Hyperinvariant subspaces for quasinilpotent operators on Hilbert spaces

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In this paper, we employ the model theory due to C. Foias and C. Pearcy and the notion of Enflo’s extremal vectors of quasinilpotent operators to study the hyperinvariant subspace problem for quasinilpotent operators. Our main result is that if a quasinilpotent quasiaffinity \( T \) has a sequence of “c-eigenvectors” \( x_n \) of \( T^n T^* \) such that the set \( \overline{\{x_n : n \in \mathbb{N}\}} \) is compact, then \( T \) has a nontrivial hyperinvariant subspace.

1. Introduction

Let \( H \) be a separable infinite dimensional complex Hilbert space and \( \mathcal{L}(H) \) be the algebra of all bounded linear operators acting on \( H \). The commutant of \( T \), denoted by \( \{T\}' \), is the algebra of all operators \( X \) in \( \mathcal{L}(H) \) such that \( XT = TX \). A subspace \( M \subset H \) is called a nontrivial hyperinvariant subspace for \( T \) if \( \{0\} \not= M \not= H \) and \( XM \subset M \) for each \( X \in \{T\}' \). In particular, if \( TM \subset M \), then the subspace \( M \) is called a nontrivial invariant subspace for \( T \). The hyperinvariant subspace problem is the question whether every operator in \( \mathcal{L}(H) \setminus \mathbb{C} \) has a nontrivial hyperinvariant subspace. This problem remains still open, especially for quasinilpotent operators in \( \mathcal{L}(H) \), i.e., operators \( T \) such that \( \sigma(T) = \{0\} \), where \( \sigma(\cdot) \) means spectrum. In this paper we consider the hyperinvariant subspace problem for quasinilpotent operators.

In 1998, S. Ansari and P. Enflo [1] introduced extremal vectors as a method of constructing hyperinvariant subspaces for a certain class of linear operators. As a consequence of this technique, P. Enflo gave the “two sequences” theorem [5, Theorem 4.8] which provided a contribution to the hyperinvariant subspace problem for quasinilpotent operators. The two sequences theorem is stated as: if \( \mathcal{A} \subset \mathcal{L}(H) \) be any commutative algebra that contains a nonzero quasinilpotent operator \( T \), then there exist two sequences \( \{s_n\} \) and \( \{t_k\} \) that converge weakly to nonzero vectors such that for every bounded sequence \( \{A_k\}_{k=1}^\infty \subset \mathcal{A} \),

\[
\lim_{k \to \infty} \langle A_k s_k, t_k \rangle = 0.
\]

In this case, if \( \{s_k\} \) or \( \{t_k\} \) converges in norm, then \( T \) has a nontrivial hyperinvariant subspace.

In 2003, I. Jung et al. [8] improved the two sequences theorem modifying Enflo’s technique mentioned in [1]. As a corollary, they obtained a result which is related to the “Pearcy–Salinas property,” which states that if \( T \) is an operator on \( H \), then there exist a sequence \( \{s_n\} \subset \{T\}' \) and a sequence \( \{K_n\} \) of compact operators such that

\[
\lim_{n \to \infty} \|s_n - K_n\| = 0.
\]

They showed that every quasinilpotent operator with the Pearcy–Salinas property has a nontrivial hyperinvariant subspace [8, Corollary 1.4].
On the other hand, in 2004, V.G. Troitsky [11] introduced a notion of a $\lambda$-extremal vector which is a generalization of Enflo’s extremal vectors to a Banach space. By using this definition, I. Chalendar and J. Partington in [3] constructed a Banach space version of the two sequences theorem and then showed that the above Jung–Ko–Pearcy’s result can be extended to reflexive Banach spaces.

Also, in 2005, C. Foiaş et al. [6] used the spectral theorem for positive operators to obtain a new result similar to the two sequences theorem. Then they got an interesting result under a “finite dimensional” condition.

**Theorem 2.1.** (See [6, Theorem 3.3].) Suppose $T \in \mathcal{L}(H)$ is a quasinilpotent quasiaffinity. If there exists a finite dimensional subspace $N \neq \{0\}$ that reduces each member of the sequence $\{T^nT^*n\}_{n \in \mathbb{N}}$ (in particular, if the operators in the sequence $\{T^nT^*n\}_{n \in \mathbb{N}}$ have a common eigenvector $x_0$), then $T$ has a nontrivial hyperinvariant subspace.

In Section 2, we introduce P. Enflo’s “extremal vectors” and raise some questions which can be strategies to solve the hyperinvariant subspace problem for quasinilpotent operators. And we construct auxiliary lemmas using the “model theory” in [7] as a preparation of Section 3. In Section 3, we define a notion of “c-eigenvector” and then present the main result which improves Theorem 1.1. We also give some examples showing that there exists a gap between Theorem 1.1 and the main result of this paper, and introduce a result which has something to do with the model theory in [7].

2. **Properties of extremal vectors**

The notion of extremal vectors was introduced by S. Ansari and P. Enflo [1]. Extremal vectors provide a tool to find hyperinvariant subspaces for compact and normal operators on Hilbert spaces. First, we introduce Enflo’s extremal vectors and consider their behavior. Assume that $T$ has dense range and choose a unit vector $x_0 \in H$ and $0 < \varepsilon < 1$. If $\mathcal{F} = \{y \in H : \|Ty - x_0\| \leq \varepsilon\}$, then $\mathcal{F}$ is a nonempty, norm closed and convex set. So there exists a unique minimal vector $y_0 = y_0(x_0, \varepsilon) \in \mathcal{F}$. We say that $y_0$ is the extremal (minimal) vector for $T$. In this case, $\|Ty_0 - x_0\| = \varepsilon$.

