$$

Since $\lambda_i(0)=0$, $\mu_i(0)=0$, we note that $\lambda_{ink}=0$ and $\mu_{ink}=0$ for $k<n$. Changing the order of summation in the latter equations gives

$$X(\lambda_i(t))=\sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \lambda_{ikn} X_k,$$

$$X'(\mu_i(t))=\sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} (k+1) \mu_{ikn} X_{k+1}.$$

However, now $\lambda_{ikn}=0$ and $\mu_{ikn}=0$ for $k>n$.

Therefore,

$$X(\lambda_i(t))=\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \lambda_{ikn} X_k,$$

$$X'(\mu_i(t))=\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (k+1) \mu_{ikn} X_{k+1}$$

and

$$A_i(t) X(\lambda_i(t)=\sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} A_{i,n-j} \sum_{k=0}^{j} \lambda_{ikj} X_k,$$

$$B_i(t) X'(\mu_i(t))=\sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} B_{i,n-j} \sum_{k=0}^{j} (k+1) \mu_{ikj} X_{k+1}.$$

Hence, the coefficients X_n of the unknown solution satisfy the equations

$$(n+1)X_{n+1} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} A_{i,n-j} \sum_{k=0}^{j} \lambda_{ikj} X_k + \sum_{i=0}^{\infty} \sum_{j=0}^{n} B_{i,n-j} \sum_{k=0}^{j} (k+1) \mu_{ikj} X_{k+1}, \qquad n \geqslant 0. \quad (4.2)$$

These relations can be written as

$$\begin{aligned}(n+1)\left(E - \sum_{i=0}^{\infty} \mu_{inn} B_{io}\right) X_{n+1} &= \sum_{i=0}^{\infty} \sum_{j=0}^{n} A_{i,n-j} \sum_{k=0}^{j} \lambda_{ikj} X_k \\ &+ \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} B_{i,n-j} \sum_{k=0}^{j} (k+1) \mu_{ikj} X_{k+1} \\ &+ \sum_{i=0}^{\infty} B_{io} \sum_{k=0}^{n-1} (k+1) \mu_{ikn} X_{k+1}.\end{aligned}$$

By virtue of the Cauchy inequalities

$$|a_n| \leqslant R^{-n} \max_{|t| \leqslant R} |f(t)|$$

for the coefficients a_n of a function $f(t)$ analytic in the disk $|t| \leqslant R$, we have

$$|\lambda_{ikj}| \leqslant \max_{|t| \leqslant 1} |(\lambda_i(t))^k| \leqslant q^k, \qquad |\mu_{ikj}| \leqslant q^k.$$

These relations together with inequality (iii) ensure the existence of the inverse matrices $(E - \sum_{i=0}^{\infty} \mu_{inn} B_{io})^{-1}$ for all n:

$$\left(E - \sum_{i=0}^{\infty} \mu_{inn} B_{io}\right)^{-1} = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} \mu_{inn} B_{io}\right)^{j},$$

$$\left\| \left(E - \sum_{i=0}^{\infty} \mu_{inn} B_{io}\right)^{-1} \right\| \leqslant \left(1 - \sum_{i=0}^{\infty} \|B_{io}\|\right)^{-1}.$$

Hence, formulas (4.2) uniquely determine the coefficients X_n and yield the inequalities

$$(n+1)\|X_{n+1}\| \leqslant cM_n \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|A_{ij}\| \sum_{k=0}^{\infty} q^k + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|B_{ij}\| \sum_{k=0}^{\infty} (k+1) q^k\right).$$

Due to (iii) and the convergence of the series $\sum q^k$ and $\sum (k+1)q^k$, we obtain

$$\|X_{n+1}\| \leqslant c_1 M_n/(n+1).$$

Thus, starting with some N,

$$\|X_{n+1}\| \leqslant M_n, \qquad M_{n+1} = M_n, \qquad M_n = M_N, \qquad n > N$$

and $\|X_n\| \leqslant c_1 M_N/n$, which proves the theorem.

Remark. If in (4.1) the number of deviations is finite, the milder restrictions $|\lambda_i(t)| < 1$, $|\mu_i(t)| < 1$ substitute for inequalities (ii).

