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COUNTING TRITANGENT PLANES OF SPACE CURVES

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LET C be a smooth simple closed curve in $\mathbb{R}^3$. A *tritangent plane of* C is a plane in $\mathbb{R}^3$ which is tangent to C at exactly three points. A *stall* x of C is a point of C at which the torsion of C is zero. We will say that a stall x is *transverse* if the curvature of C is non-zero at x, the derivative of the torsion of C is non-zero at x, and the osculating plane P of C at x is transverse to C away from x. If x is a transverse stall of C then an interval of C about x lies on one side of the osculating plane P of C at x, so P intersects C at an even number $2n$ of points other than x. The integer $n = n(x, C)$ is the *index* of the transverse stall x of C.

Let $C^\infty(S^1, \mathbb{R}^3)$ be the space of C^∞ maps α: $S^1 \to \mathbb{R}^3$ with the Whitney topology.

THEOREM. *There is an open dense subset A of $C^\infty(S^1, \mathbb{R}^3)$ such that if $\alpha \in A$ then $C = \alpha(S^1)$ is a simple curve with a finite number $T(C)$ of tritangent planes and a finite number of stalls $x_1, \ldots, x_k$, all of which are transverse. If $\alpha \in A$ then*

$$T(C) \equiv \sum_{i=1}^{k} n(x_i, C) \quad (mod\ 2).$$

An explicit description of the set A is given in Section 1 below. This theorem generalizes the result of M. Freedman that a generic smooth closed space curve with nonvanishing torsion has an even number of tritangent planes [4].

To prove the theorem, we consider the classical dual surface C^* consisting of all planes in $\mathbb{R}^3$ tangent to C. We use the theory of singularities of maps to analyze the singularities of C^*. The tritangent planes of C correspond to triple points of C^*, and the stalls of C correspond to swallowtail points of C^* (cf. [3]). Then we count the triple points of C^* using a generalization of the techniques of [1].

As this paper was in its final stages of preparation, we received a letter from Tetsuya Ozawa announcing an integer formula for the triple tangent planes. Properly indexed, the sum of the tritangent planes equals the sum of the stalls, each stall counting $\pm n(x, C)$ times. Ozawa also has a formula relating the stalls and osculating planes which are tangent to the curve at two points.

§1. THE GENERIC SET A.

Let A be the subset of $C^\infty(S^1, \mathbb{R}^3)$ consisting of all maps α: $S^1 \to \mathbb{R}^3$ satisfying the following seven conditions.

Local conditions:
1. α is regular, i.e. $\alpha'(t) \neq 0$ for all $t \in S^1$.
2. The curvature κ of α is never zero.
3. The zeros of the torsion τ of α are nondegenerate, i.e., $\tau(t) = 0$ implies $\tau'(t) \neq 0$.

Global conditions:
4. α is injective.
5. No plane P in $\mathbb{R}^3$ is tangent to α at more than three points. If P is tangent to α at three points then the three points are non-collinear.

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6. No plane is an osculating plane to the curve at two distinct points. If P is the osculating plane of α at $x = \alpha(t)$, then P is tangent to α at most at one other point y, and the point y does not lie on the tangent line to the curve at x.
7. If P is the osculating plane of α at $x = \alpha(t)$ and $\tau(t) = 0$, then P is tangent to α at no point other than x.

PROPOSITION 1. *A is an open dense set of $C^\infty(S^1, \mathbb{R}^3)$.*

Proof. The density of A follows by standard applications of the multi-jet form of the Thom transversality theorem. We will show that A is open as a corollary of Proposition 2.

§2. THE DUAL SURFACE C^*.

Let $C \subset \mathbb{R}^3$ be a smooth closed curve. We embed $\mathbb{R}^3$ in projective space $\mathbb{P}^3$ in the usual way $((x_1, x_2, x_3) \to [1, x_1, x_2, x_3])$, so that $\mathbb{P}^3$ is just $\mathbb{R}^3$ together with all the points in the projective plane at infinity. The dual projective space $(\mathbb{P}^3)^*$ is the space of all projective planes in $\mathbb{P}^3$, which correspond to all the affine planes in $\mathbb{R}^3$ plus the plane at infinity. Let $C^* \subset (\mathbb{P}^3)^*$ be the set of planes tangent to C. Then C^* is a ruled surface in $(\mathbb{P}^3)^*$ parametrized by C. A ruling of C^* consists of all the planes containing a fixed tangent line of C.

