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Discrete Numerical Solution of Coupled Mixed Hyperbolic Problems

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Abstract—This paper deals with the construction of stable discrete numerical solutions of strongly coupled mixed hyperbolic problems using difference schemes. By means of a discrete separation of variables method and solving the underlying discrete Sturm-Liouville type problem, the numerical solution of the discretized mixed problem is constructed.

Keywords—Coupled hyperbolic system, Difference schemes

1. INTRODUCTION

Coupled hyperbolic partial differential systems arise in microwave heating processes [1,2], optics [3], cardiology [4], soil flows [5,6], and others. Uncoupling techniques [5] have well-known drawbacks such as assuming unnecessary hypotheses, the increase of the order of differentiation of the system, and others [7]. In this paper, we use matrix finite difference schemes to construct discrete numerical solutions of mixed problems of hyperbolic type modeled by

$$Au_{xx}(x,t) - u_{tt}(x,t) = 0, \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \qquad t > 0, \tag{2}$$

$$Bu(1,t) + Cu_x(1,t) = 0, \qquad t > 0, \tag{3}$$

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{4}$$

$$u_t(x,0) = v(x), \qquad 0 \le x \le 1, \tag{5}$$

where A, B, C are $m \times m$ complex matrices, elements of $\mathbb{C}^{m\times m}$, and the unknown u and f, v are $\mathbb{C}^m$-valued functions.

We assume that

$$C \text{ is invertible,} \tag{6}$$

$$\text{all the eigenvalues of } A \text{ are positive.} \tag{7}$$

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It is important to note that even in the case where A is a diagonalizable matrix, the problem remains coupled if matrices B and C are not simultaneously diagonalizable with A. For the coupled parabolic case, matrix difference schemes have been recently used in [8,9].

Throughout this paper, the set of all the eigenvalues of a matrix P in $\mathbb{C}^{m\times m}$ is denoted by $\sigma(P)$ and its 2-norm, denoted by $\|P\|$ is defined by [10, p. 56]

$$\|P\| = \sup_{v\neq 0} \frac{\|Pv\|_2}{\|v\|_2},$$

where for a vector z in $\mathbb{C}^m$, $n\|zn\|_2$ is the usual Euclidean norm of z. The maximum of the set $\{|\lambda|; \lambda \in \sigma(P)\}$ is called the spectral radius of P and is denoted by $\rho(P)$. If P is Hermitian, i.e., $P = P^H$, where P^H is the conjugate transpose of P, then $\|P\| = \rho(P)$ [11, p. 23]. If P is diagonalizable and Q is an invertible matrix such that $Q^{-1}PQ$ is a diagonal matrix, then $\|P\| \leq \|Q^{-1}\|\,\|Q\|\rho(P)$.

If S is a matrix in $\mathbb{C}^{n\times m}$, we denote by $S^\dagger$ its Moore-Penrose pseudoinverse. An account of properties and applications of this concept may be found in [12,13]. The kernel of S, denoted by $\operatorname{Ker} S$, coincides with the matrix $I - S^\dagger S$, denoted by $\operatorname{Im}(I - S^\dagger S)$. We say that a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m\times m}$ if $A(E) \subset E$. The property $A(\operatorname{Ker} G) \subset \operatorname{Ker} G$ is equivalent to the condition $GA(I - G^\dagger G) = 0$ [14].

If $P \in \mathbb{C}^{m\times m}$, $f(w)$ is a holomorphic function defined on an open set Ω of the complex plane and $\sigma(P)$ lies in Ω, the holomorphic matrix functional calculus defines $f(P)$ as a matrix that may be computed as a polynomial in P of degree smaller than the minimal polynomial of P, see [15, p. 567]. In particular, if P is invertible, then $\sigma(P)$ lies in $D_\alpha = \mathbb{C} \sim H_\alpha$, $H_\alpha = \{-re^{i\alpha}; r \geq 0\}$ and considering $f(w) = \log_\alpha(w)$ a branch of the complex logarithm, holomorphic in D_α [16, p. 76], then for $w \in D_\alpha$, the function $\sqrt{w} = \exp((1/2)\log_\alpha(w))$ is holomorphic and $\sqrt{P} = \exp\left((1/2)\log_\alpha(P)\right)$ is the square root of P. Note that if J_P is the Jordan canonical form of P, $P = SJ_PS^{-1}$ and $\sqrt{J_P}$ is the square root of J_P, then $Q = S\sqrt{J_P}S^{-1}$ is a square root of P. The real line is represented by $\mathbb{R}$.