THEOREM 4.2. *The system*

$$X^{(p)}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{p} P_{ij}(t)\, X^{(j)}(\lambda_{ij} t), \qquad X^{(j)}(0) = X_j, \quad j = 0,\ldots, p-1 \tag{4.3}$$

with $r \times r$-matrices P_{ij} and X has a unique holomorphic solution, which is an entire function of zero order, if

(i) *$P_{ij}(t)$ are polynomials in t of degree not exceeding m;*

(ii) *λ_{ij} are complex numbers such that $0 < |\lambda_{ij}| \leqslant q < 1$;*

(iii) *the series $\sum P^{(i)}$ converges, where $P^{(i)} = \max_{j,k} \|P_{ijk}\|$ and P_{ijk} are the coefficients of $P_{ij}(t)$, and $\sum_{i=0}^{\infty} \|P_{im}(0)\| < 1$.*

Proof. Under the conditions of Theorem 2.1 the system

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t)\, X^{(j)}(\alpha_{ij} t) = 0$$

with real constants α_{ij} has a distributional solution (1.1) the coefficients X_n of which satisfy inequalities (1.2) and are determined with the exactness to arbitrary $X_0,\ldots, X_{p-1}$. We apply to the last equation the Laplace transformation assuming α_{ij} positive and retaining the same notation for $X(t)$ and its transform:

$$X^{(p)}(s/\alpha_{oo}) + \alpha_{oo}^{p+1} A^{-1} \sum_{i+j>0} \sum_{k=0}^{p} (-1)^{p-k} \alpha_{ij}^{-j-1} A_{ijk} (s^j X(s/\alpha_{ij}))^{(k)} = 0. \tag{4.4}$$

The substitutions $s/\alpha_{oo} = t$, $\alpha_{oo}/\alpha_{ij} = \lambda_{ij}$ reduce (4.4) to the form (4.3). This proves the theorem since the transform of $\delta^{(n)}(t)$ is s^n and the coefficients X_n satisfy inequalities (2.7). These estimates use only the moduli of α_{ij}, hence the parameters λ_{ij} may be complex.

THEOREM 4.3. *Under the assumptions of Theorem* 2.1 *there exists a polynomial* $Q(t)$ *of degree* $p-1$ *such that the system*

$$\sum_{i=0}^{k} \sum_{j=0}^{m} A_{ij}(t) X^{(j)}(\alpha_{ij} t) = Q(t)$$

with positive constants α_{ij} *has a solution* $X(t)$ *regular at* $t = \infty$ *and* $X(t^{-1})$ *is an entire function of zero order.*

Proof. We apply the unilateral Laplace transformation to (4.4).

THEOREM 4.4. *If* $A_{ij}(t)$ *are polynomials of degree not exceeding* $p+j-2$ $(p \geqslant 2)$ *and* $\alpha_{ij} > 1$, *there exists a polynomial* $Q(t)$ *of degree* $p+m-2$ *such that the system*

$$t^p X'(t) = \sum_{i=0}^{k} \sum_{j=0}^{m} A_{ij}(t) X^{(j)}(\alpha_{ij} t) + Q(t)$$

has a solution $X(t)$ *regular at infinity and* $X(t^{-1})$ *is an entire function of zero order.*

THEOREM 4.5. *Suppose the system*

$$tX'(t) = AX(t) + t \sum_{i=0}^{\infty} A_i(t) X(\lambda_i t), \tag{4.5}$$

in which A, A_i, X *are* $r \times r$*-matrices, satisfies the following hypotheses.*

(i) *A is constant,* $A_i(t)$ *are polynomials of the highest degree* m:

$$A_i(t) = \sum_{k=0}^{m} A_{ik} t^k, \qquad m \geqslant 0.$$

(ii) *The parameters* λ_i *are constant,* $0 < \lambda_i \leqslant 1$.

(iii) *None of any two eigenvalues of the matrix* A *differ by a positive integer.*

(iv) *The series* $\sum \lambda_i^{-\|A\|} A^{(i)}$ *converges, where* $A^{(i)} = \max_k \|A_{ik}\|$.

