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Representations for the Drazin inverse of the sum $P + Q + R + S$ and its applications ☆

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Abstract

Let P, Q, R and S be complex square matrices and $M = P + Q + R + S$. A quadruple (P, Q, R, S) is called a pseudo-block decomposition of M if

$$PQ = QP = \mathbf{0}, \quad PS = SQ = QR = RP = \mathbf{0} \quad \text{and} \quad R^{\mathrm{D}} = S^{\mathrm{D}} = \mathbf{0},$$

where R^{D} and S^{D} are the Drazin inverses of R and S, respectively. We investigate the problem of finding formulae for the Drazin inverse of M. The explicit representations for the Drazin inverses of M and $P + Q + R$ are developed, under some assumptions. As its application, some representations are presented for 2×2 block matrices $\begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$ and $\begin{bmatrix} A & B \\ D & C \end{bmatrix}$, where the blocks A and C are square matrices. Several results of this paper extend the well known representation for the Drazin inverse of $\begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$ given by Hartwig and Shoaf, Meyer and Rose in 1977. An illustrative example is given to verify our new representations.

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Keywords: Drazin inverse; Group inverse; Index; Pseudo-block decomposition

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1. Introduction

Let $\mathbb{C}^{m\times m}$ denote the set of $m \times m$ complex matrices and $A \in \mathbb{C}^{m\times m}$. The smallest non-negative integer k such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$, denoted by $\operatorname{Ind}(A)$, is called the index of A and $\operatorname{Ind}(A) = k$. There is a unique matrix $A^{\mathrm{D}} \in \mathbb{C}^{m\times m}$ satisfying the matrix equations [5, p. 122]

$$A^{k+1}A^{\mathrm{D}} = A^k, \quad A^{\mathrm{D}}AA^{\mathrm{D}} = A^{\mathrm{D}}, \quad AA^{\mathrm{D}} = A^{\mathrm{D}}A. \tag{1.1}$$

A^{D} is called the Drazin inverse of A. If $\operatorname{Ind}(A) = 1$, then A^{D} is called the group inverse of A and denoted by $A^{\#}$. Clearly, $\operatorname{Ind}(A) = 0$ if and only if A is nonsingular, and in this case $A^{\mathrm{D}} = A^{-1}$.

The theory of Drazin inverses has been a substantial growth over the past decades. It is of great theoretical interest as well as the applications in many diverse areas, including statistics, numerical analysis, differential equations, Markov chains, population models, cryptography, and control theory, etc. (see [5,12,14,15,17,21,22,24,27,28,30–32,37–39]). One topic of Drazin inverse of considerable interest concerns the explicit representations for the Drazin inverse of a 2×2 block matrix [2,3,6,8–11,13,18,19,23,29,34,36] and explicit representations for the Drazin inverse of the sum of two matrices (cf. [7,20,25,26,33]).

Campbell and Meyer [5] posed the research problem in 1983: find an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{bmatrix} A & B \\ D & C \end{bmatrix}$ in terms of the blocks of the partition, where the blocks A and C are assumed to be square matrices. The motivation for this open problem is the desire to find the general expressions for the solutions of the second-order system of the differential equations [4]

$$Ex''(t) + Fx'(t) + Gx(t) = \mathbf{0},$$

where the matrix E is singular.

Lots of literature devoted to this topic has sprouted [6–8,13,19,20,29,33–35]. If one takes a close look at these results, it is easy to see that a representation for the Drazin inverse of $\begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$, which was first given by Hartwig and Shoaf, Meyer and Rose in 1977 independently, employed very frequently (abbreviated as Hartwig–Shoaf–Meyer–Rose formula, in [5, Theorem 7.7.1] or [16,29]). We can see that the Hartwig–Shoaf–Meyer–Rose formula is an effective and basic tool for finding various explicit representations for the Drazin inverse of block matrix and modified matrix [35]. It would be desirable and important to extend the Hartwig–Shoaf–Meyer–Rose formula to the more general cases. This can be done in several ways. One direct and plain way is to extend Hartwig–Shoaf–Meyer–Rose formula to the case of $\begin{bmatrix} A & B \\ D & C \end{bmatrix}$, where $D \neq \mathbf{0}$. But we adopt a different approach in this paper.

Our approach is based on the following partition. For a 2×2 block matrix $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$, where A and C are square matrices and their sizes need not be the same. It can be rewritten as a block decomposition of $M = P + Q + R + S$, where

$$P = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad Q = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}, \quad R = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ D & \mathbf{0} \end{bmatrix}.$$

In this case, one can see that P, Q, R and S satisfy the following relations:

$$PQ = QP = \mathbf{0}, \quad PS = SQ = QR = RP = \mathbf{0}, \quad \text{and} \quad R^{\mathrm{D}} = S^{\mathrm{D}} = \mathbf{0}. \tag{1.2}$$

In general, we introduce the following definition.

Definition 1.1. Let M be a complex square matrix. If there exist four complex square matrices P, Q, R and S satisfying (1.2) and $M = P + Q + R + S$, then the quadruple (P, Q, R, S) is called a pseudo-block decomposition of M. Moreover, M is called a 2×2 pseudo-block matrix corresponding to (P, Q, R, S), or simply a pseudo-block matrix. i.e., M has a pseudo-block decomposition (P, Q, R, S).

It is obvious that a 2×2 matrix $\begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$ has a pseudo-block decomposition $(P_0, Q_0, R_0, \mathbf{0})$, where $P_0 = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $Q_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}$, $R_0 = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and

$$R_0^2 = \mathbf{0} \quad \text{and} \quad \operatorname{rank}(P_0 + Q_0) = \operatorname{rank}(P_0) + \operatorname{rank}(Q_0),$$

which is a special kind of a pseudo-block decomposition of M.

One main aim of our paper is to find the explicit formulas of $(P + Q + R)^{\mathrm{D}}$, where $(P, Q, R, \mathbf{0})$ is a pseudo-block decomposition of matrix $P + Q + R$. The formulae of $(P + Q + R)^{\mathrm{D}}$ given by (3.11) and (3.12), extend Hartwig–Shoaf–Meyer–Rose formula in three aspects. Firstly, the condition

$$\operatorname{rank}(P + Q) = \operatorname{rank}(P) + \operatorname{rank}(Q)$$

is removed. Secondly, the condition $\operatorname{Ind}(P + Q + R) \geqslant \max\{\operatorname{Ind}(P), \operatorname{Ind}(Q)\}$ is also relaxed. Thirdly, the condition $R^2 = \mathbf{0}$ is replaced by $R^{\mathrm{D}} = \mathbf{0}$, i.e., R is nilpotent. It is not difficult to find the formula of $(P + Q + R + S)^{\mathrm{D}}$ under some restrictions, where (P, Q, R, S) is a pseudo-block decomposition of the matrix $P + Q + R + S$. These results also generalize Hartwig–Shoaf–Meyer–Rose formula to the case $S \neq \mathbf{0}$.