§3. THE FAMILY F_C.

Counting tritangent planes to a curve is a problem of the contact of planes with a curve. This can be attacked by studying the singularities of the restriction to C of a family of mappings on $\mathbb{R}^3$ whose fibres are planes. Planes with exceptional contact with C should create singularities in the new family of maps obtained by restriction to C.

We consider the family of orthogonal projections of the curve C to all lines through the origin in $\mathbb{R}^3$.

Let $\mathbb{P}^2$ be the space of lines through the origin in $\mathbb{R}^3$, and let $L(\mathbb{P}^2)$ be the tautological line bundle over $\mathbb{P}^2$:

$$L(\mathbb{P}^2) = \{(v, l) \in \mathbb{R}^3 \times \mathbb{P}^2 \mid v \in l\}.$$

For $l \in \mathbb{P}^2$ let $\Pi_l\colon C \to l$ be the restriction to C of the orthogonal projection of $\mathbb{R}^3$ to l. Define

$$F_C\colon\quad C \times \mathbb{P}^2 \to L(\mathbb{P}^2)$$

by $F_C(x, l) = (\Pi_l(x), l)$. Let Σ be the set of critical points of F_C. Then $(x, l) \in \Sigma$ if and only if the plane P through x perpendicular to l is tangent to C at x. Let $f_C\colon \Sigma \to (\mathbb{P}^3)^*$ take (x, l) to this plane P. We have a commutative diagram

$$\begin{tikzcd}
\Sigma \arrow[rr, "f_C"] \arrow[d, phantom, "\cap"] & & (\mathbb{P}^3)^* \arrow[d, phantom, "\cup\, i"] \\
C \times \mathbb{P}^2 \arrow[rr, "F_C"] \arrow[dr] & & L(\mathbb{P}^2) \arrow[dl] \\
& \mathbb{P}^2 &
\end{tikzcd}$$

where i sends (v, l) to the plane through the point v perpendicular to the line l. Clearly $\Sigma \cong C \times \mathbb{P}^1$ and $f_C(\Sigma) = C^*$.

PROPOSITION 2. *The map F_C is stable if and only if $\alpha \in A$, where $C = \alpha(S^1)$.*

Proof. We first show that the stability of F_C as a germ at every (x, l) is equivalent to conditions 1, 2, and 3 of the definition of A. We may assume that α is parametrized by arc length t. Choosing local coordinates on $L(\mathbb{P}^2)$, F_C has the form:

$$F_C(t, l) = (\alpha(t)\cdot v, l)$$

where v is a unit vector in the line l. (We may choose either unit vector as long as we are consistent.)

Since F_C is the identity on the base $\mathbb{P}^2$, so that the rank of $DF_C \geq 2$, the only singularities which can occur are Morin singularities. The fold locus is given by $\alpha'(t)\cdot v = 0$, the cusp locus is the subset determined by the additional equation $\alpha''(t)\cdot v = 0$, while the swallowtail locus satisfies the additional equation $\alpha'''(t)\cdot v = 0$. (The next higher singularity locus would also satisfy $\alpha''''(t)\cdot v = 0$.)

The equation $\alpha'(t)\cdot v = 0$ is equivalent to the plane P with normal l through $\alpha(t)$ being tangent to C at $\alpha(t)$; if $\alpha''(t)\cdot v = \kappa(t)n(t)\cdot v = 0$ also holds, then either P is the osculating plane and v is $\pm b(t)$, the binormal of α at t, or $\kappa(t) = 0$ where $\kappa(t)$ is the curvature at t.