Then there exists a matrix solution

$$X(t) = P(t)\, t^A, \qquad P(0) = E \tag{4.6}$$

with an entire function $P(t)$ *of order not exceeding* $m+1$. *If the values* λ_i *are separated from unity:* $0 < \lambda_i \leqslant q < 1$, *the order of* $P(t)$ *is zero.*

Proof. The representation of the solution

$$X(t) = P(t)t^R, \qquad P(0) = E$$

leads to the equation

$$tP'(t) + P(t)R = AP(t) + t \sum_{i=0}^{\infty} A_i(t)\, P(\lambda_i t)\, \lambda_i^R,$$

whence at $t = 0$ we have $R = A$, and the expansion

$$P(t) = \sum_{n=0}^{\infty} P_n t^n$$

yields the relations

$$P_n(A + nE) - AP_n = \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} A_{i,n-1-k} P_k \lambda_i^k \lambda_i^A,$$

$$P_0 = E, \qquad n \geqslant 1$$

for the determination of the coefficients P_n. Since $A_{i,n-1-k} = 0$ $(n - 1 - k > m)$ we get

$$P_n(A + nE) - AP_n = \sum_{i=0}^{\infty} \sum_{k=0}^{m} A_{i,m-k} P_{n+k-m-1} \lambda_i^{n+k-m-1} \lambda_i^A. \qquad (4.7)$$

If the matrices $A + nE$ and A have no common eigenvalues for any natural n each of the Eqs. (4.7) has a unique solution P_n and all P_n can be found successively for all $n \geqslant 1$. Starting with some number the matrices $A + nE$ have inverses

$$(\mathrm{A} + nE)^{-1} = \sum_{i=0}^{\infty} (-1)^i n^{-i-1} A^i, \qquad n > N_1.$$

Therefore,

$$\|(A + nE)^{-1}\| \leqslant \sum_{i=0}^{\infty} n^{-i-1} \|A\|^i = (n - \|A\|)^{-1}, \qquad n > N_1. \qquad (4.8)$$

Taking into account that $0 < \lambda_i \leqslant 1$ it is easy to establish

$$\|\lambda_i^A\| \leqslant \sum_{j=0}^{\infty} |\ln \lambda_i|^j \|A\|^j / j! = \sum_{j=0}^{\infty} (-1)^j \ln^j \lambda_i \|A\|^j / j!,$$

$$\|\lambda_i^A\| \leqslant \lambda_i^{-\|A\|}. \qquad (4.9)$$

From (4.7) we obtain

$$P_n = \left(AP_n + \sum_{i=0}^{\infty} \sum_{k=0}^{m} A_{i,m-k} P_{n+k-m-1} \lambda_i^{n+k-m-1} \lambda_i^A \right) (A + nE)^{-1}.$$

Hence, by virtue of (4.8) and (4.9) we find

$$\|P_n\| \leqslant \sum_{i=0}^{\infty} \lambda_i^{-\|A\|} A^{(i)} \sum_{k=0}^{m} \lambda_i^{n+k-m-1} \|P_{n+k-m-1}\| \, (n - 2\|A\|)^{-1}, \qquad n > N_1.$$

Since $0 < \lambda_i \leqslant \lambda \leqslant 1$ and $2\|A\| \leqslant N_1$ for large N_1, the convergence of series (iv) implies

$$\|P_n\| \leqslant \mu \lambda^n (n - N_1)^{-1} \sum_{k=n-m-1}^{n-1} \|P_k\|.$$

With the notation

$$M_n = \max_{o \leqslant k \leqslant n} \|P_k\|$$

we have

$$\|P_n\| \leqslant \mu(m+1)\, \lambda^n (n - N_1)^{-1} M_{n-1}$$

and from $N > N_1$ onwards

$$\mu(m+1)\, \lambda^n (n - N)^{-1} \leqslant 1, \qquad \|P_n\| \leqslant M_{n-1}, \qquad M_n = M_{n-1}.$$

By employing the procedure of Theorem 2.1 we evaluate

$$\|P_{N+1+n(m+1)+k}\| \leqslant \mu^{n+1} (m+1)\, \lambda^{n(n+1)(m+1)/2} M_N / n!. \tag{4.10}$$

When $\lambda \leqslant 1$,

$$\|P_{N+1+n(m+1)+k}\| \leqslant (m+1)\, \mu^{n+1} / n!.$$

Thus, $P(t)$ is an entire function whose order of growth does not exceed $m+1$. If we supplement the conditions of the theorem by the restriction $\lambda \leqslant q < 1$ it is obvious from (4.10) that $P(t)$ is of zero order. This conclusion generalizes the results of [18] and [19] obtained for Eq. (1.3).

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