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A nonexterior Hopf algebra and loop spaces

Yutaka Hemmi [1]

Department of Mathematics, Faculty of Science, Kochi University, Kochi, 780, Japan

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Abstract

Let A be the Hopf algebra over $\mathbb{Z}/p$ for a prime p given by $A = \Lambda(x_i, y_i \mid 0 \leqslant i \leqslant p-2) \otimes (\mathbb{Z}/p[z]/(z^p))$ ($\deg x_i = 2(p+1)i+3$, $\deg y_i = 2(p+1)(i+1)-1$, $\deg z = 2(p+1)$). Kane showed that A is a minimum candidate for the mod p cohomology of a simply connected mod p finite loop space with p-torsion. In fact, if $p = 2, 3, 5$, then for $X = G_2, F_4, E_8$, we have $H^*(X; \mathbb{Z}/p) \cong A$. We prove that if $p \geqslant 7$, then there are no such loop spaces X.

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1. Introduction

Let X be a simply connected loop space with finite mod p cohomology $H^*(X;\mathbb{Z}/p)$ for a prime p. Then, $H^*(X;\mathbb{Z}/p)$ is isomorphic as an algebra to a tensor product of monogenic Hopf algebras [1, Theorem 3.2]. If $H^*(X;\mathbb{Z}/p)$ is an exterior algebra, then for a space BX with $\Omega(BX) \simeq X, H^*(BX;\mathbb{Z}/p)$ is a polynomial algebra. Lots of facts are known about the realization problem of polynomial algebras over the mod p Steenrod algebra, and hence about loop spaces with exterior mod p cohomologies. On the other hand, very little is known for the case that $H^*(X;\mathbb{Z}/p)$ is not an exterior algebra. Kane [2, Theorem 1.3] showed that if the number of odd degree generators of $H^*(X;\mathbb{Z}/p)$ is less than $2(p-1)$, then $H^*(X;\mathbb{Z}/p)$ is an exterior algebra, while if it is just $2(p-1)$ and $H^*(X;\mathbb{Z}/p)$ is not an exterior algebra, then, as a Hopf algebra over the mod p Steenrod algebra, $H^*(X;\mathbb{Z}/p)$ is isomorphic to a Hopf algebra

$$A = \Lambda(x_i, y_i \mid 0 \leqslant i \leqslant p-2) \otimes \left(\mathbb{Z}/p[z]/(z^p)\right).$$

[1] E-mail: hemmi@math.kochi-u.ac.jp.

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Here $\deg x_i = 2(p+1)i + 3$, $\deg y_i = 2(p+1)(i+1) - 1$, $\deg z = 2(p+1)$, and for the coproduct $\mu^* : A \to A \otimes A$,

$$\mu^*(x_i) = x_i \otimes 1 + 1 \otimes x_i + \sum_{j=0}^{i-1} \frac{1}{(i-j)!} x_j \otimes z^{i-j},$$

$$\mu^*(y_i) = y_i \otimes 1 + 1 \otimes y_i + \sum_{j=0}^{i-1} \frac{1}{(i-j)!} y_j \otimes z^{i-j},$$

$$\mu^*(z) = z \otimes 1 + 1 \otimes z.$$

Furthermore, the action of the mod p Steenrod algebra is given by

$$\mathscr{P}^1 x_i = y_i, \qquad \beta y_0 = z.$$

We notice that if $p = 2$, then $\mathscr{P}^1$ is considered as Sq^2, and so we have $z = \mathrm{Sq}^1 \mathrm{Sq}^2(x_0) = \mathrm{Sq}^3(x_0) = x_0^2$. Thus, A is considered as

$$A = \mathbb{Z}/2[x_0]/(x_0^4) \otimes \Lambda(y_0).$$

Recently, Yagita [3] showed that even if the number of odd generators is greater than $2(p-1)$, $H^*(X;\mathbb{Z}/p)$ has A as a sub-Hopf algebra over $\mathbb{Z}/p$ on the condition that $H^*(X;\mathbb{Z}/p)$ has only one even degree generator in degree $2(p+1)$.

For $p = 2, 3, 5$, the Lie groups G_2, F_4, E_8 realize A. On the other hand, if $p \geqslant 7$, there are no known simply connected loop spaces realizing A.

In this paper, we show the following fact.

Theorem 1.1. *If $p \geqslant 7$, then there is no simply connected loop space X with the homotopy type of a CW-complex of finite type, so that $H^*(X;\mathbb{Z}/p) \cong A$ as a Hopf algebra over the* mod p *Steenrod algebra.*

Our method can also be applied to prove that A is not realized by an A_p-space.