The paper is organized as follows. In Section 2, we will introduce preliminary. Section 3 is devoted to the study of explicit representations of the group inverse and Drazin inverse for the matrix $P + Q + R$, where $(P, Q, R, \mathbf{0})$ is a pseudo-block decomposition of $P + Q + R$. We find the necessary and sufficient condition for the existence of group inverse of the pseudo-block matrix $P + Q + R$ and present the explicit representation of $(P + Q + R)^{\#}$ (if it exists) under the condition $\operatorname{Ind}(P) = \operatorname{Ind}(Q) = 1$. Based on it, the explicit representation of Drazin inverse for the pseudo-block matrix $P + Q + R$ is given, which is one of our main formulae. Our another main results are included in Section 4. There are some explicit representations of $(P + Q + R + S)^{\mathrm{D}}$ under some conditions, where (P, Q, R, S) is a pseudo-block decomposition of the matrix $P + Q + R + S$. Some 2×2 block matrix version results are also given in this section, which extend Hartwig–Shoaf–Meyer–Rose formula and a key lemma of [19]. Finally, in Section 5, we present an illustrative example to show the concept of pseudo-block decomposition of a matrix is useful to find the Drazin inverse of a square matrix.

Throughout this paper we denote zero matrix by $\mathbf{0}$, the identity matrix by I. If the lower limit of a sum is bigger than its upper limit, we always define the sum to be $\mathbf{0}$. We denote the transpose and conjugate transpose of a matrix A by A^{T} and A^*, respectively. We also define $G^0 = I$ for any square matrix G hereinafter.

2. Preliminary

In this section, we recall the following properties of Drazin inverse (cf. [1,5,32]), which will be used later on.

Let A be complex square matrix and let l and m be integers, then

$$A^{\mathrm{D}} = \mathbf{0}, \text{ if and only if } A \text{ is nilpotent.} \tag{2.1}$$

$$\text{If } l > m > 0, \text{ then } A^m (A^{\mathrm{D}})^l = (A^{\mathrm{D}})^{l-m}. \tag{2.2}$$

$$\text{If } l > 0, \text{ then } (A^l)^{\mathrm{D}} = (A^{\mathrm{D}})^l,\ A^l (A^l)^{\mathrm{D}} = AA^{\mathrm{D}}. \tag{2.3}$$

$$\text{If } l - \max\{\mathrm{Ind}(A), 1\} \geqslant m > 0, \text{ then } A^l (A^{\mathrm{D}})^m = A^{l-m}. \tag{2.4}$$

$$(A^{\mathrm{T}})^{\mathrm{D}} = (A^{\mathrm{D}})^{\mathrm{T}}. \tag{2.5}$$

An additive result for the sum of two matrices which gives a representation for $(P+Q)^{\mathrm{D}}$ under the condition $PQ = \mathbf{0}$.

Lemma 2.1 [16, Lemma 4, 20, Corollary 2.1]. *Let P, Q and R be complex matrices and $R^k = \mathbf{0}$:*

(1) *If $RP = \mathbf{0}$, then* $(R+P)^{\mathrm{D}} = P^{\mathrm{D}} + \sum_{i=1}^{k-1} (P^{\mathrm{D}})^{i+1} R^i$,

(2) *If $QR = \mathbf{0}$, then* $(Q+R)^{\mathrm{D}} = Q^{\mathrm{D}} + \sum_{i=1}^{k-1} R^i (Q^{\mathrm{D}})^{i+1}$.

3. Explicit representations of group inverse and Drazin inverse of pseudo-block matrix $P + Q + R$

In this section, we investigate the group inverse of a pseudo-block matrix $P+Q+R$ corresponding to $(P, Q, R, \mathbf{0})$ under some conditions. The results are presented in the following theorem in which we establish the representation for the Drazin inverse of the pseudo-block matrix $P+Q+R$.

Theorem 3.1. *Let P, Q and R be complex m-square matrices and $M = P+Q+R$. Let $(P, Q, R, \mathbf{0})$ be a pseudo-block decomposition of M. Suppose that* $\mathrm{Ind}(P) \leqslant 1, \mathrm{Ind}(Q) \leqslant 1$ *and* $\mathrm{Ind}(R) \geqslant 2$. *Then the following assertions hold*:

(1) $\mathrm{Ind}(M) \leqslant \mathrm{Ind}(R)$,

(2) $\mathrm{Ind}(M) \leqslant 1$ *if and only if*

$$\sum_{i=1}^{k-2} (P^{\#})^i R^{i+1} (I - QQ^{\#}) + \sum_{j=1}^{k-2} (I - PP^{\#}) R^{j+1} (Q^{\#})^j$$
$$= (I - PP^{\#}) R (I - QQ^{\#}) + \sum_{i=1}^{k-3} \sum_{j=1}^{k-i-2} (P^{\#})^i R^{i+j+1} (Q^{\#})^j. \tag{3.1}$$

Furthermore, if $M^{\#}$ exists, then it is given by

$$M^{\#} = \sum_{i=1}^{k} (P^{\#})^i R^{i-1} (I - QQ^{\#}) + \sum_{j=1}^{k} (I - PP^{\#}) R^{j-1} (Q^{\#})^j$$
$$- \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} (P^{\#})^i R^{i+j-1} (Q^{\#})^j, \tag{3.2}$$

where $k = \mathrm{Ind}(R)$.