Let $y_n = y_n(x_0, \varepsilon)$ be the extremal vector for $T^n$ and for a notational convenience, write $y_1 := y_0$. In [1], Enflo established an important equation on extremal vectors called “Orthogonality Equation:” if $T$ has dense range, then there exists a constant $r < 0$ satisfying

$$T^*(Ty_0 - x_0) = ry_0.$$  (1)

For fixed $r$, the vector $y_0$ satisfying (1) is unique. Indeed, if $T^*(T_0 - x_0) = rz_0$ for some $z_0 \in H$, then

$$T^*T(z_0 - y_0) = r(z_0 - y_0).$$

But since $T^*T$ is positive and $r < 0$, it follows $y_0 = z_0$. It was also shown in [1, Lemma 1] that if $T$ is a quasinilpotent operator with dense range, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k \to \infty} \frac{\|y_{n_k}\|}{\|y_{n_{k+1}}\|} = 0.$$  (2)

In [11], V.G. Troitsky defined a notion of $\lambda$-extremal vectors which is a generalization of that of Enflo’s extremal vectors to a Banach space. Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$. Assume that $T$ has dense range and choose a unit vector $x_0 \in X$ and $0 < \varepsilon < 1$. For $\lambda \geq 1$, a vector $y_0 = y_0(x_0, \varepsilon, T)$ is called a $\lambda$-extremal (minimal) vector if

$$\|Ty_0 - x_0\| \leq \varepsilon \quad \text{and} \quad \|y_0\| \leq \lambda \inf\{\|y\| : \|Ty - x_0\| \leq \varepsilon\}.$$  (3)

For $\lambda > 1$, the existence of $\lambda$-extremal vectors is clear and they are not unique. If $\lambda = 1$, then the $\lambda$-extremal vector in the Hilbert space setting is exactly the Enflo’s extremal vector.

Before we go on, we first discuss the stability of image of extremal vectors mentioned in [9]. Suppose $T$ has dense range. Then $T$ is said to be strongly stable (for $x_0$) if there exists a unit vector $x_0 \in H$ and $0 < \varepsilon < 1$ such that $T^n y_n \in \vee \{x_0\}$ for all $n$, where the $y_n = y_n(x_0, \varepsilon)$ are extremal vectors for $T^n$. In this case, $T^n y_n = (1 - \varepsilon)x_0$ for all $n$. The following theorem is a characterization of strongly stable operators.

**Theorem 2.1.** (See [9, Theorem 3.4].) Suppose $T$ has dense range. Then $T$ is strongly stable for $x_0$ if and only if the operators in the family $\{T^nT^*n\}_{n \in \mathbb{N}}$ (some $n \in \mathbb{N}$) have a common eigenvector $x_0$.

More generally, suppose that $T^n y_n = z_0$ for all $n$, i.e., $T^n y_n$ are fixed. Then by (1), we have

$$T^nT^*n(x_0 - z_0) = \delta_n z_0 \quad (\delta_n > 0).$$

Write $\lambda_n := \frac{\delta_n}{\delta_1}$. Then

$$T(T^{-1}T^{n-1} - \lambda_n)T^*(x_0 - z_0) = 0.$$
Therefore if \( T \) is one-one, then the nonzero vector \( T^*(x_0 - z_0) \) is a common eigenvector of \( \{T^nT^*\}_{n=1}^\infty \), and hence \( T \) is strongly stable by Theorem 2.1. So it can be redefined that a quasiaffinity \( T \) is strongly stable if there exist a unit vector \( x_0 \) in \( H \) and \( 0 < \varepsilon < 1 \) such that \( T^n y_n \) are fixed for all \( n \), where the \( y_n = y_n(x_0, \varepsilon) \) are extremal vectors for \( T^n \).

An operator \( T \in \mathcal{L}(H) \) is called a quasiaffinity if it is a one-one mapping having dense range. In [9], we introduced two interesting questions. Firstly, if \( T \) is a strongly stable quasiaffinity, is \( T^* \) strongly stable? We were unable to answer this question. However, with some condition, we can get an affirmative answer: If \( T \) is a quasinormal operator, that is, \( T \) satisfies the condition \( T^*TT = TT^*T \), then the answer is true. Secondly, let \( T \) be a quasiaffinity and \( A \) be a restriction of \( T \) onto \( M \). If \( A \) is a quasiaffinity, is it true that

\[
T \text{ is strongly stable } \iff A \text{ is strongly stable? (3)}
\]

To examine the above question we suppose \( M \) is a Hilbert space and write \( H := M \oplus M \). Let \( T = \frac{1}{\sqrt{2}} \begin{pmatrix} I & B \end{pmatrix} \) on \( H \). Then \( I = T|M \) is clearly strongly stable for every unit vector in \( M \). Note that

\[
T^n T^* = \frac{1}{4^n} \begin{pmatrix} (n^2 + 1)I & nI \\ nI & I \end{pmatrix}.
\]

Let \( x \in H \) be a common eigenvector of \( \{T^nT^*\} \). Then we have \( x \in M \) because \( x \) is an eigenvector of

\[
TT^* - 2T^2T^* = \frac{1}{8} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.
\]

But any vector in \( M \) cannot be an eigenvector of \( TT^* \) which leads a contradiction. Therefore \( T \) is not strongly stable by Theorem 2.1. This shows that the backward implication of (3) fails. On the other hand, the forward implication does not also hold in general. To see this, take an operator \( T = A \oplus B \), where \( B \) is strongly stable, but \( A \) is not. From this viewpoint, we might ask the following question:

\[
T \text{ is strongly stable for } x_0 \in M \implies A \text{ is strongly stable for } x_0? \quad (4)
\]

This question is nontrivial and remains still open. However, if the answer to the question (3) is true with some constraint, then the answer to the question (4) would be true. Here is such a case.