The results by Kane [2] and Yagita [3] are not for loop spaces but for homotopy associative H-spaces. The author does not know if A is realized by a homotopy associative H-space.

2. 3-connected covering

In the rest of this paper, the cohomology is assumed to have coefficients in $\mathbb{Z}/p$ for a fixed odd prime p unless otherwise stated.

In this section we assume that X is a simply connected loop space with the homotopy type of a CW-complex of finite type, so that

$$H^*(X) \cong A$$

as a Hopf algebra over the mod p Steenrod algebra $\mathscr{A}$. Let $r : X\langle 3\rangle \to X$ be the 3-connected cover of X. Then we can prove the following:

Lemma 2.1. *We have*

$$H^*(X\langle 3\rangle) \cong \Lambda(r^*x_i, r^*y_i, v, w \mid 1 \leqslant i \leqslant p-2) \otimes \mathbb{Z}/p[u],$$

where $\deg r^*x_i = 2(p+1)i+3$, $\deg r^*y_i = 2(p+1)(i+1)-1$, $\deg v = 2p^2+1$, $\deg w = 2p^2+2p-1$, $\deg u = 2p^2$, *and*

$$\mathscr{P}^1 r^*x_i = r^*y_i, \quad \beta u = v, \quad \mathscr{P}^1 v = w.$$

Furthermore, $r^*x_i, r^*y_i, v, w \in H^*(X\langle 3\rangle)$ *are all primitive, and* $r^*x_0 = r^*y_0 = r^*z = 0$.

The above lemma can be proved by using the Serre spectral sequence, and we omit the proof.

Let $BX\langle 3\rangle$ be a space with $\Omega(BX\langle 3\rangle) \simeq X\langle 3\rangle$. Then we have the cobar spectral sequence $\{E_r^{s,t}, d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}\}$ converging to $H^*(BX\langle 3\rangle)$. The E_1-term is the cobar complex of $\widetilde{H}^*(X\langle 3\rangle)$, i.e.,

$$E_1^{s,*} = \widetilde{H^*}(X\langle 3\rangle)^{\otimes s},$$

$$d_1|E_1^{s,*} = \sum_{j=1}^{s}(-1)^j \operatorname{id}^{\otimes j-1} \otimes \widetilde{\mu} \otimes \operatorname{id}^{\otimes s-j},$$

where we use the notation $\Phi^{\otimes t}$ for the t-fold tensor product of Φ for any module or map Φ. Then, we have

$$E_2^{*,*} = \operatorname{Cotor}^{*,*}_{H^*(X\langle 3\rangle)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

We can prove the following fact.

Lemma 2.2. *If u is primitive, then for total degree less than or equal to $2p^3$, we have*

$$E_2^{*,*} \cong \mathbb{Z}/p[X_i, Y_i, V, W \mid 1 \leqslant i \leqslant p-2] \otimes \Lambda(U).$$

If u is not primitive, then for total degree less than $2p^3$, we have

$$E_2^{*,*} \cong \mathbb{Z}/p[X_i, Y_i, V, W \mid 1 \leqslant i \leqslant p-2]/(U').$$

Here $X_i = [r^*x_i]$, $Y_i = [r^*y_i]$, $U = [u]$, $V = [v]$, $W = [w]$, *and* (U') *is an ideal generated by some class* U' *of degree* $(2, 2p^2)$.

Proof. The first half is clear.

Suppose that u is not primitive. We define a filtration $\{F_n\}$ of $H^*(X\langle 3\rangle)$ by

$$F_0 = \Lambda(r^*x_i, r^*y_i, v, w),$$

$$F_n = F_{n-1} + u \cdot F_{n-1} \quad (n \geqslant 1).$$

Then the filtration $\{F_n\}$ is compatible with the coalgebra structure on $H^*(X\langle 3\rangle)$, that is, if μ^* is the coproduct on $H^*(X\langle 3\rangle)$, then $\mu^*(F_n) \subset \sum F_i \otimes F_{n-i}$. Thus, we have a filtration on the cobar complex of $H^*(X\langle 3\rangle)$. Let $\{\widetilde{E}_r^{*,*}, \widetilde{d}_r\}$ be the spectral sequence associated to the above filtration. Then, $\widetilde{E}_\infty^{*,*}$ is the graded associated algebra of $\operatorname{Cotor}^{*,*}_{H^*(X\langle 3\rangle)}(\mathbb{Z}/p, \mathbb{Z}/p)$, and

