Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Math. Anal. Appl. 303 (2005) 492–498

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On explicit conditions for the asymptotic stability of linear higher order difference equations

Eduardo Liz [1]

Departamento de Matemática Aplicada II, ETSI Telecomunicación, Campus Marcosende, Universidad de Vigo, 36280 Vigo, Spain

Received 6 April 2004

Submitted by R.P. Agarwal

Abstract

We derive some explicit sufficient conditions for the asymptotic stability of the zero solution in a general linear higher order difference equation, and compare our estimations with other related results in the literature. Our main result also applies to some nonlinear perturbations satisfying a kind of sublinearity condition.

Keywords: Linear difference equations; Asymptotic stability

1. Introduction

As it is well known, the asymptotic stability of the zero solution of the linear difference equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k}, \quad n \geqslant 0, \tag{1.1}$$

is determined by the location of the roots of the associated characteristic equation

E-mail address: eliz@dma.uvigo.es.

[1] Supported in part by MCT (Spain) and FEDER, under project BFM2001-3884-C02-02.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.08.048

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0. \tag{1.2}$$

Thus, for each particular choice of the coefficients a_i, $i = 0, \ldots, k$, one can use the so-called Schur–Cohn criterion (see, e.g., [3,8]). However, with this method, it is very difficult to get explicit conditions for a general form of Eq. (1.1) depending on the coefficients. This kind of explicit conditions are of special importance in the applications, where the coefficients are meaningful parameters of the model. A famous example is the Clark model [1],

$$x_{n+1} = \alpha x_n + f(x_{n-k}), \quad n \geqslant 0, \tag{1.3}$$

where $\alpha \in (0, 1)$ and f is a continuous real function. This equation is of special interest in population dynamics (see also [4,7,8] and references therein). If $\bar{x}$ is a positive equilibrium of (1.3) and f is differentiable at $\bar{x}$, then the linearization about $\bar{x}$ is

$$x_{n+1} = \alpha x_n + \beta x_{n-k}, \quad n \geqslant 0, \tag{1.4}$$

where $\beta = f'(\bar{x})$. Clark proved that $\bar{x}$ is asymptotically stable if $|\beta| < 1 - \alpha$, that is, if $|\alpha| + |\beta| < 1$. In fact, this condition is generalized to Eq. (1.1) (see, e.g., [8, Remark 1.3.1]): the zero solution of (1.1) is asymptotically stable if

$$\sum_{i=0}^{k} |a_i| < 1. \tag{1.5}$$

Actually, it can be also extended to the case of variable coefficients

$$x_{n+1} = a_0(n)x_n + a_1(n)x_{n-1} + \cdots + a_k(n)x_{n-k}, \quad n \geqslant 0. \tag{1.6}$$

Indeed, the zero solution of (1.6) is asymptotically stable if $\sup_{n\geqslant 0} \sum_{i=0}^{k} |a_i(n)| < 1$ (see [12, p. 658]).

While condition (1.5) is optimal when $a_i > 0$ for all $i = 0, 1, \ldots, k$ (see, e.g., [15]), it is far from being sharp when not all the coefficients are positive. For Eq. (1.4), the necessary and sufficient condition for the asymptotic stability of the zero solution was found in [10] (see also [11] for $\alpha = 1$). However, this task for the general case of Eq. (1.1) seems to be extremely complicated.

In this paper, we derive some explicit sufficient conditions for the asymptotic stability of the zero solution in (1.1) when some coefficients are negative. Moreover, our main theorem also applies to some nonlinear perturbations of (1.1), complementing the results in [14]. We discuss the relation of our results with other papers in the literature, and establish some comparisons for concrete examples.

2. Main results

First, we assume that $a_0 = 1$, that is, (1.1) has the form

$$x_{n+1} - x_n = a_1 x_{n-1} + \cdots + a_k x_{n-k}, \quad n \geqslant 0. \tag{2.1}$$

Clearly, (1.5) does not work here. In the literature, we can find some results for (2.1) when it has only two terms of different sign in the right-hand side, that is, for equation

$$x_{n+1} - x_n = qx_{n-m} - px_{n-k}, \quad n \geqslant 0, \tag{2.2}$$

where p, q are positive real constants, and $m, k \geqslant 1$ are integers. In [6] (see also [5]), Győri et al. proved that under condition

$$0 < p < \frac{k^k}{(k+1)^{k+1}}, \tag{2.3}$$

the zero solution of (2.2) is asymptotically stable if and only if $q - p < 0$. As noticed in [14], condition (2.3) is required to guarantee that the fundamental solution of (2.2) is positive for all $n \geqslant 0$. (We recall that the fundamental solution $\{v_n\}$ is defined as the solution of (2.2) with initial conditions $v_0 = 1$, $v_{-i} = 0$, $i = 1, \ldots, \max\{k, m\}$.) The result in [6] was generalized in [14] to the general case of Eq. (2.1). In concrete, the following result is a direct consequence of Corollary 2 and Remark 2 in [14, p. 1198] (see also [9]).