**Proposition 2.2.** Let \( T \) be a quasiaffinity and \( A \) be a restriction of \( T \) onto \( M \). Suppose \( A \) is a quasiaffinity and \( T \) is strongly stable for \( x_0 \in M \). If the adjoint of a quasiaffinity which is strongly stable for a unit vector \( x \) is also strongly stable for same \( x \), then \( A \) is also strongly stable for \( x_0 \).

**Proof.** Write

\[
T := \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} M \oplus M^\perp
\]

and \( B_n := P_M T^n(1 - P_M) \), where \( P_M \) is the orthogonal projection onto \( M \). Then we have

\[
T^*T^n := \begin{pmatrix} A^* & A^*B \\ B_A^* & B_A^* + C^*C \end{pmatrix} M \oplus M^\perp.
\]

Since \( T^* \) is strongly stable for \( x_0 \in M \) by assumption, there exists a common eigenvector \( x_0 \) of \( \{T^*T^n\} \) by Theorem 2.1. Since \( x_0 \in M \), it follows that \( A^*A^n x_0 = \lambda_n x_0 \) (\( \lambda_n > 0 \)) for all \( n \). Therefore \( A^* \) is strongly stable for \( x_0 \), so that \( A \) is strongly stable for \( x_0 \) by assumption. \( \square \)

Next, let us introduce a new definition which is a natural extension of the strong stability redefined in the below of Theorem 2.1.

**Definition 2.3.** Suppose \( T \) has dense range. Then \( T \) is said to be strongly quasi-stable (for \( x_0 \)) if there exist a unit vector \( x_0 \) in \( H \) and \( 0 < \varepsilon < 1 \) such that the sequence \( \{T^n y_n\} \) converges in norm, where the \( y_n = y_n(x_0, \varepsilon) \) are extremal vectors for \( T^n \).

It is obvious that every strongly stable operator is strongly quasi-stable. And by [9, Definition 3.11], strongly quasi-stable operators are finitely quasi-stable with one-dimensional quasi-stable space. In particular, [2, Proposition 2.1] states that if \( T \) is a normal operator with dense range, then \( \{T^n y_n\} \) converges in norm for any \( x_0 \) and \( \varepsilon \). So normal operators can be a good example of strongly quasi-stable operators.

Next, we study a part of a strongly quasi-stable quasinilpotent operator. In a similar manner to the strongly stable case, we can ask a question about strong quasi-stability.

**Question 2.4.** Let \( T \) be a quasiaffinity, let \( A \) be a restriction of \( T \) onto \( M \), and let \( T \) and \( A \) be quasiaffinities. Does there exist a nonzero vector \( x_0 \in M \) such that if \( T \) is strongly quasi-stable for \( x_0 \), then \( A \) is strongly quasi-stable for \( x_0 \)?
This question is closely related to the model of quasinilpotent operators due to C. Foias and C. Pearcy [7,10]. Let \( \alpha = \{\alpha_n\} \) be a positive sequence decreasing to zero. Define \( K = K_\alpha \in \mathcal{L}(\bigoplus H) \) by

\[
K_\alpha := \begin{pmatrix}
0 & \alpha_11_H \\
0 & \alpha_21_H \\
& \ddots \\
& & 0
\end{pmatrix}.
\]

Then evidently \( K_\alpha \) is a quasinilpotent operator. The following is the model theory for quasinilpotent operators.

**Lemma 2.5.** (See [7, Theorem 1], [10, Theorem 9.6].) If \( T \) is a quasinilpotent operator, then there exist a decreasing sequence \( \alpha = \{\alpha_n\} \) of nonnegative numbers converging to zero, an invariant subspace \( M \) of the operator \( K_\alpha \), and an invertible operator \( S : H \to M \) such that \( STS^{-1} = K_\alpha|_M \).

In this case, the weight \( \alpha \) depends on the norm of \( T \). In view of Lemma 2.5, every quasinilpotent operator can be considered as a part of some quasinilpotent backward weighted shift with an infinite multiplicity. On the other hand, in [2], I. Chalendar and J. Partington have shown that if \( W \) is the bilateral weighted shift with weights \( \{\alpha_n\} \) such that \( \lim_{n \to -\infty} \alpha_n = 0 \), then \( [W^n]_\alpha \) converges in norm for any \( x_0 \) and \( \varepsilon \), where the \( y_n = y_n(x_0, \varepsilon) \) are extremal vectors for \( W^n \). The operator \( K_\alpha \) also enjoys the same property.

**Lemma 2.6.** For any \( x_0 \) and \( \varepsilon \), let \( y_n = y_n(x_0, \varepsilon) \) be extremal vectors for \( K_\alpha^n \). Then \( \{K_\alpha^n y_n\} \) converges in norm.

**Proof.** This follows from a slight change of the proof of [2, Theorem 3.1]. □

If the answer of Question 2.4 is affirmative for \( K_\alpha \) in the model theory of Foias and Pearcy, the hyperinvariant subspace problem for quasinilpotent operators is solved completely. To see this, let \( K_\alpha \) and \( M \) be associated with a quasinilpotent quasiaffinity \( T \) as in Lemma 2.5. Then by Lemma 2.6, the operator \( K_\alpha \) is strongly quasi-stable regardless of the choice of \( x_0 \) and \( \varepsilon \). Moreover, in [9, Theorem 3.12] it was shown that every strongly quasi-stable quasinilpotent operator has a nontrivial hyperinvariant subspace. So, if we can find \( x_0 \in M \) such that \( K_\alpha|_M \) is strongly quasi-stable, then \( K_\alpha|_M \) has a nontrivial hyperinvariant subspace, and so does \( T \) since similarity preserves the existence of hyperinvariant subspaces.