Proof. Let the right-hand side of (3.2) be X. We will show that $X = M^{\mathrm{D}}$ by the definition of Drazin inverse. Firstly

$$MX = \left[\left(PP^{\#} + \sum_{i=1}^{k-1}(P^{\#})^i R^i (I - QQ^{\#})\right) - \left(\sum_{j=1}^{k-1} PP^{\#} R^j (Q^{\#})^j \right.\right.$$
$$\left.\left. + \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}(P^{\#})^i R^{i+j}(Q^{\#})^j\right)\right] + QQ^{\#} + \sum_{j=1}^{k-1} R^j (Q^{\#})^j$$
$$= PP^{\#} + QQ^{\#} + \sum_{i=1}^{k-1}(P^{\#})^i R^i (I - QQ^{\#}) + \sum_{j=1}^{k-1}(I - PP^{\#})R^j (Q^{\#})^j$$
$$- \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}(P^{\#})^i R^{i+j}(Q^{\#})^j. \tag{3.3}$$

Similarly

$$XM = PP^{\#} + QQ^{\#} + \sum_{i=1}^{k-1}(P^{\#})^i R^i (I - QQ^{\#}) + \sum_{j=1}^{k-1}(I - PP^{\#})R^j (Q^{\#})^j$$
$$- \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}(P^{\#})^i R^{i+j}(Q^{\#})^j, \tag{3.4}$$

which imply

$$MX = XM. \tag{3.5}$$

We now prove that $XMX = X$. Let us denote the first, second and third terms on the right-hand side of (3.2) by X_1, X_2 and X_3, respectively. Then $X = X_1 + X_2 - X_3$. Expanding XM as (3.4), we obtain that

$$XMX_2 = \left(Q^{\#} + \sum_{j=1}^{k-1}(I - PP^{\#})R^j (Q^{\#})^{j+1} - \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}(P^{\#})^i R^{i+j}(Q^{\#})^{j+1}\right)$$
$$+ \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}(P^{\#})^i R^{i+j}(Q^{\#})^{j+1} = \sum_{j=1}^{k}(I - PP^{\#})R^{j-1}(Q^{\#})^j.$$

Hence

$$XMX = \sum_{i=1}^{k}(P^{\#})^i R^{i-1}(I - QQ^{\#}) + \sum_{j=1}^{k}(I - PP^{\#})R^{j-1}(Q^{\#})^j$$
$$- \sum_{i=1}^{k-1}\sum_{j=1}^{k-i}(P^{\#})^i R^{i+j-1}(Q^{\#})^j,$$

namely

$$XMX = X. \tag{3.6}$$

Now we claim that

$$M^{k+1}X = M^k. \tag{3.7}$$

In fact, by induction on $l > 1$, one can see that

$$M^l = P^l + Q^l + R^l + \sum_{i=1}^{l-1}\sum_{j=0}^{l-i} P^j R^i Q^{l-i-j}. \tag{3.8}$$

Notice that $P(I - PP^{\#}) = \mathbf{0}$, $Q(I - QQ^{\#}) = \mathbf{0}$, and $R^k = R^{k+1}R^{\mathrm{D}} = \mathbf{0}$, combining (3.8) and (3.3), we obtain

$$\begin{aligned}
M^{k+1}X &= \left(P^k + \sum_{i=1}^{k-1} P^{k-i}R^i(I - QQ^{\#}) - \sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1} P^{k-i}R^{i+j}(Q^{\#})^j\right)\\
&\quad + Q^k + \left(\sum_{i=1}^{k-1}\sum_{j=0}^{k-i} P^j R^i Q^{k-i-j}QQ^{\#} + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} P^{k-i}R^{i+j}(Q^{\#})^j\right)\\
&= P^k + Q^k + \sum_{i=1}^{k-1}\sum_{j=0}^{k-i} P^j R^i Q^{k-i-j}\\
&= M^k
\end{aligned} \tag{3.9}$$

as claimed in (3.7). Also, we have

$$\begin{aligned}
M^2X &= \left[P + \left(PP^{\#}R(I - QQ^{\#}) + \sum_{i=1}^{k-2}(P^{\#})^i R^{i+1}(I - QQ^{\#})\right)\right.\\
&\quad \left. - \left(\sum_{j=1}^{k-2} PP^{\#}R^{j+1}(Q^{\#})^j + \sum_{i=1}^{k-3}\sum_{j=1}^{k-i-2}(P^{\#})^i R^{i+j+1}(Q^{\#})^j\right)\right] + Q\\
&\quad + \left[(R - R(I - QQ^{\#})) + \sum_{j=1}^{k-1} R^{j+1}(Q^{\#})^j\right]\\
&= (P + Q + R)\\
&\quad + \left(\sum_{i=1}^{k-2}(P^{\#})^i R^{i+1}(I - QQ^{\#}) + \sum_{j=1}^{k-2}(I - PP^{\#})R^{j+1}(Q^{\#})^j\right)\\
&\quad - \left((I - PP^{\#})R(I - QQ^{\#}) + \sum_{i=1}^{k-3}\sum_{j=1}^{k-i-2}(P^{\#})^i R^{i+j+1}(Q^{\#})^j\right).
\end{aligned} \tag{3.10}$$

If $M^{\#}$ exists, then (3.2) holds. We can use $M^{\#} = X$ and (3.10), it is not difficult to see that $\mathrm{Ind}(M) \leqslant 1$ if and only if (3.1) holds. This completes the proof. □

Remark 3.1. Theorem 3.1 is not only important for the representation of Drazin inverse, but also useful for the existence of group inverse. If we assume $\mathrm{Ind}(R) < 2$ (equivalently $R = \mathbf{0}$, if $(P, Q, R, \mathbf{0})$ is a pseudo-block decomposition of M), and keep other assumptions of Theorem 3.1, then it is still true for $\mathrm{Ind}(M) \leqslant 1$ and the formula (3.2).

Combining the above remark and Theorem 3.1, we obtain the following corollary, which will be useful.

Corollary 3.1. *Let P, Q and R be complex m-square matrices and $M = P + Q + R$. Let $(P, Q, R, \mathbf{0})$ be a pseudo-block decomposition of M. Suppose that* $\mathrm{Ind}(P) \leqslant 1$, $\mathrm{Ind}(Q) \leqslant 1$, $\mathrm{Ind}(R) \leqslant 3$, *and* $\mathrm{Ind}(M) \leqslant 1$. *Then*

$$\begin{aligned} M^{\#} = {} & P^{\#} + Q^{\#} + (P^{\#})^2 R(I - QQ^{\#}) + (P^{\#})^3 R^2 (I - QQ^{\#}) \\ & + (I - PP^{\#}) R (Q^{\#})^2 + (I - PP^{\#}) R^2 (Q^{\#})^3 - P^{\#} R Q^{\#} \\ & - (P^{\#})^2 R^2 Q^{\#} - P^{\#} R^2 (Q^{\#})^2. \end{aligned}$$

Now we present our main result: the explicit representation for the Drazin inverse of a pseudo-block matrix $P + Q + R$ corresponding to $(P, Q, R, \mathbf{0})$. This result is based on Corollary 3.1.