This paper is organized as follows. Section 2 deals with the discretization of problem (1)–(5) using central difference approximations for the second derivatives u_{xx},u_{tt}, forward difference approximations for u_t and backward difference for u_x. Section 3 deals with the construction of nontrivial solutions of the boundary value problem resulting from the discretization of problem (1)–(3). Section 4 deals with the construction of stable solutions of the discretized mixed problem using a discrete separation of the variables method. Finally, Section 5 includes a projection method that permits us to extend the results of Section 4 to a wider class of initial value functions $f(x)$ and $v(x)$.

2. ON THE DISCRETIZATION PARTIAL DIFFERENCE PROBLEM

Let us divide the domain $[0,1] \times [0,+\infty[$ into equal rectangles of sides $\Delta x = h$, $\Delta t = k$, and introduce coordinates of a typical mesh point (ih, jk) with $U(i,j) = u(ih, jk)$. Using central difference approximations for both u_{tt} and u_{xx} [17,18] forward differences for u_t and backward for u_x:

$$u_t\,(ih,jk) \simeq \frac{U(i,j+1) - U(i,j)}{k}, \qquad u_{tt}\,(ih,jk) \simeq \frac{U(i,j+1) - 2U(i,j) + U(i,j-1)}{k^2},$$

$$u_x\,(ih,jk) \simeq \frac{U(i,j) - U(i-1,j)}{h}, \qquad u_{xx}\,(ih,jk) \simeq \frac{U(i+1,j) - 2U(i,j) + U(i-1,j)}{h^2};$$

discretization of problem (1)–(5) yields

$$r^2A\left[U(i+1,j) + U(i-1,j)\right] + 2\left(I - r^2A\right)U(i,j) - \left[U(i,j+1) + U(i,j-1)\right] = 0, \quad 0 < i < N, \quad j > 0, \tag{8}$$

$$U(0,j) = 0, \qquad j > 0, \tag{9}$$

$$BU(N,j) + NC\left[U(N,j) - U(N-1,j)\right] = 0, \qquad j > 0, \tag{10}$$

$$U(i,0) = F(i) = f\left(\frac{i}{N}\right), \qquad 0 \le i \le N, \tag{11}$$

$$\frac{U(i,1) - U(i,0)}{k} = V(i) = v\left(\frac{i}{N}\right), \qquad 0 \le i \le N, \tag{12}$$

where

$$r = \frac{k}{h}, \qquad Nh = 1. \tag{13}$$

3. THE DISCRETIZED BOUNDARY DIFFERENCE PROBLEM

Let us seek solutions of problem (8)–(10) of the form

$$U(i,j) = T(j)H(i), \qquad T(j) \in \mathbb{C}^{m\times m}, \qquad H(i) \in \mathbb{C}^m. \tag{14}$$

Equation (8) for sequences of form (14) gives

$$r^2 AT(j)\left[H(i+1) + H(i-1)\right] + 2\left(I - r^2 A\right)T(j)H(i) - \left[T(j+1) - T(j-1)\right]H(i) = 0. \tag{15}$$

Let ρ be a real number and let us write equation (15) in the form

$$\begin{aligned} &r^2 AT(j)\left[H(i+1) - \left(2 + \frac{\rho}{r^2}\right)H(i) + H(i-1)\right] \\ &- \left[T(j+1) - (2I + \rho A)\,T(j) + T(j-1)\right]H(i) = 0. \end{aligned} \tag{16}$$