Now, let us move on auxiliary lemmas to need in the next section. Suppose \( T \) is a quasinilpotent quasiaffinity. Write \( K = K_\alpha \) as in Lemma 2.5 such that \( K_\alpha \cong \left( \begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix} \right) 
\)

where \( A \) is similar to \( T \). Write \( w_k := \frac{1}{\alpha_1\alpha_2 \cdots \alpha_k} \) \((k \in \mathbb{N})\). Then by the proof of Lemma 2.5, the subspace \( M \) is of the form

\[
M = \vee \{b, w_1Tb, w_2T^2b, \ldots, w_nT^nb, \ldots) : b \in H\} \subset \bigoplus H.
\]

And the operator \( S : H \to \bigoplus H \) written by \( Sb := (b, w_1Tb, w_2T^2b, \ldots, w_nT^nb, \ldots) \) is invertible and hence has closed range. So we can write \( M = \text{ran} S \) by (5). Since \( S \) is bounded, there exists \( d > 0 \) such that for \( b \in H \),

\[
\|b\| \leq \|Sb\| \leq d\|b\|.
\]

Also by a straightforward calculation, we have

\[
K_Sb = STb \quad \implies \quad K^nSb = ST^nb, \quad n \in \mathbb{N}.
\]

Choose a unit vector \( x_0 \in \ker K \) and \( 0 < \varepsilon < 1 \). Then \( x_0 \) is the element in the first summand of \( \bigoplus H \) and \( K^nK^*x_0 = \|K^n\|^2x_0 \). Let \( y_n = y_n(x_0, \varepsilon) \) be extremal vectors for \( K^n \) and write \( y_n := (y_n)_{M^\perp} \). By a direct calculation, we can show that

\[
y_n = (0, \ldots, 0, (1 - \varepsilon)w_nx_0, 0, \ldots) \in \bigoplus H,
\]

where the vector \( (1 - \varepsilon)w_nx_0 \) is in the \((n+1)\)th summand of \( \bigoplus H \), and hence \( K^n y_n = (1 - \varepsilon)x_0 \). Moreover, we have

\[
\lim_{n \to \infty} \frac{\|y_n\|}{\|y_{n+1}\|} = 0
\]

since \( \|y_n\| = \alpha_{n+1} \|y_{n+1}\| \) and \( \{\alpha_n\} \) converges to zero. By the orthogonality equation (1), we have

\[
K^*n(x_0 - K^n y_n) = \delta_n y_n \quad (\delta_n > 0).
\]

In this case, since \( \langle K^*n(x_0 - K^n y_n), y_n \rangle = \delta_n \|y_n\|^2 \), we have

\[
\delta_n = \frac{\varepsilon (1 - \varepsilon)}{\|y_n\|^2} = \frac{\varepsilon}{(1 - \varepsilon)w_n^2}.
\]

We now have:
Lemma 2.7. $x_0 \notin M^\perp$.

Proof. Suppose that $x_0 \in M^\perp$. Then $\langle x_0, x \rangle = 0$ for $x = Sx_0 \in M$. But it contradicts the fact that $\langle x_0, x \rangle = \|x_0\|^2 = 1$. ⊓⊔

Lemma 2.8. $x_n \neq 0$ for all $n \geq 1$.

Proof. Suppose that $x_n = 0$ for some fixed $n$. Let $P_M$ be the orthogonal projection onto $M$ and write $x_1 := P_Mx_0$. Then by Lemma 2.7 $x_1$ is nonzero. Write $K^\perp := (K^\perp b_n)^T_{M^\perp}$. Then by the orthogonality equation (1), we have

$$K^\perp(x_0 - K^\perp y_n) = \begin{pmatrix} A^\perp(x_1 - B_nv_n) \\ 0 \end{pmatrix} = \delta_n \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

But since $A = K|_M$ is quasiaffinity, it follows that $B_nv_n = x_1$. On the other hand, by the fact $K^\perp y_n = (1 - \epsilon)x_0$, we have $B_nv_n = (1 - \epsilon)x_1$ and hence $x_1 = 0$, which is a contradiction. ⊓⊔

Lemma 2.9. $\frac{\|x_n\|}{\|y_n\|} \to 0$ as $n \to \infty$.

Proof. Write $x_n := Sb_n \in M$. Since $\|x_n\|^2 = \langle x_n, y_n \rangle$, it follows that

$$\frac{\|x_n\|}{\|y_n\|} = \left( \frac{x_n}{\|x_n\|} \right) \frac{y_n}{\|y_n\|} \leq \frac{1}{\|x_n\|} \|w_n T^n b_n\|.$$ 

Observe that by (6) we have $\|x_n\| > \|b_n\|$. Moreover, we can notice that $\|w_n T^n\| \leq \|T^n\|^2$ in the proof of Lemma 2.5. Therefore

$$\frac{\|x_n\|}{\|y_n\|} < \frac{1}{\|b_n\|} \|T^n\|^2 \|b_n\| = \|T^n\|^2,$$

which converges to zero since $T$ is a quasinilpotent operator. ⊓⊔

3. The main results

We now begin with a notion of $c$-eigenvector.