Theorem 3.2. *Let P, Q, R be complex m-square matrices and $M = P + Q + R$. Let $k = \mathrm{Ind}(R)$, $l_P = \mathrm{Ind}(P)$, and $l_Q = \mathrm{Ind}(Q)$. Suppose $(P, Q, R, \mathbf{0})$ is a pseudo-block decomposition of M. Then*

$$\begin{aligned} M^{\mathrm{D}} = {} & P^{\mathrm{D}} + Q^{\mathrm{D}} - \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} (P^{\mathrm{D}})^i R^{i+j-1} (Q^{\mathrm{D}})^j \\ & + \left(\sum_{i=1}^{k-1} \sum_{j=0}^{i_Q} (P^{\mathrm{D}})^{i+j+1} R^i Q^j + \sum_{i=1}^{k-2} \sum_{j=1}^{k-i-1} (P^{\mathrm{D}})^{m+i+1} R^{i+j} Q^{m-j} \right) (I - QQ^{\mathrm{D}}) \\ & + (I - PP^{\mathrm{D}}) \left(\sum_{i=1}^{k-1} \sum_{j=0}^{i_P} P^j R^i (Q^{\mathrm{D}})^{i+j+1} \right. \\ & \left. + \sum_{i=1}^{k-2} \sum_{j=1}^{k-i-1} P^{m-i} R^{i+j} (Q^{\mathrm{D}})^{m+j+1} \right), \end{aligned} \tag{3.11}$$

where $i_P = \min\{l_P - 1, m - i\}$ and $i_Q = \min\{l_Q - 1, m - i\}$. Alternatively

$$\begin{aligned} M^{\mathrm{D}} = {} & P^{\mathrm{D}} + Q^{\mathrm{D}} - \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} (P^{\mathrm{D}})^i R^{i+j-1} (Q^{\mathrm{D}})^j \\ & + \left(\sum_{i=1}^{m-1} \sum_{j=0}^{m-i} (P^{\mathrm{D}})^{i+j+1} R^i Q^j + \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (P^{\mathrm{D}})^{m+i+1} R^{i+j} Q^{m-j} \right) (I - QQ^{\mathrm{D}}) \\ & + (I - PP^{\mathrm{D}}) \left(\sum_{i=1}^{m-1} \sum_{j=0}^{m-i} P^j R^i (Q^{\mathrm{D}})^{i+j+1} + \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} P^{m-i} R^{i+j} (Q^{\mathrm{D}})^{m+j+1} \right). \end{aligned} \tag{3.12}$$

Proof. Using an additive result for the sum of two matrices included in [20] or Lemma 2.1, which gives a representation for $(A+B)^{\mathrm{D}}$ under the condition $AB=\mathbf{0}$.

The steps of the proof are the following:

$$\begin{aligned} M^{\mathrm{D}} &= [P+(R+Q)]^{\mathrm{D}} \\ &= \sum_{t=0}^{l-1}(P^{\mathrm{D}})^{t+1}(R+Q)^t\Pi + \sum_{t=0}^{l_P-1}(I-PP^{\mathrm{D}})P^t[(R+Q)^{\mathrm{D}}]^{t+1}, \end{aligned}$$

where $\Pi = I-(R+Q)(R+Q)^{\mathrm{D}}$ and $l=\mathrm{Ind}(R+Q)$.

We also have

$$(R+Q)^{\mathrm{D}} = \sum_{j=0}^{k-1}R^j(Q^{\mathrm{D}})^{j+1}, \quad \Pi = I-QQ^{\mathrm{D}}+\sum_{j=0}^{k-1}R^{j+1}(Q^{\mathrm{D}})^{j+1}.$$

Prove by induction that

$$(R+Q)^t = \sum_{i=0}^{t}R^{t-i}Q^i, \quad [(R+Q)^{\mathrm{D}}]^{t+1} = \sum_{j=0}^{k-1}R^i(Q^{\mathrm{D}})^{t+j+1}$$

and substitute the above expressions in the formula for M^{D} to obtain the desired result. □

The following corollary of Theorem 3.2 is obvious.

Corollary 3.2. *Let P, Q and R be complex m-square matrices and $M=P+Q+R$. Let $(P,Q,R,\mathbf{0})$ be a pseudo-block decomposition of M. Suppose* $\mathrm{Ind}(R)\leqslant 2$ (*i.e.*, $R^2=\mathbf{0}$), $l_P=\mathrm{Ind}(P)$ *and* $l_Q=\mathrm{Ind}(Q)$, *then*

$$\begin{aligned} (P+Q+R)^{\mathrm{D}} &= P^{\mathrm{D}}+Q^{\mathrm{D}}-P^{\mathrm{D}}RQ^{\mathrm{D}}+\left(\sum_{j=0}^{m-1}(P^{\mathrm{D}})^{j+2}RQ^j\right)(I-QQ^{\mathrm{D}}) \\ &\quad + (I-PP^{\mathrm{D}})\left(\sum_{j=0}^{m-1}P^jR(Q^{\mathrm{D}})^{j+2}\right) \\ &= P^{\mathrm{D}}+Q^{\mathrm{D}}-P^{\mathrm{D}}RQ^{\mathrm{D}}+\left(\sum_{j=0}^{l_Q-1}(P^{\mathrm{D}})^{j+2}RQ^j\right)(I-QQ^{\mathrm{D}}) \\ &\quad + (I-PP^{\mathrm{D}})\left(\sum_{j=0}^{l_P-1}P^jR(Q^{\mathrm{D}})^{j+2}\right). \end{aligned} \tag{3.13}$$

Remark 3.2. Let $P=\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, $Q=\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C\end{bmatrix}$ and $R=\begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$. If we take the block decomposition $(P,Q,R,\mathbf{0})$ of $M=P+Q+R=\begin{bmatrix} A & B \\ \mathbf{0} & C\end{bmatrix}$, then the famous Hartwig–Shoaf–Meyer–Rose formula follows immediately from Corollary 3.2, as $P^{\mathrm{D}}=\begin{bmatrix} A^{\mathrm{D}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$ and $Q^{\mathrm{D}}=\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C^{\mathrm{D}}\end{bmatrix}$, where A and C are square matrices.

Corollary 3.3 (Hartwig–Shoaf–Meyer–Rose formula [5, Theorem 7.7.1, 16, 29]). *If* $M = \begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix} \in \mathbb{C}^{m\times m}$ *(A and C are square)*, $l_A = \operatorname{Ind}(A)$ *and* $l_C = \operatorname{Ind}(C)$, *then*

$$M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} & X \\ \mathbf{0} & C^{\mathrm{D}} \end{bmatrix},$$

where

$$X = \left(\sum_{j=0}^{l_C-1} (A^{\mathrm{D}})^{j+2} B C^j\right)(I - CC^{\mathrm{D}}) + (I - AA^{\mathrm{D}})\left(\sum_{j=0}^{l_A-1} A^j B (C^{\mathrm{D}})^{j+2}\right) - A^{\mathrm{D}} B C^{\mathrm{D}}.$$

4. Explicit representations of Drazin inverse of pseudo-block matrix $P + Q + R + S$

Our next goal is to present some explicit representations for the Drazin inverse of a pseudo-block matrix $P + Q + R + S$ corresponding to (P, Q, R, S) under some assumptions.