Proposition 2.1. *Assume that*

$$\sum_{i=1}^{k} a_i^- \frac{(k+1)^{i+1}}{k^i} \leqslant 1, \tag{2.4}$$

where $a_i^- = \max\{0, -a_i\}$. *Then the zero solution of Eq.* (2.1) *is asymptotically stable if* $\sum_{i=1}^k a_i < 0$.

Coming back to Eq. (2.2), when (2.3) does not hold, it was proven in [8, Theorem 6.9.2] that the zero solution is still asymptotically stable if

$$kp < 1 \quad \text{and} \quad p\frac{1-kp}{1+kp} > q. \tag{2.5}$$

Our main result in this paper allows us to generalize this result to Eq. (2.1) when (2.4) does not hold. We also give a partial answer to the open problem 6.9.1 in [8, p. 171].

We are now in a position to state the main result of this section, which is valid for the following perturbation of (2.1):

$$x_{n+1} - x_n = \sum_{i=1}^{m} (-a_i) x_{n-k_i} + f(n, x_n, \ldots, x_{n-r}), \quad n \geqslant 0, \tag{2.6}$$

where $a_i > 0$, $i = 1, \ldots, m$, are real constants, $k_m > k_{m-1} > \cdots > k_1 \geqslant 1$ are integers, and f satisfies

$$\bigl|f(n, x_0, \ldots, x_r)\bigr| \leqslant b \max\bigl\{|x_0|, \ldots, |x_r|\bigr\}, \tag{2.7}$$

for some constant $b \geqslant 0$ and all $n \in \mathbb{N}$, $(x_0, \ldots, x_r) \in \mathbb{R}^{r+1}$.

Remark 2.2. Notice that (2.1) is a particular case of (2.6), taking $-a_{k_1}, \ldots, -a_{k_m}$ the negative coefficients in the right-hand side of (2.1) and $f(n, x_n, \ldots, x_{n-r})$ the part corresponding to the positive coefficients. In this case, $b = \sum_{i=1}^k a_i^+$, where $a_i^+ = \max\{0, a_i\}$.

For the sake of completeness, we enunciate here a result from [12].

Theorem 2.3 [12, Theorem 2]. *Assume that* $0 < a \leqslant 1$ *and there exists a positive constant* $c < a$ *such that*

$$\bigl|g(n,\dots,n-l,x_0,\dots,x_l)\bigr| \leqslant c\max\bigl\{|x_0|,\dots,|x_l|\bigr\}, \tag{2.8}$$

for all $n \in \mathbb{N}$, $(x_0,\dots,x_l) \in \mathbb{R}^{l+1}$. *Then there exists* $\lambda_0 \in (0,1)$ *such that*

$$|x_n| \leqslant \Bigl(\max_{-l\leqslant i\leqslant 0}\bigl\{|x_i|\bigr\}\Bigr)\lambda_0^n, \quad n \geqslant 0,$$

for every solution $\{x_n\}$ *of*

$$x_{n+1} = (1-a)x_n + g(n,\dots,n-l,x_n,x_{n-1},\dots,x_{n-l}). \tag{2.9}$$

Theorem 2.4. *Assume that* (2.7) *holds for some* $b \geqslant 0$, *and*

$$\frac{a(1-\sum_{i=1}^m k_i a_i)}{1+\sum_{i=1}^m k_i a_i} > b, \tag{2.10}$$

where $a = \sum_{i=1}^m a_i > 0$. *Then, the zero solution of* (2.6) *is globally exponentially stable.*

Proof. Notice that (2.6) can be rewritten in the form

$$\begin{aligned} x_{n+1} &= (1-a)x_n + \sum_{i=1}^m a_i(x_n - x_{n-k_i}) + f(n,x_n,\dots,x_{n-r}) \\ &= (1-a)x_n + \sum_{i=1}^m a_i \sum_{j=n-k_i}^{n-1}(x_{j+1}-x_j) + f(n,x_n,\dots,x_{n-r}). \end{aligned}$$