Definition 3.1. Let $T \in \mathcal{L}(H)$ be a positive operator. A unit vector $x_0$ is said to be a $c$-eigenvector of $T$ if $x_0$ satisfies that

$$\langle T x_0, x_0 \rangle \geq c \|T x_0\|.$$ 

Evidently, $c \leq 1$ and also if $d < c$, then every $c$-eigenvector of $T$ is a $d$-eigenvector of $T$. Moreover, for fixed $0 < c < 1$, every positive operator $T$ always has a $c$-eigenvector. To see this, let $T = A^* A$ for some $A \in \mathcal{L}(H)$. Write $\epsilon := (1 - c)\|A\| > 0$ and choose a unit vector $x_0$ which satisfies $\|A x_0\| > \|A\| - \epsilon$. Then

$$\langle T x_0, x_0 \rangle = \|A x_0\|^2 > c \|A\| \|A x_0\| \geq c \|A^* A x_0\| = c \|T x_0\|,$$

so that $x_0$ is a $c$-eigenvector of $T$. Moreover, this shows that the $c$-eigenvector of $T$ need not be unique. In particular, if $T$ is a one-one positive operator, then a unit vector $x_0$ is an eigenvector of $T$ if and only if $x_0$ is a 1-eigenvector of $T$. Indeed, if $T x_0 = \lambda x_0$ for some $\lambda > 0$, then $\langle T x_0, x_0 \rangle = \lambda = \|T x_0\|$. Conversely if $\langle T x_0, x_0 \rangle \geq \|T x_0\|$ then $T x_0 = \|T x_0\| x_0$. Therefore $x_0$ can be considered some generalized eigenvector corresponding to the eigenvalue $\|T x_0\|$.

Suppose that $T$ is a quasinilpotent quasiaffinity on $H$. Theorem 1.1 which is one of the main results in [6] states that if there exists a finite dimensional subspace $N$ of $H$ satisfying $T^n T^* N \subseteq N$ for all $n$, then $T$ has a nontrivial hyperinvariant subspace. In this case, for each $n \in \mathbb{N}$, choose a $c$-eigenvector $x_n$ of $T^n T^* n$. Then $\dim \{x_n\} < \infty$ because $N$ is finite dimensional, and hence the set $\mathcal{C}(x_n; n \in \mathbb{N})$ is compact. In this sense, our main theorem is a generalization of Theorem 1.1.

Theorem 3.2. Let $T$ be a quasinilpotent quasiaffinity on $H$ and let $x_0$ be a $c$-eigenvector of $T^n T^* n$ with $c > 0$ for each $n$. If the set $\mathcal{C}(x_n; n \in \mathbb{N})$ is compact then $T$ has a nontrivial hyperinvariant subspace.

Proof. Let $b_n = c_0 T^n x_n$, $(c_0 > 0)$ such that $\|T^n b_n\| = 1$. Then by assumption, we have $c \leq (T^n b_n, x_n) \leq 1$. We now claim that there exists a subsequence $\{n_k\}$ such that

$$\lim_{k \to \infty} \frac{\|b_{n_k}\|}{\|b_{n_k+1}\|} = 0. \quad (11)$$ 

Assume to the contrary that there exist $t > 0$ and $N \in \mathbb{N}$ such that

$$\inf_{n \geq N} \frac{\|b_n\|}{\|b_{n+1}\|} = t.$$
Then $\|b_N\| \geq t^{n-N} \|b_n\|$ for all $n \geq N$. On the other hand, we have

$$\frac{\langle T^N b_n, x_N \rangle}{\langle T^N b_n, x_N \rangle} = \frac{\|b_n\| \|T^n x_n\|}{\|b_n\| \|T^n x_N\|} \geq c.$$  

Thus

$$\|T^n\| = \|T^n\| \geq \|T^n x_n\| \geq ct^{n-N} \|T^n x_N\|$$

and hence $\|T^n\|^\frac{1}{n} \geq (ct^{n-N} \|T^n x_N\|)^{\frac{1}{n}} \rightarrow t > 0$ as $n \rightarrow \infty$, which contradicts the fact that $T$ is a quasinilpotent operator. Now, let $K$ be an backward weighed shift mentioned in Lemma 2.5 such that $T$ is similar to $T_M$, where $M$ is a subspace of $\bigoplus H$. Write $z_n := Sb_n \in M$ and $\tilde{x}_n := (x_0, 0, 0, \ldots) \in \bigoplus H$. Then by (7), $K^n z_n = ST^n b_n$, and hence

$$\{K^n z_n, \tilde{x}_n\} = \{T^n b_n, x_n\} \geq c.$$  

(12)

Since $\|T^n b_n\| = 1$, we obtain, by (6),

$$1 \leq \|K^n z_n\| \leq d. \tag{13}$$

Choose a subsequence $(n_j)$ satisfying Eq. (11) such that $(K^n z_{n_j})$ converges weakly to $z_0$. Since the set $cl\{x_n: n \in \mathbb{N}\}$ is compact and each $x_n$ has norm of 1, there exists a subsequence $(n_k)$ of $(n_j)$ such that $(x_{n_k})$ converges in norm to a nonzero vector $x_0$ which is an element in $ker K$. Therefore $(K^n z_{n_k}, \tilde{x}_{n_k}) \rightarrow (z_0, x_0)$, and hence $(z_0, x_0) \geq c$ by (12) so that $z_0$ is nonzero. Let $s_k := K^{n_k+1} z_{n_k} \in M$ and $t_k := \tilde{x}_{n_k+1} - K^{n_k+1} y_{n_k+1}$, where $y_n = y_n(\tilde{x}_n, \varepsilon)$ are extremal vectors for $K^n$. We now claim that

$$\langle \tilde{X}_k, t_k \rangle \rightarrow 0 \quad \text{for all } X \in \{A\}'.$$  

(14)

where $\tilde{X}$ is an extension of $X$ such that $\tilde{X} := \{X^0 0\} \subset M^1$. Suppose that $X$ is a contraction. Indeed, for $X \in \{A\}'$, choose $b_n' \in H$ such that $Xz_n = Sb_n' \in M = ran S$. Then since $\|X\| \leq 1$ and $z_n = Sb_n$, it follows from (6) that $\|b_n'\| \leq d\|b_n\|$, and hence

$$\lim_{k \rightarrow \infty} \frac{\|b_{n_k+1}\|}{\|b_{n_k+1}\|} = 0. \tag{15}$$