We begin with a lemma.

Lemma 4.1. *Let* P, Q *and* R *be complex m-square matrices and* $M = P + Q + R$. *Suppose* $(P, Q, R, \mathbf{0})$ *is a pseudo-block decomposition of* M, *then*

$$\begin{aligned}
\left(M^{\mathrm{D}}\right)^l &= \left((P+Q+R)^{\mathrm{D}}\right)^l \\
&= (P^{\mathrm{D}})^l + (Q^{\mathrm{D}})^l \\
&\quad + \sum_{k=1}^{l-1}\sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1} (P^{\mathrm{D}})^{k+i} R^{i+j} (Q^{\mathrm{D}})^{l-k+j} \\
&\quad - \sum_{k=1}^{l}\sum_{i=1}^{m-1}\sum_{j=1}^{m-i} (P^{\mathrm{D}})^{k+i-1} R^{i+j-1} (Q^{\mathrm{D}})^{l-k+j} \\
&\quad + \left(\sum_{i=1}^{m-1}\sum_{j=0}^{m-i} (P^{\mathrm{D}})^{l+i+j} R^i Q^j + \sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1} (P^{\mathrm{D}})^{l+m+i} R^{i+j} Q^{m-j}\right)(I - QQ^{\mathrm{D}}) \\
&\quad + (I - PP^{\mathrm{D}})\left(\sum_{i=1}^{m-1}\sum_{j=0}^{m-i} P^j R^i (Q^{\mathrm{D}})^{l+i+j} + \sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1} P^{m-i} R^{i+j} (Q^{\mathrm{D}})^{l+m+j}\right)
\end{aligned}$$

for $l = 2, 3, \ldots$

Proof. By (3.12) in Theorem 3.2 and induction on l, the desired result follows after carefully verification. □

Theorem 4.1. *Let* P, Q, R *and* S *be complex m-square matrices and* $M = P + Q + R + S$. *Suppose* (P, Q, R, S) *is a pseudo-block decomposition of* M *and* $SP = SR = \mathbf{0}$, *then*

$$
\begin{aligned}
M^{\mathrm{D}} = {} & P^{\mathrm{D}} + Q^{\mathrm{D}} + \sum_{l=2}^{l_S} (Q^{\mathrm{D}})^l S^{l-1} \\
& + \sum_{l=2}^{l_S} \sum_{k=1}^{l-1} \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} (P^{\mathrm{D}})^{k+i} R^{i+j} (Q^{\mathrm{D}})^{l-k+j} S^{l-1} \\
& + \sum_{l=1}^{l_S} \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} (P^{\mathrm{D}})^{l+i+j} R^i Q^j (I - QQ^{\mathrm{D}}) S^{l-1} \\
& + \sum_{l=1}^{l_S} \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} (I - PP^{\mathrm{D}}) P^j R^i (Q^{\mathrm{D}})^{l+i+j} S^{l-1} \\
& + \sum_{l=1}^{l_S} \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (P^{\mathrm{D}})^{l+m+i} R^{i+j} Q^{m-j} (I - QQ^{\mathrm{D}}) S^{l-1} \\
& + \sum_{l=1}^{l_S} \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (I - PP^{\mathrm{D}}) P^{m-i} R^{i+j} (Q^{\mathrm{D}})^{l+m+j} S^{l-1} \\
& - \sum_{l=1}^{l_S} \sum_{k=1}^{l} \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} (P^{\mathrm{D}})^{k+i-1} R^{i+j-1} (Q^{\mathrm{D}})^{l-k+j} S^{l-1},
\end{aligned} \tag{4.1}
$$

where $l_S = \mathrm{Ind}(S)$. Alternatively, replacing l_S by an integer n ($m \geqslant n \geqslant l_S$) in (4.1), the above explicit representation still holds.

Proof. It follows from Lemma 2.1 that

$$
M^{\mathrm{D}} = (P + Q + R + S)^{\mathrm{D}} = \sum_{l=1}^{l_S} ((P + Q + R)^{\mathrm{D}})^l S^{l-1}, \tag{4.2}
$$

since that $SP = SQ = SR = \mathbf{0}$. Clearly, $(P, Q, R, \mathbf{0})$ is also a pseudo-block decomposition of the matrix $P + Q + R$. Combining (4.2) with Lemma 4.1, (4.1) is easy to obtain. This completes the proof. $\square$

Now, replacing the condition $SP = SR = \mathbf{0}$ in the above theorem by $RS = QS = \mathbf{0}$, we derive an analogous result.

Theorem 4.2. *Let P, Q, R and S be complex m-square matrices and $M = P + Q + R + S$. Suppose (P, Q, R, S) is a pseudo-block decomposition of M and $RS = QS = \mathbf{0}$, then*

$$
\begin{aligned}
M^{\mathrm{D}} = {} & P^{\mathrm{D}} + Q^{\mathrm{D}} + \sum_{l=2}^{l_S} S^{l-1} (P^{\mathrm{D}})^l \\
& + \sum_{l=2}^{l_S} \sum_{k=1}^{l-1} \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} S^{l-1} (P^{\mathrm{D}})^{k+i} R^{i+j} (Q^{\mathrm{D}})^{l-k+j} \\
& + \sum_{l=1}^{l_S} \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} S^{l-1} (P^{\mathrm{D}})^{l+i+j} R^i Q^j (I - QQ^{\mathrm{D}})
\end{aligned}
$$

$$+\sum_{l=1}^{l_S}\sum_{i=1}^{m-1}\sum_{j=0}^{m-i} S^{l-1}(I-PP^{\mathrm{D}})P^jR^i(Q^{\mathrm{D}})^{l+i+j}$$
$$+\sum_{l=1}^{l_S}\sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1} S^{l-1}(P^{\mathrm{D}})^{l+m+i}R^{i+j}Q^{m-j}(I-QQ^{\mathrm{D}})$$
$$+\sum_{l=1}^{l_S}\sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1} S^{l-1}(I-PP^{\mathrm{D}})P^{m-i}R^{i+j}(Q^{\mathrm{D}})^{l+m+j}$$
$$-\sum_{l=1}^{l_S}\sum_{k=1}^{l}\sum_{i=1}^{m-1}\sum_{j=1}^{m-i} S^{l-1}(P^{\mathrm{D}})^{k+i-1}R^{i+j-1}(Q^{\mathrm{D}})^{l-k+j}, \tag{4.3}$$

where $l_S = \mathrm{Ind}(S)$. Alternatively, replacing l_S by any integer n ($m \geqslant n \geqslant l_S$) in (4.3), the above explicit representation still holds.