Using again (2.6) for each j in the last expression, we have

$$\begin{aligned} x_{n+1} &= (1-a)x_n - \sum_{i=1}^m a_i \sum_{j=n-k_i}^{n-1}\Biggl[\sum_{s=1}^m a_s x_{j-k_s} - f(j,x_j,\dots,x_{j-r})\Biggr] \\ &\quad + f(n,x_n,\dots,x_{n-r}) \\ &= (1-a)x_n + g(n,\dots,n-l,x_n,\dots,x_{n-l}), \end{aligned}$$

where $l = \max\{2k_m, k_m + r\}$, and g satisfies

$$\bigl|g(n,\dots,n-l,x_0,\dots,x_l)\bigr| \leqslant \Biggl[b + (a+b)\sum_{i=1}^m k_i a_i\Biggr]\max\bigl\{|x_0|,\dots,|x_l|\bigr\},$$

for all $n \in \mathbb{N}$ and $(x_0,\dots,x_l) \in \mathbb{R}^{l+1}$.

The result follows from Theorem 2.3, by observing that

$$b + (a+b)\sum_{i=1}^m k_i a_i < a$$

if and only if (2.10) holds. (Notice that (2.10) also implies that $a < 1$.) □

Taking into account Remark 2.2, we get

Corollary 2.5. *The zero solution of Eq.* (2.1) *is asymptotically stable if*

$$\sum_{i=1}^{k} a_i^+ < \frac{\sum_{i=1}^{k} a_i^- (1 - \sum_{i=1}^{k} i a_i^-)}{1 + \sum_{i=1}^{k} i a_i^-}. \tag{2.11}$$

Clearly, condition (2.11) becomes (2.5) for the particular case of Eq. (2.2). Moreover, a direct application of Theorem 2.4 shows that the zero solution of equation

$$x_{n+1} - x_n + p x_{n-k} = f(x_{n-m}), \quad n \geqslant 0,$$

with f a continuous real function, is globally exponentially stable if $|f(x)| \leqslant qx$ for all $x \in \mathbb{R}$, and (2.5) holds. This gives a partial answer to Open problem 6.9.1 in [8, p. 171].

On the other hand, if all coefficients a_i in (2.1) are negative, Corollary 2.5 gives the asymptotic stability of the zero solution if

$$\sum_{i=1}^{k} i|a_i| < 1, \tag{2.12}$$

which is the result obtained in [2, Remark 1, p. 83]. As noticed there, for the particular case of equation

$$x_{n+1} - x_n + p x_{n-1} = 0, \quad n \geqslant 0, \tag{2.13}$$

condition (2.12) is sharp, since the zero solution of (2.13) is asymptotically stable if and only if $p < 1$. Notice that the application of Proposition 2.1 gives the asymptotic stability in (2.13) only for $p \leqslant 1/4$.

Next we briefly discuss the case when $a_0 \neq 1$ in (1.1). In this case, the result of Proposition 2.1 remains valid if condition (2.4) is replaced by the existence of a constant $\mu \in (0, 1)$ such that

$$\mu + \sum_{i=1}^{k} a_i^- \mu^{-i} \leqslant a_0 \tag{2.14}$$

(see, for example, [9,14]).

When (2.14) does not hold, we can apply Theorem 2.4 by observing that (1.1) can be written in the form

$$x_{n+1} - x_n = (a_0 - 1)x_n + \sum_{i=1}^{k} a_i x_{n-i}, \quad n \geqslant 0.$$

Hence, we have the following

Corollary 2.6. *The zero solution of Eq.* (1.1) *is asymptotically stable if*

$$|a_0 - 1| + \sum_{i=1}^{k} a_i^+ < \frac{\sum_{i=1}^{k} a_i^- (1 - \sum_{i=1}^{k} i a_i^-)}{1 + \sum_{i=1}^{k} i a_i^-}. \tag{2.15}$$

In the particular case when $a_0 > 0$ and $a_i < 0$ for all $i = 1, \dots, k$, we can also apply Theorem 4 in [13]. For it, observe that (1.1) can be written as

$$x_{n+1} - x_n = -(1 - a_0)x_n - bf(x_{n-1}, \dots, x_{n-k}), \quad n \geqslant 0, \tag{2.16}$$

where $b = \sum_{i=1}^k |a_i|$, and $f(x_1, \dots, x_k) = \sum_{i=1}^k (|a_i|/b)x_i$ satisfies condition

$$\min\{x_1, \dots, x_k\} \leqslant f(x_1, \dots, x_k) \leqslant \max\{x_1, \dots, x_k\},$$

for all $(x_1, \dots, x_k) \in \mathbb{R}^k$. Thus, we have the following result.