Let

$$b_{n_k} := \beta_k b_{n_k+1} + \omega_k, \quad \text{where } \omega_k \perp b_{n_k+1}.$$  

Then

$$\|b_{n_k}^2\| \geq |\beta_k|^2 \|b_{n_k+1}\|^2 + \|\omega_k\|^2,$$

which gives

$$|\beta_k| \leq \frac{\|b_{n_k}^2\|}{\|b_{n_k+1}\|^2} \rightarrow 0 \tag{16}$$

by (15). Note that

$$Xz_{n_k} = Sb_{n_k}' = \beta_k \tilde{x}_{n_k+1} + S\omega_k.$$  

Since $\tilde{X}A = A\tilde{X}$ and $K^n|_M = \tilde{A}^n|_M$ for all $n$, we have $\tilde{X}z_{n_k} = \tilde{A}^{n_k+1} \tilde{x}_{n_k} = K^{n_k+1} Xz_{n_k}$ and hence

$$\langle \tilde{X}z_{n_k}, t_k \rangle = \beta_k \{K^{n_k+1} z_{n_k+1}, t_k\} + \{K^{n_k+1} S\omega_k, t_k\}.$$  

By the orthogonality equation (9), we have $K^{n_k+1} t_k = \delta_{n_k+1} y_{n_k+1}$, and hence $(K^{n_k+1} S\omega_k, t_k) = \delta_{n_k+1} (S\omega_k, y_{n_k+1})$. But since $y_{n_k+1} = (0, \ldots, 0, w_{n_k+1}(1-\varepsilon)x_{n_k+1}, 0, \ldots)$, where the vector $w_{n_k+1}(1-\varepsilon)x_{n_k+1}$ is in the $(n_k+2)$th summand of $\bigoplus H$, we have

$$\langle S\omega_k, y_{n_k+1} \rangle = w_{n_k+1}^2 (1-\varepsilon)(T^{n_k+1} \omega_k, x_{n_k+1}) = \frac{1}{c_{n_k+1}} w_{n_k+1}^2 (1-\varepsilon)(\omega_k, b_{n_k+1}) = 0.$$  

Therefore

$$\langle \tilde{X}z_{n_k}, t_k \rangle = \beta_k \{K^{n_k+1} z_{n_k+1}, t_k\}.$$  

Observe that $K^{n_k+1} y_{n_k+1} = (1-\varepsilon)\tilde{x}_{n_k+1}$, thus $t_k = \varepsilon \tilde{x}_{n_k+1}$. Since $\|K^{n_k+1} z_{n_k+1}\| \leq d$ by (13) and $\|t_k\| = \varepsilon$, it follows from (16) that

$$\langle \tilde{X}z_{n_k}, t_k \rangle \leq d \varepsilon |\beta_k| \rightarrow 0 \quad \text{for all } X \in \{A\}'.$$
Theorem 3.2 is not only a generalized version of Theorem 1.1, but more useful than Theorem 1.1. Since, for each $x \in A'$, let $\tilde{X}x_0 = \tilde{A}^*x_0$. Then by (14) we have
\[
\langle \tilde{X}x_0, \tilde{A}^*x_0 \rangle = 0 \quad \text{for all } x \in A'.
\]
Write $x_1 := P_Mx_0$, where $P_M$ is the orthogonal projection onto $M$. Then $x_1$ is nonzero by Lemma 2.7 because $x_0 \in \ker K$. Since $\tilde{X}x_0 \in M$ and $P_M\tilde{A}^*x_0 = \tilde{A}^*x_1$, it follows that
\[
\langle \tilde{X}x_0, \tilde{A}^*x_1 \rangle = 0 \quad \text{for all } x \in A'.
\]
Observe that $\tilde{A}^*$ is one–one and hence $\tilde{A}^*x_1$ is nonzero. Since $z_0$ is a nonzero vector, we can say that $N \equiv \cl[A]'z_0$ is a nontrivial hyperinvariant subspace for $A$. Since $T$ is similar to $A$, $T$ has a nontrivial hyperinvariant subspace. □

Corollary 3.3. Let $T$ be a quasinilpotent quasiaffinity on $H$. If there exist $x_0 \in H$ and $c > 0$ such that $\langle T^nT^*x_0, x_0 \rangle \geq c\|T^nT^*x_0\|$ for all $n$, then $T$ has a nontrivial hyperinvariant subspace.

Proof. Put $x_0 := x_0$ (all $n$) in Theorem 3.2. □

In the assumption of Theorem 3.2, the closed subspace $\vee\{x_0\}$ need not be finite dimensional. Indeed, if $x_n := \sqrt[2]{1 - \frac{1}{n^2}}e_1 + \frac{1}{n}e_n$, where $\{e_n\}$ is an orthonormal basis of $H$, then clearly $\cl[x_n : n \in \mathbb{N}]$ is compact but $\dim\vee\{x_0\} = \infty$. Theorem 3.2 is not only a generalized version of Theorem 1.1, but more useful than Theorem 1.1. Since, for each $n \in \mathbb{N}$, a $c$-eigenvector of $T^nT^*$ is not unique, it is much easier to construct $\cl[x_n : n \in \mathbb{N}]$ and check the compactness of it, than to find a finite dimensional common reducing subspace for $T^nT^*$. Moreover, the following example shows that there exists a gap between Theorem 3.2 and Theorem 1.1.