Note that equation (16) is satisfied in $\{H(i)\}$,$\{T(j)\}$ satisfy

$$H(i+1) - \left(2 + \frac{\rho}{r^2}\right)H(i) + H(i-1) = 0, \qquad 0 < i < N, \tag{17}$$

$$T(j+1) - (2I + \rho A)\,T(j) + T(j-1) = 0, \qquad j > 0. \tag{18}$$

Let us take $\rho \in \mathbb{R}$ such that

$$-4r^2 < \rho < 0. \tag{19}$$

Then $-1 < (\rho + 2r^2)/2r^2 < 1$ and there exists $\theta \in]0, 2\pi[\sim \{\pi\}$ such that

$$\cos\theta = \frac{2r^2 + \rho}{2r^2}, \qquad \rho = 2r^2(\cos\theta - 1) = -4r^2 \sin^2\left(\frac{\theta}{2}\right). \tag{20}$$

Let

$$z_0 = \frac{2r^2+\rho}{2r^2} + i\sqrt{1 - \left(\frac{2r^2+\rho}{2r^2}\right)^2} = e^{i\theta} \qquad \text{and} \qquad z_1 = \frac{2r^2+\rho}{2r^2} - i\sqrt{1 - \left(\frac{2r^2+\rho}{2r^2}\right)^2} = e^{-i\theta}$$

be the solutions of the scalar equation

$$z^2 - \left(2 + \frac{\rho}{r^2}\right)z + 1 = 0, \tag{21}$$

and note that

$$z_0^n = \cos(n\theta) + i\sin(n\theta), \qquad z_1^n = \cos(n\theta) - i\sin(n\theta). \tag{22}$$

The general solution of equation (17) satisfying $H(0) = 0$ takes the form

$$H(i) = \sin(i\theta)E, \qquad E \in \mathbb{C}^m. \tag{23}$$

Thus, the boundary condition (9) is satisfied and condition (10) holds if

$$M(\theta)T(j)E = 0, \qquad j > 0, \tag{24}$$

where $\{T(j)\}$ is a solution of (18) and

$$M(\theta) = C^{-1}B\sin(N\theta) + N\left[\cos(N\theta) - \sin((N-1)\theta)\right]I. \tag{25}$$

In order to solve (18), let us consider the algebraic matrix equation

$$W^2 - (2I + \rho A)W + I = 0, \qquad W \in \mathbb{C}^{m\times m}. \tag{26}$$

If A has positive eigenvalues, taking

$$r < \frac{1}{\sqrt{\rho(A)}} = \min\left\{a^{-1/2};\ a \in \sigma(A)\right\} \tag{27}$$

and using that $\rho = -4r^2\sin^2(\theta/2)$, it follows that matrices

$$W_0 = I + \frac{\rho A}{2} + \sqrt{\left(I + \frac{\rho A}{2}\right)^2 - I}, \qquad W_1 = I + \frac{\rho A}{2} - \sqrt{\left(I + \frac{\rho A}{2}\right)^2 - I} \tag{28}$$

are solutions of (20) such that

$$W_0 - W_1 = 2\sqrt{\left(I + \frac{\rho A}{2}\right)^2 - I} \text{ is invertible.} \tag{29}$$

In fact, by the spectral mapping theorem [15, p. 524], one gets

$$\sigma(W_0 - W_1) = \left\{2\sqrt{\left(1 + \frac{\rho a}{2}\right)^2 - 1};\ a \in \sigma(A)\right\} \tag{30}$$

and

$$\left(1 + \frac{\rho a}{2}\right)^2 - 1 = \frac{\rho^2 a^2}{4} + \rho a = \rho a\left(1 + \frac{\rho a}{4}\right) \neq 0. \tag{31}$$

By [19,20], condition (29) means that the pair $\{W_0, W_1\}$ is a complete set of solutions of (26). Hence, the general solution of (18) is given by

$$T(j) = W_0^j P + W_1^j Q, \qquad P, Q \in \mathbb{C}^{m\times m}. \tag{32}$$

By the properties of the matrix functional calculus, both matrices W_0 and W_1 are polynomials in the matrix A of degree $p - 1$, where

$$p \text{ is the degree of the minimal polynomial of } A. \tag{33}$$

Hence, condition (24) is equivalent to the condition

$$M(\theta)A^j(P, Q) = 0, \qquad 0 \le j < p, \quad P, Q \in \mathbb{C}^m. \tag{34}$$