$$\widetilde{E}_2^{*,*} \cong \operatorname{Cotor}^{*,*}_B(\mathbb{Z}/p, \mathbb{Z}/p),$$

where $B = \sum F_n/F_{n-1}$. By definition, B is isomorphic to $\Lambda(r^*x_i, r^*y_i, v, w) \otimes \mathbb{Z}/p[u]$ with all generators primitive. Thus we have

$$\widetilde{E}_2^{*,*} \cong \mathbb{Z}/p[\widetilde{X}_i, \widetilde{Y}_i, \widetilde{V}, \widetilde{W}] \otimes \Lambda(\widetilde{U})$$

for total degree $\leqslant 2p^3$, where $\widetilde{X}_i = [r^*x_i]$, $\widetilde{Y}_i = [r^*y_i]$, $\widetilde{U} = [u]$, $\widetilde{V} = [v]$, and $\widetilde{W} = [w]$.

Now $\widetilde{X}_i, \widetilde{Y}_i, \widetilde{V}, \widetilde{W}$ are all infinite cycles. Let $\widetilde{d}_r$ be the first nontrivial differential for total degree $\leqslant 2p^3$. Put $\widetilde{U}' = \widetilde{d}_r(\widetilde{U}) \neq 0$. Then we have

$$\widetilde{E}_{r+1}^{*,*} \cong \mathbb{Z}/p[\widetilde{X}_i, \widetilde{Y}_i, \widetilde{V}, \widetilde{W}]/(\widetilde{U}')$$

for total degree $\leqslant 2p^3$. Since $\widetilde{E}_{r+1}^{s,t}$ has no nontrivial classes of odd total degree for $s + t \leqslant 2p^3$, we have $\widetilde{E}_\infty^{s,t} = \widetilde{E}_{r+1}^{s,t}$ for $s + t < 2p^3$. This shows that there is an epimorphism

$$\mathbb{Z}/p[X_i, Y_i, V, W] \to \mathrm{Cotor}_{H^*(X\langle 3\rangle)}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

for total degree $< 2p^3$. Let U' be a nontrivial class of minimal total degree in the kernel of the above map. Since U' is determined by $d_1(u) = \widetilde{\mu}(u)$ in the cobar complex of $H^*(X\langle 3\rangle)$, we have $\deg U' = (2, 2p^2)$. It is clear that the induced epimorphism

$$\mathbb{Z}/p[X_i, Y_i, V, W]/(U') \to \mathrm{Cotor}_{H^*(X\langle 3\rangle)}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is also an isomorphism for total degree $< 2p^3$ by considering the rank. Thus the second half is proved. □

If u is not primitive, the above lemma shows that $E_2^{s,t}$ with $s + t < 2p^3$ has no nontrivial classes of odd total degree, which shows that the spectral sequence collapses for total degree less than $2p^3 - 1$. We have the same conclusion in the first case provided that u is transgressive, for dimensional reasons. If u is primitive and not transgressive, then we have only one nontrivial differential d_r with $0 \neq d_r(u) \in E_r^{1+r, 2p^2-r+1}$. Then, by the same method as in the proof of the above lemma, we have the following result.

Lemma 2.3. *If u is transgressive, then for degree less than $2p^3$, we have*

$$H^*(BX\langle 3\rangle) \cong \mathbb{Z}/p[X_i, Y_i, V, W \mid 1 \leqslant i \leqslant p-2] \otimes \Lambda(U).$$

If u is not transgressive, then for degree less than $2p^3 - 1$, we have

$$H^*(BX\langle 3\rangle) \cong \mathbb{Z}/p[X_i, Y_i, V, W \mid 1 \leqslant i \leqslant p-2]/(U'')$$

for some U'' with $\deg U'' = 2p^2+2$. Here $\deg X_i = 4+2(p+1)i$, $\deg Y_i = 2(p+1)(i+1)$, $\deg U = 2p^2 + 1$, $\deg V = 2p^2 + 2$, $\deg W = 2p^2 + 2p$, and $\mathscr{P}^1 X_i = Y_i$, $\mathscr{P}^1 V = W$, $\beta U = V$.

Let I be the ideal of $H^*(BX\langle 3\rangle)$ generated by the following factors:

(i) $\sum_{t \geqslant 2p^3-1} H^t(BX\langle 3\rangle)$,

(ii) $\{X_i, Y_i, V, W \mid i \neq (p-3)/2, p-2\}$,

(iii) U if u is transgressive.