Theorem 2.7. *The zero solution of Eq.* (1.1) *is asymptotically stable if $a_i < 0$ for all $i = 1, \dots, k$, and either one of the following conditions holds*:

(C1) $\sum_{i=1}^k |a_i| \leqslant a_0 \leqslant 1$ *and* $0 < k \sum_{i=1}^k |a_i| < 1$;
(C2) $1 < a_0 < 1 + \sum_{i=1}^k |a_i|$ *and*

$$k \sum_{i=1}^k |a_i| < \frac{1 - a_0 + \sum_{i=1}^k |a_i|}{-1 + a_0 + \sum_{i=1}^k |a_i|}.$$

Example 2.8. As a sort of comparison among the above stated results, we consider the linearized Clark equation with $f'(\bar{x}) < 0$ (the usual situation in its applications to population dynamics), that is,

$$x_{n+1} = \alpha x_n - \beta x_{n-k}, \quad n \geqslant 0, \tag{2.17}$$

where $\alpha \in (0, 1)$ and $\beta > 0$.

It is not difficult to check that, in this case, (2.14) holds if and only if

$$\alpha^{k+1} \geqslant \beta \frac{(k+1)^{k+1}}{k^k}. \tag{2.18}$$

On the other hand, Corollary 2.6 ensures the asymptotic stability of the zero solution in (2.17) if

$$\beta < \frac{1}{k} \quad \text{and} \quad 1 - \alpha < \beta \frac{1 - k\beta}{1 + k\beta}. \tag{2.19}$$

One can check that (2.18) and (2.19) are not comparable. In this case, the application of condition (C1) in Theorem 2.7 gives a better result: the zero solution of Eq. (2.17) is asymptotically stable if $\beta \leqslant \alpha$ and $\beta k < 1$. As it was already noted, this result is sharp for $k = 1$ and $\alpha = 1$.

References

[1] C.W. Clark, A delayed recruitment model of population dynamics with an application to baleen whale populations, J. Math. Biol. 3 (1976) 381–391.
[2] K. Cooke, I. Győri, Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments, Comput. Math. Appl. 28 (1994) 81–92.
[3] S. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.

[4] H.A. El-Morshedy, The global attractivity of difference equations of nonincreasing nonlinearities with applications, Comput. Math. Appl. 45 (2003) 749–758.
[5] I. Győri, G. Ladas, Oscillation Theory of Delay Differential Equations, Oxford Science, New York, 1991.
[6] I. Győri, G. Ladas, P.N. Vlahos, Global attractivity in a delay differential equation, Nonlinear Anal. 17 (1991) 473–479.
[7] I. Győri, S. Trofimchuk, Global attractivity and persistence in a discrete population model, J. Differ. Equations Appl. 6 (2000) 647–665.
[8] V.L. Kocić, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Math. Appl., vol. 256, Kluwer Academic, Dordrecht, 1993.
[9] U. Krause, M. Pituk, Boundedness and stability for higher order difference equations, J. Differ. Equations Appl. 10 (2004) 343–356.
[10] S.A. Kuruklis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, J. Math. Anal. Appl. 188 (1994) 719–731.
[11] S.A. Levin, R. May, A note on difference delay equations, Theor. Popul. Biol. 9 (1976) 178–187.
[12] E. Liz, J.B. Ferreiro, A note on the global stability of generalized difference equations, Appl. Math. Lett. 15 (2002) 655–659.
[13] E. Liz, A. Ivanov, J.B. Ferreiro, Discrete Halanay-type inequalities and applications, Nonlinear Anal. 55 (2003) 669–678.
[14] M. Pituk, Global asymptotic stability in a perturbed higher-order linear difference equation, Comput. Math. Appl. 45 (2003) 1195–1202.
[15] S. Stević, A note on the recursive sequence $x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \cdots + p_1 x_{n-k+1}$, Ukrainian Math. J. 55 (2003) 691–697.