Thus, the boundary value problem (8)–(10) admits nontrivial solutions of the form (14) if there are vectors P, Q nonsimultaneously zero, satisfying (34). A necessary condition to have eigenfunctions is that matrix $M(\theta)$ be singular and, by (25), this occurs if the matrix

$$C^{-1}B + \frac{N\left[\sin(N\theta) - \sin((N-1)\theta)\right]}{\sin(N\theta)}I \text{ is singular.} \tag{35}$$

Let us assume that

$$\text{there exists } \mu \in \sigma\left(-C^{-1}B\right) \cap \mathbb{R}. \tag{36}$$

Note that condition (35) holds if there are solutions θ of the scalar equation

$$\frac{N\left[\sin(N\theta) - \sin((N-1)\theta)\right]}{\sin(N\theta)} = \mu. \tag{37}$$

Equation (37) is equivalent to

$$(2N - \mu)\, tg\left(\frac{\theta}{2}\right) = \mu tg\left(\left(N - \frac{1}{2}\right)\theta\right). \tag{38}$$

If $\mu \neq 0$, it is easy to show that there exists a root θ_l of (38) in the interval $I(l) =](2l-1)\pi/(2N-1), (2l+1)\pi/(2N-1)[$ for $1 \leq l \leq N-1$. If $\mu = 0$, then B is singular and equation (37) is equivalent to the equation $\sin(N\theta) = \sin((N-1)\theta)$, having solutions $\theta_l = (2l-1)\pi/(2N-1)$ for $l = 1, 2, \dots, N-1$. Let $G(\mu) = C^{-1}B + \mu I$ and note that condition (34) is equivalent to

$$G(\mu)A^j(P,Q) = 0, \qquad 0 \leq j < p. \tag{39}$$

If we define the block matrix $\tilde{G}(\mu)$ by

$$\tilde{G}(\mu) = \begin{bmatrix} G(\mu) \\ G(\mu)A \\ \vdots \\ G(\mu)A^{p-1} \end{bmatrix}, \tag{40}$$

then (39) can be written in the form

$$\tilde{G}(\mu)(P,Q) = 0. \tag{41}$$

By [12, p. 24], the algebraic system (41) admits nonzero solutions (P,Q) if $\operatorname{rank}\tilde{G}(\mu) < m$, and in this case, the solution set of (41) is expressed by

$$(P,Q) = \left(I - \tilde{G}(\mu)^{\dagger}\tilde{G}(\mu)\right)(P_0, Q_0), \qquad P_0, Q_0 \in \mathbb{C}^m.$$

Summarizing, the following result has been established.

THEOREM 1. *Let A, C be matrices satisfying (6),(7) and let μ be a real number satisfying (36). Let $r > 0$ satisfy (27) and let θ_l be solutions of (38) in $I(l) =](2l-1)\pi/(2N-1), (2l+1)\pi/(2N-1)[$, for $1 \leq l \leq N-1$, if $\mu \neq 0$; and $\theta_l = (2l-1)\pi/(2N-1)$, for $1 \leq l \leq N-1$, if $\mu = 0$. Let $\rho_l = -4r^2\sin^2(\theta_l/2)$, $M(\theta_l)$ be defined by (25), $G(\mu) = C^{-1}B + \mu I$, and $\tilde{G}(\mu)$ defined by (40). If $\operatorname{rank}\tilde{G}(\mu) < m$, then the boundary problem (8)–(10) admits nonzero solutions defined by*

$$U_l(i,j) = \left(W_0^j P_l + W_1^j Q_l\right)\sin(i\theta_l), \qquad 1 \leq i \leq N-1, \quad j > 0, \tag{42}$$

where $1 \leq l \leq N-1$,

$$\begin{aligned} W_0 &= I - 2Ar^2\sin^2\left(\frac{\theta_l}{2}\right) + \sqrt{\left(I - 2Ar^2\sin^2\left(\frac{\theta_l}{2}\right)\right)^2 - I}, \\ W_1 &= I - 2Ar^2\sin^2\left(\frac{\theta_l}{2}\right) - \sqrt{\left(I - 2Ar^2\sin^2\left(\frac{\theta_l}{2}\right)\right)^2 - I}, \end{aligned} \tag{43}$$

$$(P_l, Q_l) = \left(I - \tilde{G}(\mu)^{\dagger}\tilde{G}(\mu)\right)(P,Q), \qquad P, Q \in \mathbb{C}^m. \tag{44}$$