Proof. The proof is similar to that of Theorem 4.1. □

Specializing Theorem 4.1 to the case $\mathrm{Ind}(S) \leqslant 2$ and $\mathrm{Ind}(R) \leqslant 2$ (i.e., $R^2 = S^2 = \mathbf{0}$), we have the corollary.

Corollary 4.1. *Let P, Q, R and S be complex m-square matrices and $M = P + Q + R + S$. Suppose (P, Q, R, S) is a pseudo-block decomposition of M, $SP = SR = \mathbf{0}$ and $S^2 = R^2 = \mathbf{0}$, then*

$$M^{\mathrm{D}} = P^{\mathrm{D}} + Q^{\mathrm{D}} - P^{\mathrm{D}}RQ^{\mathrm{D}} + ((Q^{\mathrm{D}})^2 - P^{\mathrm{D}}R(Q^{\mathrm{D}})^2 - (P^{\mathrm{D}})^2RQ^{\mathrm{D}})S$$
$$+\sum_{j=0}^{m-1}(P^{\mathrm{D}})^{j+2}RQ^j(I-QQ^{\mathrm{D}}) + \sum_{j=0}^{m-1}(I-PP^{\mathrm{D}})P^jR(Q^{\mathrm{D}})^{j+2}$$
$$+\sum_{j=0}^{m-1}((P^{\mathrm{D}})^{j+3}RQ^j(I-QQ^{\mathrm{D}}) + (I-PP^{\mathrm{D}})P^jR(Q^{\mathrm{D}})^{j+3})S \tag{4.4}$$

or alternatively

$$M^{\mathrm{D}} = P^{\mathrm{D}} + Q^{\mathrm{D}} - P^{\mathrm{D}}RQ^{\mathrm{D}} + ((Q^{\mathrm{D}})^2 - P^{\mathrm{D}}R(Q^{\mathrm{D}})^2 - (P^{\mathrm{D}})^2RQ^{\mathrm{D}})S$$
$$+\sum_{j=0}^{l_Q-1}(P^{\mathrm{D}})^{j+2}RQ^j(I-QQ^{\mathrm{D}}) + \sum_{j=0}^{l_P-1}(I-PP^{\mathrm{D}})P^jR(Q^{\mathrm{D}})^{j+2}$$
$$+\sum_{j=0}^{l_Q-1}(P^{\mathrm{D}})^{j+3}RQ^j(I-QQ^{\mathrm{D}})S + \sum_{j=0}^{l_P-1}(I-PP^{\mathrm{D}})P^jR(Q^{\mathrm{D}})^{j+3}S, \tag{4.5}$$

where $l_P = \mathrm{Ind}(P)$ and $l_Q = \mathrm{Ind}(Q)$. Furthermore

$$\mathrm{Ind}(M) \leqslant \mathrm{Ind}(P) + \mathrm{Ind}(Q) + 2. \tag{4.6}$$

Proof. The formulas (4.4) and (4.5) are immediate from Theorem 4.1. All we need to do now is to prove (4.6).

By induction on l, we have

$$M^{l+1} = P^{l+1} + Q^{l+1} + Q^l S + \sum_{i=0}^{l} P^{l-i} R Q^i + \sum_{i=0}^{l-1} P^{l-i-1} R Q^i S$$

for $l \geqslant 1$. Let Y_i be the ith terms on the right-hand side of above equality for $i = 1, 2, \ldots, 5$, and $X = M^{\mathrm{D}}$. Combining the above equality with (4.5), for $l \geqslant l_P + l_Q + 2$, we obtain

$$\begin{aligned} M^{l+1} M^{\mathrm{D}} &= Y_1 X + Y_2 X + Y_3 X + Y_4 X + Y_5 X = Y_1 X + Y_2 X + Y_4 X \\ &= \Bigg[P^l - P^l R Q^{\mathrm{D}} - P^l R Q^{\mathrm{D}} - (P^l R (Q^{\mathrm{D}})^2 + P^{l-1} R Q^{\mathrm{D}}) S \\ &\quad + \sum_{j=0}^{l-1} P^{l-j-1} R Q^j (I - Q Q^{\mathrm{D}}) + \sum_{j=0}^{l-2} P^{l-j-2} R Q^j (I - Q Q^{\mathrm{D}}) S \Bigg] \\ &\quad + (Q^l + Q^{l-1} S) + \Bigg[\left(P^l R Q^{\mathrm{D}} + \sum_{i=0}^{l-1} P^{l-i+1} R Q^i \cdot Q Q^{\mathrm{D}} \right) \\ &\quad + \left(P^l R (Q^{\mathrm{D}})^2 S + P^{l-1} R Q^{\mathrm{D}} S + \sum_{i=0}^{l-2} P^{l-i-2} R Q^i \cdot Q Q^{\mathrm{D}} S \right) \Bigg] \\ &= M^l, \end{aligned} \tag{4.7}$$

since that $S X_3 = \mathbf{0}$, $S X_5 = \mathbf{0}$, $\sum_{j=l_Q}^{l-1} P^{l-j-1} R Q^j (I - Q Q^{\mathrm{D}}) = \mathbf{0}$, $P^l (P^{\mathrm{D}})^j = P^{l-j}$ for $l - \max\{\mathrm{Ind}(P), 1\} \geqslant j > 0$, and $\sum_{j=l_Q}^{l-2} P^{l-j-2} R Q^j (I - Q Q^{\mathrm{D}}) S = \mathbf{0}$. Clearly, (4.6) follows from (4.7). $\square$

By restricting our attention to the cases $P^{\mathrm{D}} = \mathbf{0}$ (i.e., P is nilpotent) in Corollary 4.1, then we can obtain the following Corollary.