Example 3.4. Suppose $H = \mathbb{C} \oplus \mathbb{C}$. Let, on $H$, $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let
\[
\alpha_k = \begin{cases} 2^{-k} & \text{if } k \geq 0, \\ 3^k & \text{if } k < 0. \end{cases}
\]
Write $H := \bigoplus H_i$, where $H_i = H$. Define $T \in \mathcal{L}(H)$ by
\[
T := \begin{pmatrix} \cdots & B & \alpha_{-1}A \\
& [B] & \alpha_0A \\
& & [B] & \alpha_1A \\
& & & [B] & \alpha_2A \\
& & & & \ddots \end{pmatrix}.
\]
Since $B^2 = 0$, $AB + BA = 0$ and $A^2 = I$, it follows that
\[
T^2 := \begin{pmatrix} \cdots & 0 & 0 & \alpha_{-1}\alpha_0I \\
& 0 & 0 & \alpha_0\alpha_1I \\
& 0 & 0 & \alpha_1\alpha_2I \\
& & \ddots & \ddots \end{pmatrix}.
\]
Therefore $T^2$ is a quasinilpotent quasiaffinity, and so is $T$ by the spectral mapping theorem. Write $w_{0,k} := 1$ and $w_{n,k} := \alpha_k^2\alpha_{k+1}^2 \cdots \alpha_k^2$ for $n \geq 1$ and $k \in \mathbb{Z}$. Then by a straightforward calculation, $T^nT^*$ is of the form, if $n$ is even,
\[
\begin{pmatrix} \cdots & w_{n-1,0}I \\
& [w_{n,0}I] & c_{n,1}I \\
& & [w_{n,1}I] & c_{n,2}I \\
& & & \ddots \end{pmatrix}.
\]
(17)
and, if \( n \) is odd,

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\alpha_{-1} w_{n-1, 0} AB^* \\
[ w_{n-1, 0} B B^* + w_{n, 0} AA^* ] \\
\alpha_0 w_{n-1, 1} AB^* \\
\alpha_0 w_{n-1, 1} BA^* \\
w_{n-1, 1} B B^* + w_{n, 1} AA^* \\
\alpha_1 w_{n-1, 2} BA^* \\
\alpha_1 w_{n-1, 2} B A^* \\
\vdots \\
\vdots
\end{pmatrix}
\]

Choose a unit vector \( x_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H_0 \). Then by some elementary calculation, we have

\[
\begin{pmatrix} T^n T^n x_0 \\ \| T^n T^n x_0 \| \end{pmatrix} x_0 = \begin{cases} 1 & \text{if } n \text{ is even,} \\
\frac{1}{\sqrt{3}} & \text{if } n \text{ is odd.}
\end{cases}
\]

and hence \( x_0 \) is a \( \frac{1}{\sqrt{3}} \)-eigenvector of \( \{ T^n T^n \} \), so that \( T \) satisfies the assumption of Corollary 3.3. On the other hand, suppose \( N \) is a finite dimensional subspace of \( H \) which satisfies \( T^n T^n N \subseteq N \) for each \( n \). By (17), if \( n = 2 \), we have \( T^2 T^{-2} \) is a block diagonal operator whose block entries are \( w_{2, k} \). Let \( \eta A \) be the “polynomially convex hull” of a set \( A \). Since \( \alpha_{k} \neq \alpha_{k+1} \) if \( m \neq n \) and \( \alpha_{n+1} \alpha_{n+1} \) is positive for each \( n \), it follows that \( \eta \sigma (w_{2, m}) \cap \eta \sigma (w_{2, n}) = \emptyset \). Hence, by [4, Theorem 1] the finite dimensional space \( N \) is a finite direct sum of \( N_i \) which is subspace of \( H_i \), that is, \( N \) is of the form

\[
N = \bigoplus_{i \in \mathbb{J}} N_i,
\]

where \( N_i \subseteq H_i \) and \( \mathbb{J} \) is an index set in \( \mathbb{Z} \). Now let \( k \) be the smallest number in \( \mathbb{J} \). Since for any nonzero vector \( x \in N_k \), \( T T x \in \mathbb{N}_{k-1} \), it follows \( x \in \ker A B^* \) from the matrix (18). Thus \( x \) is of the form \( x = c(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in H_k \) for some \( c \in \mathbb{C} \), and hence \( N_k \) is one-dimensional subspace of \( H_k \). However, for \( x := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H_k \), we have \( T T x = \alpha_k^2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \alpha_k \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in N_k \oplus N_{k+1} \) by a direct calculation, so that \( \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in N_k \). Therefore \( N_k \) is a two-dimensional space, which is a contradiction. So the operators in the family \( \{ T^n T^n \} \) have no finite dimensional common reducing subspace in \( H \). Therefore the operator \( T \) does not satisfy the assumption of Theorem 1.1.

Let us now find a gap in another case. If the operators in the family \( \{ T^n T^n \} \) have a common eigenvector \( x_0 \), then \( x_0 \) satisfies the assumption of Corollary 3.3. The following example says that the converse is not true. Moreover, it is interesting that the example also shows that the existence of a common eigenvector of \( A^n A^n \) does not imply the assumption of Theorem 1.1.