4. THE MIXED PROBLEM

Let us assume the hypotheses and the notation of Section 3. By superposition, we seek a candidate solution of problem (8)–(12) of the form

$$U(i,j) = \sum_{l=1}^{N-1} \left(W_0^j P_l + W_1^j Q_l \right) \sin(i\theta_l), \qquad 1 \le i \le N-1, \quad j > 0, \tag{45}$$

where vectors P_l, Q_l lie in $\operatorname{Ker} \tilde{G}(\mu)$ and must be chosen so that conditions (11) and (12) hold.

In order to identify vectors P_l, Q_l appearing in (45), let us consider the scalar discrete Sturm-Liouville problem

$$\begin{aligned} & h(i+1) - \left(2 + \frac{\rho}{r^2}\right) h(i) + h(i-1) = 0, \qquad 0 < i < N, \\ & h(0) = 0, \qquad \mu h(N) - N\left[h(N) - h(N-1)\right] = 0, \end{aligned} \tag{46}$$

whose eigenfunctions set is $\{\sin(i\theta_l)\}_{l=1}^{N-1}$. By imposing the initial conditions (11) and (12) to the sequence $\{U(i,j)\}$ defined by (45), it follows that

$$F(i) = \sum_{l=1}^{N-1} (P_l + Q_l) \sin(i\theta_l), \tag{47}$$

$$kV(i) + F(i) = \sum_{l=1}^{N-1} \{W_0 P_l + W_1 Q_l\} \sin(i\theta_l). \tag{48}$$

By the theory of discrete Fourier series [21, Chapter 11], working component by component, from (47), it follows that

$$P_l + Q_l = \frac{\sum\limits_{l=1}^{N-1} \sin(i\theta_l) F(i)}{\sum\limits_{i=1}^{N-1} \sin^2(i\theta_l)}, \tag{49}$$

$$W_0 P_l + W_1 Q_l = \frac{\sum\limits_{l=1}^{N-1} \{kV(i) + F(i)\} \sin(i\theta_l)}{\sum\limits_{i=1}^{N-1} \sin^2(i\theta_l)}. \tag{50}$$

Premultiplying (49) by W_0 and subtracting (50), one gets

$$(W_0 - W_1) Q_l = \frac{\sum\limits_{l=1}^{N-1} \{(W_0 - I) F(i) - kV(i)\} \sin(i\theta_l)}{\sum\limits_{i=1}^{N-1} \sin^2(i\theta_l)}. \tag{51}$$

Premultiplying (49) by $(-W_1)$ and adding (50), it follows that

$$(W_0 - W_1) P_l = \frac{\sum\limits_{l=1}^{N-1} \{kV(i) - (W_1 - I) F(i)\} \sin(i\theta_l)}{\sum\limits_{i=1}^{N-1} \sin^2(i\theta_l)}. \tag{52}$$

Since $W_0 - W_1$ is invertible, from (51),(52), one gets

$$P_l = \frac{(W_0 - W_1)^{-1} \sum_{l=1}^{N-1} \{kV(i) - (W_1 - I) F(i)\} \sin(i\theta_l)}{\sum_{i=1}^{N-1} \sin^2(i\theta_l)}, \tag{53}$$

$$Q_l = \frac{(W_0 - W_1)^{-1} \sum_{l=1}^{N-1} \{(W_0 - I) F(i) - kV(i)\} \sin(i\theta_l)}{\sum_{i=1}^{N-1} \sin^2(i\theta_l)}. \tag{54}$$