Corollary 4.2. *Let P, Q, R and S be complex m-square matrices and $M = P + Q + R + S$. Suppose (P, Q, R, S) is a pseudo-block decomposition of M, $SP = SR = \mathbf{0}$, $S^2 = R^2 = \mathbf{0}$ and P is nilpotent, then*

$$\begin{aligned} M^{\mathrm{D}} &= Q^{\mathrm{D}} + (Q^{\mathrm{D}})^2 S + \sum_{j=0}^{m-1} P^j R (Q^{\mathrm{D}})^{j+2} + \sum_{j=0}^{m-1} P^j R (Q^{\mathrm{D}})^{j+3} S \\ &= Q^{\mathrm{D}} + (Q^{\mathrm{D}})^2 S + \sum_{j=0}^{l_p-1} P^j R (Q^{\mathrm{D}})^{j+2} + \sum_{j=0}^{l_p-1} P^j R (Q^{\mathrm{D}})^{j+3} S, \end{aligned} \tag{4.8}$$

where $l_P = \mathrm{Ind}(P)$. Furthermore

$$\mathrm{Ind}(M) \leqslant \mathrm{Ind}(P) + \mathrm{Ind}(Q) + 1. \tag{4.9}$$

Proof. The formula (4.8) follows from Corollary 4.1. The proof of (4.9) is analogous to that of (4.6) in Corollary 4.1. $\square$

We now pay our attention to obtain the Drazin inverse of a 2×2 block matrix, which also extend Hartwig–Shoaf–Meyer–Rose formula.

Let $P = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $Q = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}$, $R = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $S = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ D & \mathbf{0} \end{bmatrix}$, where A and C are square matrices. Taking the block decomposition (P, Q, R, S) of $M = P + Q + R + S = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$, the desired results are derived from Corollary 4.1 and Theorem 4.2, respectively.

Corollary 4.3. *Let* $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{m \times m}$, $l_A = \operatorname{Ind}(A)$ *and* $l_C = \operatorname{Ind}(C)$, *where A and C are square. Suppose* $DA = \mathbf{0}$ *and* $DB = \mathbf{0}$, *then*

$$M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} + X_2 D & X_1 \\ (C^{\mathrm{D}})^2 D & C^{\mathrm{D}} \end{bmatrix},$$

where

$$\begin{aligned} X_i &= \sum_{j=0}^{m-1} ((A^{\mathrm{D}})^{i+j+1} B C^j (I - CC^{\mathrm{D}}) + (I - AA^{\mathrm{D}}) A^j B (C^{\mathrm{D}})^{i+j+1}) \\ &\quad - \sum_{j=0}^{i-1} (A^{\mathrm{D}})^{j+1} B (C^{\mathrm{D}})^{i-j} \\ &= \left(\sum_{j=0}^{l_C - 1} (A^{\mathrm{D}})^{i+j+1} B C^j \right) (I - CC^{\mathrm{D}}) + (I - AA^{\mathrm{D}}) \left(\sum_{j=0}^{l_A - 1} A^j B (C^{\mathrm{D}})^{i+j+1} \right) \\ &\quad - \sum_{j=0}^{i-1} (A^{\mathrm{D}})^{j+1} B (C^{\mathrm{D}})^{i-j} \quad (i = 1, 2). \end{aligned} \tag{4.10}$$

Furthermore

$$\operatorname{Ind}(M) \leqslant \operatorname{Ind}(A) + \operatorname{Ind}(C) + 2.$$

Corollary 4.4 [11, Theorem 2.1]. *Let* $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{m \times m}$, $l_A = \operatorname{Ind}(A)$ *and* $l_C = \operatorname{Ind}(C)$, *where A and C are square. Suppose* $BD = \mathbf{0}$ *and* $CD = \mathbf{0}$, *then*

$$M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} & X_1 \\ D(A^{\mathrm{D}})^2 & C^{\mathrm{D}} + DX_2 \end{bmatrix},$$

where X_1 and X_2 are defined in (4.10). *Furthermore*

$$\operatorname{Ind}(M) \leqslant \operatorname{Ind}(A) + \operatorname{Ind}(C) + 2.$$

The following results hold from Corollaries 4.2 and 4.4.

Corollary 4.5 [13]. *Let* $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{m \times m}$, *where A and C are square matrices. Suppose* $BD = \mathbf{0}$, $CD = \mathbf{0}$ *and* $BC = \mathbf{0}$, *then*

$$M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} & (A^{\mathrm{D}})^2 B \\ D(A^{\mathrm{D}})^2 & C^{\mathrm{D}} + D(A^{\mathrm{D}})^3 B \end{bmatrix}.$$

Corollary 4.6 [19]. *Let* $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{m\times m}$, $l_A = \mathrm{Ind}(A)$ *and* $l_C = \mathrm{Ind}(C)$, *where A and C are square. Suppose* $BD = \mathbf{0}$, $CD = \mathbf{0}$ *and C is nilpotent, then*

$$M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} & \sum_{j=0}^{l_C-1} (A^{\mathrm{D}})^{j+2} BC^j \\ D(A^{\mathrm{D}})^2 & \sum_{j=0}^{l_C-1} D(A^{\mathrm{D}})^{j+3} BC^j \end{bmatrix} = \begin{bmatrix} I \\ DA^{\mathrm{D}} \end{bmatrix} A^{\mathrm{D}} \begin{bmatrix} I & \sum_{j=0}^{l_C-1} (A^{\mathrm{D}})^{j+1} BC^j \end{bmatrix}.$$

Furthermore

$$\mathrm{Ind}(M) \leqslant \mathrm{Ind}(A) + \mathrm{Ind}(C) + 1.$$

We conclude this section with the following remark.

Remark 4.1. It is easy to know that if (P, Q, R, S) is a pseudo-block decomposition for the matrix $P + Q + R + S$, then $(P^{\mathrm{T}}, Q^{\mathrm{T}}, S^{\mathrm{T}}, R^{\mathrm{T}})$ is a pseudo-block decomposition for the matrix $P^{\mathrm{T}} + Q^{\mathrm{T}} + S^{\mathrm{T}} + R^{\mathrm{T}}$. On the other hand, by the property (2.5), we have

$$(P + Q + R + S)^{\mathrm{D}} = ((P^{\mathrm{T}} + Q^{\mathrm{T}} + S^{\mathrm{T}} + R^{\mathrm{T}})^{\mathrm{D}})^{\mathrm{T}}. \tag{4.11}$$

Under similar conditions, we can express $(P^{\mathrm{T}} + Q^{\mathrm{T}} + S^{\mathrm{T}} + R^{\mathrm{T}})^{\mathrm{D}}$ by Theorems 4.1 and 4.2 and Corollaries 4.2 and 4.3, respectively. So, from (4.11), one can see that there are many results similar to Theorems 4.1 and 4.2 and Corollaries 4.2 and 4.3. It is trivial to derive these results and we omit them here.

5. Example

Now we present an example to show that the block forms of M and its block decompositions fail to apply to find M^{D} by previous formulae directly, but one can work with a special pseudo-block decomposition of M.