**Example 3.5.** Suppose \( H = \mathbb{C} \oplus \mathbb{C} \). Let, on \( H \), \( A = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \alpha_k = 2^{-|k|} \) (\( k \in \mathbb{Z} \)). Define \( T \in \mathcal{L}(\oplus H) \) by

\[
T := \begin{pmatrix}
\vdots \\
\vdots \\
\alpha_{-1} A \\
\alpha_{0} A \\
0 \\
\alpha_{1} A \\
\vdots \\
\vdots
\end{pmatrix}
\]

Then \( T \) is a quasinilpotent quasiaffinity since \( A \) is invertible. Observe that the \( T^n T^n \) are block diagonal operators, whose block diagonal entries are of the form \( w_{n, k} A^n A^n \) (\( k \in \mathbb{Z} \)), where \( w_{n, k} \) is as in Example 3.4. Then a two-dimensional subspace \( H \) reduces \( T^n T^n \) for each \( n \), so that \( T \) satisfies the assumption of Theorem 1.1. Observe that \( A^n A^n = \frac{1}{n+1}(\begin{pmatrix} n+1 \\ -n \end{pmatrix}) \). Choose a unit vector \( x_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) in an arbitrary subspace \( H \) of \( \oplus H \). Then by a straightforward calculation, it follows that

\[
\begin{pmatrix} T^n T^n x_0 \\ \| T^n T^n x_0 \| \end{pmatrix} x_0 = \frac{n^2 + 1}{\sqrt{(n^2 + 1)^2 + n^2}} \geq \frac{1}{\sqrt{2}}
\]

which satisfies the assumption of Corollary 3.3. However, the operators in the family \( \{ T^n T^n \} \) have no common eigenvector. Indeed, if the operators in the family \( \{ T^n T^n \} \) have a common eigenvector, then so do \( A^n A^n \). Let \( x \in H \) be a common eigenvector of \( A^n A^n \). Then \( x \) is either \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), because \( x \) must be an eigenvector of

\[
AA^* - 2A^2 A^2 = \frac{1}{8} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.
\]

But any such \( x \) cannot be an eigenvector of \( AA^* \), which leads to a contradiction.
We conclude with a theorem related to the model theory introduced in Section 2.

**Theorem 3.6.** Suppose $T$ is a quasinilpotent quasi-affinity on $H$ and $K \equiv K_\alpha$ is a shift operator in Lemma 2.5 such that $T$ is similar to $A := K_M$. Choose a unit vector $x_0 \in \ker K$ and $0 < \varepsilon < 1$. Let $y_n = y_n(x_0, \varepsilon)$ be extremal vectors for $K^n$ and write $x_n := P_M y_n$. If there exists $c > 0$ such that $(K^n x_n, x_0) \geq c \|K^n x_n\|$ for all $n$, then $T$ has a nontrivial hyperinvariant subspace.

**Proof.** Since $A = K_M$ is a quasi-affinity, by Lemma 2.8 we can define $z_n := c_n x_0$ satisfying $\|K^n z_n\| = 1$. Note that by (9) it follows $K^n x_0 = \frac{1}{\varepsilon} y_n$ because $K^n y_n = (1 - \varepsilon) x_0$. Thus by the assumption, we have

$$c \leq \langle K^n z_n, x_0 \rangle = \frac{\delta_n}{\varepsilon} \|z_n\| \|x_0\| < 1. \quad (19)$$

Write $a_n := \frac{\|z_n\|}{\|y_n\|}$. By Lemma 2.9, there exists a subsequence $(n_k)$ of $(n)$ such that $a_{n_k} \geq a_{n_k} + 1$ for all $k$. Since $\delta_n = \frac{\varepsilon(1-\varepsilon)}{\|y_n\|^2}$ for all $n \geq 1$ by (10), it follows from (19) that

$$\frac{\langle K^n z_n, x_0 \rangle}{\|K^n+1 z_{n+1}, x_0\|} = \frac{\|z_n\| \|x_0\| \|y_{n+1}\|^2}{\|z_{n+1}\| \|x_{n+1}\| \|y_n\|^2} \leq \frac{1}{c}. $$

And hence

$$\|z_n\| < \frac{1}{c} \|y_{n+1}\| \|x_{n+1}\|. $$

Since $\frac{a_{n_k}+1}{a_{n_k}}$ is bounded, by (8) we have

$$\lim_{k \to \infty} \|z_{n_k}\| = 0. \quad (20)$$

Since $\langle K^n z_n, x_0 \rangle \geq c$ and $\|K^n z_0\| = 1$, there exists a subsequence $(n_k)$ such that $\{K^{n_k+1} z_{n_k}\}$ converges weakly to a nonzero vector $z_0$. Choose such a subsequence $(n_k)$ of $(n)$ which satisfies Eq. (20) at once. Let $s_k := K^{n_k+1} z_{n_k} \in M$ and $t_k := x_0 - K^{n_k+1} y_{n_k+1}$. If we can show that

$$\langle X s_k, t_k \rangle \to 0 \quad \text{for all } X \in \{A\}', \quad (21)$$

where $X$ is an extension of $X$ in the proof of Theorem 3.2, then the proof is complete by the same argument of the proof of Theorem 3.2. Suppose that $X$ is a contraction. For $X \in \{A\}'$, let

$$X z_{n_k} := \beta_k z_{n_k+1} + \omega_k, \quad \text{where } \omega_k \perp x_{n_k+1}. $$

Then, as a similar manner in the proof of Theorem 3.2, by (20), we have $|\beta_k| \to 0$. Moreover, by (9) we have $\langle K^{n_k+1} \omega_k, t_k \rangle = \delta_{n_k+1} \langle \omega_k, x_{n_k+1} \rangle = 0$, and consequently,

$$\langle X s_k, t_k \rangle = \beta_k (K^{n_k+1} z_{n_k+1}, t_k). $$

Since $\|K^{n_k+1} z_{n_k+1}\| = 1$ and $\|t_k\| = \varepsilon$, it follows that

$$|\langle X s_k, t_k \rangle| \leq \varepsilon |\beta_k| \to 0 \quad \text{for all } X \in \{A\}', \quad (21)$$

which proves (21). Therefore $T$ has a nontrivial hyperinvariant subspace. $\square$

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**References**