Since by (28),(29), matrices W_0, W_1, and $(W_0 - W_1)^{-1}$ are polynomials in the matrix A of degree $p-1$, by (40), (53), and (54), vectors P_l, Q_l satisfy (41) if

$$\{F(i), V(i),\ 1 \le i \le N-1\} \subset \operatorname{Ker} G(\mu) \tag{55}$$

and

$$\operatorname{Ker} G(\mu) \text{ is an invariant subspace of } A. \tag{56}$$

Condition (56) can be expressed in the form

$$G(\mu)A\left(I - G(\mu)^{\dagger}G(\mu)\right) = 0. \tag{57}$$

THEOREM 2. *Under hypotheses (55) and (57), together with those of Theorem 1, the sequence $\{U(i,j)\}$ given by (45) defines a solution of the mixed problem (8)–(12).*

By the spectral mapping theorem [15, p. 569], the spectrum of matrices W_0 and W_1 defined by (43) are given by

$$1 - 2ar^2 \sin^2\left(\frac{\theta_l}{2}\right) \pm \sqrt{\left(1 - 2ar^2 \sin^2\left(\frac{\theta_l}{2}\right)\right)^2 - 1}, \qquad a \in \sigma(A).$$

Note that $|1 - 2ar^2 \sin^2(\theta_l/2) \pm \sqrt{(1 - 2ar^2 \sin^2(\theta_l/2))^2 - 1}| = 1$, because $1 - 2ar^2 \sin^2(\theta_l/2) < 1$. Hence, $\rho(W_0) = \rho(W_1) = 1$.

We are now concerned with the stability of the solution given by (45), (53), and (54). This means that given (X, T), where $X = i/N = ih_0$, $h_0 = 1/N$ fixed, $T = Jk$ finite, we are concerned with the behaviour of $\{U(i,j)\}$ as $k \to 0$, i.e., $j \to \infty$, but with $Jk = T$ fixed. By (7) and (43), it follows that

$$\begin{aligned} &\|W_0\| \le 1 + O(r), \quad \|W_0 - I\| = O(r), \quad \left\|(W_0 - W_1)^{-1}\right\| = O\left(r^{-1}\right), \quad r \to 0, \\ &\|W_1\| \le 1 + O(r), \quad \|W_1 - I\| = O(r), \quad r \to 0. \end{aligned} \tag{58}$$

Fixed $h_0 = 1/N$, since $r = k/h$, (58) means

$$\begin{aligned} &\|W_0\| \le 1 + O(k), \quad \|W_0 - I\| = O(k), \quad \left\|(W_0 - W_1)^{-1}\right\| = O\left(k^{-1}\right), \quad k \to 0, \\ &\|W_1\| \le 1 + O(k), \quad \|W_1 - I\| = O(k), \quad k \to 0. \end{aligned} \tag{59}$$

By (53), (54), and (59), it follows that

$$\|P_l\| = O(1), \quad \|Q_l\| = O(1), \qquad k \to 0. \tag{60}$$

By (60), it follows that $\{U(i,j)\}$ remains bounded as j increases, if the numbers (see [18, p. 106])

$$\left\|W_0^j\right\|, \left\|W_l^j\right\| \text{ remains bounded as } j \to \infty,\ k \to 0,\ 0 < j \le J,\ Jk = T.$$

Note that since $\|W_0\| \le 1 + O(k)$, let $\|W_0\| \le 1 + kS$, for some positive constant, then, for $0 \le j \le J$, one gets

$$\left\|W_0^j\right\| \le \|W_0\|^j \le (1+O(k))^j \le (1+O(k))^J \le e^{JO(k)} \le e^{JkS} = e^{TS}.$$

The same occurs for $\|W_1^j\|$. Hence, by (60), the solution defined by (45), (53), and (54) is stable, i.e.,

$$\begin{aligned} &\|U(i,j)\| = O(1), && k \to 0, && h_0 = \frac{1}{N} \text{ fixed}, \\ &1 < i < N, && j \to \infty, && t = jk \text{ finite}. \end{aligned} \tag{61}$$

Summarizing, the following result has been established.