Example 5.1. Let

$$M = \begin{bmatrix} 8 & 0 & 8 & 8 \\ 0 & 0 & -6 & 6 \\ 4 & -4 & -7 & 15 \\ 4 & -4 & -7 & 15 \end{bmatrix} = \begin{bmatrix} A & B \\ D & C \end{bmatrix}.$$

There exist three cases of the block form as follows:

$$\text{Case (1):} \quad A = \begin{bmatrix} 8 \end{bmatrix}, \; B = \begin{bmatrix} 0, & 8, & 8 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -6 & 6 \\ -4 & -7 & 15 \\ -4 & -7 & 15 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}.$$

$$\text{Case (2):} \quad A = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 8 & 8 \\ -6 & 6 \end{bmatrix}, \; C = \begin{bmatrix} -7 & 15 \\ -7 & 15 \end{bmatrix}, \; D = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}.$$

$$\text{Case (3):} \quad A = \begin{bmatrix} 8 & 0 & 8 \\ 0 & 0 & -6 \\ 4 & -4 & -7 \end{bmatrix}, \; B = \begin{bmatrix} 8 \\ 6 \\ 15 \end{bmatrix}, \; C = \begin{bmatrix} 15 \end{bmatrix}, \; D = \begin{bmatrix} 4, & -4, & -7 \end{bmatrix}.$$

In each case, one can verify that $DB \neq \mathbf{0}$ and $BD \neq \mathbf{0}$. Thus the block matrix version results Corollaries 4.3 and 4.4 fail to find M^{D}. Similarly, let (P_1, Q_1, R_1, S_1) be the block decompositions of M, where

$$P_1 = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}, \quad R_1 = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad S_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ D & \mathbf{0} \end{bmatrix}.$$

We can see that Theorems 4.1, 4.2 and Corollary 4.1, also fail to find M^{D}, since $S_1R_1 \neq \mathbf{0}$ and $R_1S_1 \neq \mathbf{0}$. If we take a special pseudo-block decomposition (P_2, Q_2, R_2, S_2) of M as follows:

$$P_2 = \begin{bmatrix} 8 & -8 & 8 & 8 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 2 & 6 \\ 4 & -4 & 2 & 6 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 8 & 0 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & -8 & 8 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

then we can compute M^{D} by Theorem 4.1. In fact, one can verify that (P_2, Q_2, R_2, S_2) is a pseudo-block decomposition of M, as $S_2P_2 = \mathbf{0}$ and $S_2R_2 = \mathbf{0}$.

Let $\alpha_1^{\mathrm{T}} = (0, 0, 1, 1), \alpha_2^{\mathrm{T}} = (1, 0, 0, 0), \alpha_3^{\mathrm{T}} = (-1, 1, 1, 1), \beta_1^{\mathrm{T}} = (4, -4, 2, 6), \beta_2^{\mathrm{T}} = (8, -8, 8, 8)$, and $\beta_3^{\mathrm{T}} = (0, 0, -2, 2)$. Note that $(UV)^{\mathrm{D}} = U[(VU)^{\mathrm{D}}]^2V$ (cf. [10]), where U and V are $r \times s$ and $s \times r$ matrices, respectively, we have

$$\begin{aligned} P_2^{\mathrm{D}} &= (\alpha_1\beta_1^{\mathrm{T}} + \alpha_2\beta_2^{\mathrm{T}})^{\mathrm{D}} = \left([\alpha_1, \alpha_2]\begin{bmatrix} \beta_1^{\mathrm{T}} \\ \beta_2^{\mathrm{T}} \end{bmatrix}\right)^{\mathrm{D}} \\ &= [\alpha_1, \alpha_2]\left[\left(\begin{bmatrix} \beta_1^{\mathrm{T}} \\ \beta_2^{\mathrm{T}} \end{bmatrix}[\alpha_1, \alpha_2]\right)^{\mathrm{D}}\right]^2 \begin{bmatrix} \beta_1^{\mathrm{T}} \\ \beta_2^{\mathrm{T}} \end{bmatrix} = \frac{1}{256}\begin{bmatrix} 8 & -8 & 6 & 10 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 3 & 5 \\ 4 & -4 & 3 & 5 \end{bmatrix} \end{aligned}$$

and $Q_2^{\mathrm{D}} = (\alpha_3\beta_3^{\mathrm{T}})^{\mathrm{D}} = \alpha_3[(\beta_3^{\mathrm{T}}\alpha_3)^{\mathrm{D}}]^2\beta_3^{\mathrm{T}} = \mathbf{0}$. Hence by Theorem 4.1, we have

$$\begin{aligned} M^{\mathrm{D}} &= P_2^{\mathrm{D}} + \sum_{l=1}^{l_{S_2}} \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} (P_2^{\mathrm{D}})^{l+i+j} R_2^i Q_2^j S_2^{l-1} \\ &\quad + \sum_{l=1}^{l_{S_2}} \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (P_2^{\mathrm{D}})^{l+m+i} R_2^{i+j} Q_2^{m-j} S_2^{l-1} \\ &= P_2^{\mathrm{D}} + (P_2^{\mathrm{D}})^2 R_2 + (P_2^{\mathrm{D}})^3 (R_2Q_2 + R_2^2) \\ &= \frac{1}{256}\begin{bmatrix} 8 & -8 & 6 & 10 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 3 & 5 \\ 4 & -4 & 3 & 5 \end{bmatrix} + \frac{1}{256}\begin{bmatrix} 0 & 4 & -6 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 3 \\ 0 & 2 & -3 & 3 \end{bmatrix} \end{aligned}$$

$$+\frac{3}{1024}\begin{bmatrix}0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}$$
$$=\frac{1}{1024}\begin{bmatrix}32 & -16 & -6 & 70\\ 0 & 0 & 0 & 0\\ 16 & -8 & -3 & 35\\ 16 & -8 & -3 & 35\end{bmatrix},$$

now $Q_2^{\mathrm{D}} = \mathbf{0}$, $l_{S_2} = \mathrm{Ind}(S_2) = 2$ (since $S_2^2 = \mathbf{0}$ and $S_2 \neq \mathbf{0}$), $m = 4$, $R_2^3 = \mathbf{0}$, $Q_2S_2 = \mathbf{0}$, $R_2S_2 = \mathbf{0}$, $Q_2^2 = \mathbf{0}$, and $R_2^2Q_2 = \mathbf{0}$.

6. Concluding remarks

This paper is devoted to present various representation formulae for the Drazin inverses of $M = P + Q + R + S$, and derive representations of the Drazin inverse of a 2×2 block matrix $\begin{bmatrix}A & B\\ D & C\end{bmatrix}$. It is still very difficult to obtain the most general expressions of the Drazin inverse of a 2×2 block matrix without any restrictions, which is a future research topic.

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