We prove the following:

Lemma 2.4. *Let $p \geqslant 5$. Put $\alpha = X_{(p-3)/2}$, $\beta = Y_{(p-3)/2}$, $\gamma = X_{p-2}$ and $\delta = Y_{p-2}$. Then*

$$H^*(BX\langle 3\rangle)/I \cong \mathbb{Z}/p[\alpha,\beta,\gamma,\delta]/(\deg \geqslant 2p^3 - 1).$$

Furthermore, I is closed under the action of $\mathscr{A}$. Thus $\mathbb{Z}/p[\alpha,\beta,\gamma,\delta]/(\deg \geqslant 2p^3 - 1)$ is an unstable algebra over $\mathscr{A}$ with the following relations:

$$\mathscr{P}^1\alpha = \beta, \qquad \mathscr{P}^1\gamma = \delta.$$

Proof. For the case that u is transgressive, the first half of the lemma is clear.

For the case that u is not transgressive, $H^*(BX\langle 3\rangle)/I$ is clearly a quotient algebra of $C = \mathbb{Z}/p[\alpha,\beta,\gamma,\delta]/(\deg \geqslant 2p^3 - 1)$. Here, every element in C has degree congruent to $0 \bmod 2(p-1)$. Since $\deg U'' = 2p^2 + 2 \not\equiv 0 \mod 2(p-1)$, there are no more relations for $H^*(BX\langle 3\rangle)/I$, and we have $H^*(BX\langle 3\rangle)/I \cong C$.

Next note that $\deg X_i \equiv \deg Y_i \not\equiv 0 \mod 2(p-1)$ for $i \neq (p-3)/2,\ p-2$, and $\deg V \not\equiv 0 \mod 2(p-1)$, $\deg W \not\equiv 0 \mod 2(p-1)$. Thus we have $\mathscr{P}^k(I) \subset I$ for dimensional reasons. Since there are no odd degree elements in $H^*(BX\langle 3\rangle)/I$, I is also closed under the Backstein operation, and we have the results. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The proof is by contradiction.

Assume that $p \geqslant 7$, and that X is a simply connected loop space with the homotopy type of a CW-complex of finite type so that

$$H^*(X) \cong A.$$

Then we have an unstable $\mathscr{A}$ algebra

$$H^*(BX\langle 3\rangle)/I = C = \mathbb{Z}/p[\alpha,\beta,\gamma,\delta]/(\deg \geqslant 2p^3 - 1)$$

by Lemma 2.4.

Lemma 3.1. *Let J be the ideal of C generated by $\{\beta,\gamma,\delta\}$. Then $\mathscr{P}^1(\delta)$ does not belong to J.*

Proof. Suppose contrarily that $\mathscr{P}^1\delta \in J$. Since $p \geqslant 7$, we have $\mathscr{P}^1\beta \in J$ for dimensional reason. So we have $\mathscr{P}^1(J) \subset J$. Here $\alpha^p = \mathscr{P}^n\alpha = \mathscr{P}^1(n^{-1}\mathscr{P}^{n-1}\alpha)$. Since $\mathscr{P}^{n-1}\alpha \in J$ for dimensional reason, we have a contradiction. □

Lemma 3.2. *$p = 7$ and $\mathscr{P}^1(\delta) = \alpha$.*

Proof. First we note that $\deg \mathscr{P}^1\delta = \deg \alpha^t$ for some t by Lemma 3.1. Since $\deg \mathscr{P}^1\delta = 2p^2 - 2 + 2(p-1)$ and $\deg \alpha^t = t(p^2 - 2p + 1)$, we have $t \equiv -4 \mod p$. Thus $t = p - 4$, and we have $2p^2 - 2 + 2(p-1) = (p-4)(p^2 - 2p + 1)$, which implies that $p = 7$ and $\mathscr{P}^1\delta = e\alpha^3$ for some $e \neq 0$. By substituting $e^{-1}\gamma, e^{-1}\delta$ for γ, δ, if necessary, we have $\mathscr{P}^1\delta = \alpha^3$. □

Now

$$\alpha^7 = \mathscr{P}^{18}\alpha = \mathscr{P}^4\mathscr{P}^{14}\alpha = \mathscr{P}^4(a\alpha^3\beta^2 + b\alpha^3\delta + c\alpha^2\beta\gamma + d\alpha\gamma^2),$$

for some $a, b, c, d \in \mathbb{Z}/p$. Then, by comparing the coefficients of α^7 and $\alpha^3\beta\delta$ in the above equation, we have two contradictory equations $d \neq 0$ and $d = 0$. Thus, for $p \geqslant 7$, there is no such loop space X, and Theorem 1.1 is proved.

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