THEOREM 3. *Under the hypothesis of Theorem 2, the solution $\{U(i,j)\}$ defined by (45), (53), and (54) is stable in the sense of (61), for $r > 0$ satisfying*

$$r < [\rho(A)]^{-1/2}. \tag{62}$$

REMARK 1. If matrix A is symmetric, then W_0 and W_1 are also symmetric. Then $\|W_0\| = \rho(W_0) = 1 = \rho(W_1) = \|W_1\|$ and independently of t, the solution given by Theorem 3 remains bounded as $j \to \infty$.

EXAMPLE 1. Consider problem (8)–(12) where $m = 3$ and matrices

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

The matrix C is invertible and $-C^{-1}B$ is given by

$$-C^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

In this case, we have $\sigma(A) = \{1, 2\}$ and $\mu = 1 \in \sigma(-C^{-1}B)$. The matrix $G(1) = C^{-1}B + I$ and $G(1)^\dagger$ take the form

$$G(1) = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}, \qquad G(1)^\dagger = \begin{bmatrix} 0 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}.$$

The subspace $\operatorname{Ker} G(1)$ is invariant by A because

$$G(1)A\left[I - G(1)^\dagger G(1)\right] = 0.$$

If we consider problem (8)–(12) with the above data and initial functions $F(i), V(i)$ lying in $\operatorname{Ker} G(1)$, then the solution given by (45), (53), and (54) is stable for $r < [\rho(A)]^{-12} = 1/\sqrt{2}$. Let $f = (f_1, f_2, f_3)^\top$, $v = (v_1, v_2, v_3)^\top$, then

$$\left\{f\left(\frac{i}{N}\right), v\left(\frac{i}{N}\right); 1 \le i \le N\right\} \subset \operatorname{Ker} G(1)$$

means that

$$f_1\left(\frac{i}{N}\right) = f_3\left(\frac{i}{N}\right), \quad v_1\left(\frac{i}{N}\right) = v_3\left(\frac{i}{N}\right), \qquad 1 \le i \le N.$$

Thus, if f and v satisfy the last condition, the solution of problem (8)–(12) given by Theorem 3 is stable for $r < 1/\sqrt{2}$.

5. THE PROJECTION METHOD

This section is concerned with the construction of solutions of problem (8)–(12) for functions F, V, satisfying more general conditions than those of Section 4. Suppose that

$$\{\mu(1), \dots, \mu(q)\} \subset \sigma\left(-C^{-1}B\right) \cap \mathbb{R}, \tag{63}$$

where $\mu(i) \neq \mu(j)$ for $1 \leq i,\, j \leq q,\, i \neq j$, and let $G(\mu(h))$ be the matrix

$$G(\mu(h)) = \left(C^{-1}B + \mu(h)I\right), \qquad 1 \leq h \leq q. \tag{64}$$

As polynomials $x - \mu(h)$ are mutually coprime, by the decomposition theorem [22, p. 536], if $R(x)$ is the polynomial

$$R(x) = (x - \mu(1))\,(x - \mu(2)) \cdots (x - \mu(q)), \tag{65}$$

then

$$S = \operatorname{Ker} R\left(-C^{-1}B\right) = \ker G(\mu(1)) \oplus \cdots \oplus \operatorname{Ker} G(\mu(q)). \tag{66}$$

Assume that

$$\{F(i),\ V(i),\ 1 \leq i \leq N-1\} \subset S. \tag{67}$$

Now we define the projection of functions F, V on the subspace $\operatorname{Ker} G(\mu(h))$. Since polynomial

$$q_h(x) = \prod_{\substack{k=1\\ k\neq h}}^{q} (x - \mu(k)) \tag{68}$$

is coprime, by Bezout's theorem [22, p. 538], taking

$$\alpha_h = \left(\prod_{\substack{s=1\\ s\neq h}}^{q} (\mu(h) - \mu(s))\right)^{-1}, \qquad 1 \leq h \leq q, \tag{69}$$

one gets

$$1 = \sum_{k=1}^{q} \alpha_k q_k(x). \tag{70}$$

By applying the matrix functional calculus on matrix $(-C^{-1}B)$, by (64), (65), and (70), it follows that

$$R(\mu(s)) = G(\mu(1)) \cdots G(\mu(s-1))\, G(\mu(s+1)) \cdots G(\mu(q)), \tag{71}$$

$$I = (-1)^q \sum_{s=1}^{q} \alpha_s R(\mu(s)). \tag{72}$$

Hence, the projections of $F(i) = f(i/N)$, $V(i) = v(i/N)$ on $\operatorname{Ker} G(\mu(s))$ take the form

$$\begin{aligned} F_s &= (-1)^s \alpha_s R(\mu(s))\, F(i) \in \operatorname{Ker} G(\mu(s)), \\ V_s &= (-1)^s \alpha_s R(\mu(s))\, V(i) \in \operatorname{Ker} G(\mu(s)), \qquad 1 \leq s \leq q, \end{aligned} \tag{73}$$

where

$$F(i) = \sum_{h=1}^{q} F_h(i), \quad V(i) = \sum_{h=1}^{q} V_h(i), \qquad 1 \leq i \leq N-1. \tag{74}$$

Let us assume that projections F_s, V_s, $1 \le s \le q$, satisfy

$$\begin{gathered} \{F_s(i), G_s(i),\ 1 \le i \le N-1\} \subset \operatorname{Ker} G(\mu(s)), \\ G(\mu(s))\, A\left(I - G(\mu(s))^{\dagger}\, G(\mu(s))\right) = 0, \end{gathered} \tag{75}$$

and let $U(\cdot,\cdot,s)$ be the solution of the mixed problem given by Seotion 4 and associated to the eigenvalue $\mu(s)$ instead of μ; i.e.,

$$U(i,j,s) = \sum_{l=1}^{N-1} \left\{ W_{0,s}^{j} P_l(s) + W_{1,s}^{j} Q_l(s) \right\} \sin(i\theta_l(s)), \tag{76}$$

where $\{\theta_l(s);\ 1 \le l \le N-1\}$ are solutions of equation

$$\begin{gathered} (2N - \mu(s))tg\left(\frac{\theta(s)}{2}\right) = \mu(s) tg\left(\left(N - \frac{1}{2}\right)\theta(s)\right), \\ \theta(s) \in \left[\frac{(2l-1)\pi}{2N-1}, \frac{(2l+1)\pi}{2N-1}\right], \qquad 1 \le l \le N-1, \end{gathered} \tag{77}$$

$$\begin{aligned} P_l(s) &= \frac{(W_{0,s} - W_{1,s})^{-1} \sum_{i=1}^{N-1} \{kV_s(i) - (W_{1,s} - I)\, F_s(i)\} \sin(i\theta_l(s))}{\sum_{i=1}^{N-1} \sin^2(i\theta_l(s))}, \\ Q_l(s) &= \frac{(W_{0,s} - W_{1,s})^{-1} \sum_{i=1}^{N-1} \{(W_{0,s} - I)\, F_s(i) - kV_s(i)\} \sin(i\theta_l(s))}{\sum_{i=1}^{N-1} \sin^2(i\theta_l(s))}, \end{aligned} \tag{78}$$

$$\begin{aligned} W_{0,s} &= I - 2Ar^2 \sin^2\left(\frac{\theta_l(s)}{2}\right) + \sqrt{\left(I - 2Ar^2 \sin^2\left(\frac{\theta_l(s)}{2}\right)\right)^2 - I}, \\ W_{1,s} &= I - 2Ar^2 \sin^2\left(\frac{\theta_l(s)}{2}\right) - \sqrt{\left(I - 2Ar^2 \sin^2\left(\frac{\theta_l(s)}{2}\right)\right)^2 - I}. \end{aligned} \tag{79}$$

By construction,

$$U(i,j) = \sum_{s=1}^{q} U(i,j,s) \tag{80}$$

is a solution of (8)–(12), that is, stable if

$$r < [\rho(A)]^{-1/2}, \quad h_0 = \frac{1}{N} \text{ fixed}, \quad j \to \infty, \qquad 1 \le i \le N-1. \tag{81}$$

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