The point (t, l) is in the swallowtail locus if $\alpha'''(t)\cdot v = (\kappa' n(t) - \kappa^2\alpha'(t) - \kappa\tau b(t))\cdot v = 0$. If $\kappa(t) \neq 0$ this holds if $\tau(t) = 0$. (If $\kappa(t) = 0$, then this holds for all v in the plane spanned by the binormal vector and tangent vector.) It is easy to check that $\alpha'''(t)\cdot v = 0$ at a swallowtail of F_C if and only if $\tau'(t) = 0$.

These singularities will be exhibited transversely provided that at a cusp $D_{\mathbb{P}^2}(\alpha'(t)\cdot v) \neq 0$ and at a swallowtail $D_{\mathbb{P}^2}(\alpha'(t)\cdot v)$, $D_{\mathbb{P}^2}(\alpha''(t)\cdot v)$ are independent.

To check these statements we may choose local coordinates on S^1, $\mathbb{P}^2$ and $L(\mathbb{P}^2)$ so that F_C has the form

$$(t, v_1, v_2) \to (\alpha(t)\cdot(v_1, v_2, 1), v_1, v_2),$$

$$\alpha(t)\cdot(v_1, v_2, 1) = tv_1 - \tfrac{1}{6}\kappa^2 t^3 v_1 + \tfrac{1}{2}\kappa t^2 v_2 + \tfrac{1}{6}\kappa t^3 v_2 - \tfrac{1}{6}\kappa\tau t^3 + H,$$

where H consists of terms of order $\geqq 4$ in t.

To achieve this form, rotate so that the Frenet frame of α at $t = 0$ is $((1, 0, 0), (0, 1, 0), (0, 0, 1))$. The point of interest is given by $t = 0$, $v_1 = v_2 = 0$.

Then $$D_{\mathbb{P}^2}(\alpha'(t)\cdot v)|_{t=0} = (1, 0) \quad\text{and}\quad D_{\mathbb{P}^2}(\alpha''(t)\cdot v)|_{t=0} = (0, \kappa(0)).$$

Thus cusps appear transversely as long as α is an immersion (condition 1 in the definition of A), and swallowtails appear transversely as long as $\kappa(0) \neq 0$ (condition 2). To exclude higher order (and non-stable) singularities we require $\tau'(t) \neq 0$ if $\tau(t) = 0$ (condition 3).

These conditions are then equivalent to the local stability of F_C [6, Prop. 6.2., p. 191].

The additional conditions that the mapping F_C must satisfy to be stable are global. We must have that the image of $F_C|\Sigma(F_C)$ is in general position with itself ([6, prop 6.3, p. 192]).

Suppose at $F_C(t_0, l) = F_C(t_1, l)$, C^* has an ordinary double point, l orthogonal to $\alpha'(t_0)$, $\alpha'(t_1)$. We may assume $\alpha(t_0) = (0, 0, 0)$, and l determined by $(0, 0, 1)$. The images of DF_C at (t_0, l) and (t_1, l) give the tangent planes to the two branches of C^* at $F_C(t_0, l)$, since DF_C has rank 2 at these points and the kernel of DF_C is not tangent to $\Sigma(F_C)$ at these points. Let $a = (1, 0, 0)$ and $b = (0, 1, 0)$, vectors orthogonal to v which determines l. The image of DF_C at (t, l) is spanned by $(\alpha(t)\cdot a, 1, 0)$ and $(\alpha(t)\cdot b, 0, 1)$ given $(t, l) \in \Sigma(F_C)$. Hence, the normal to the image is given by $(1, -\alpha(t)\cdot a, -\alpha(t)\cdot b)$.

Returning to (t_0, l) and (t_1, l), the respective normals are $(1, 0, 0)$ and $(1, -\alpha(t_1)\cdot a, -\alpha(t_1)\cdot b)$. These are not parallel if $a\cdot\alpha(t_1)$ or $b\cdot\alpha(t_1)$ are different from zero if and only if $\alpha(t_1) \neq \alpha(t_0)$.

Similar reasoning shows that C^* has an ordinary triple point at $F(t, l)$ iff C has a tritangent plane with normal l at $\alpha(t_0)$, $\alpha(t_1)$, $\alpha(t_2)$ with $\alpha(t_0)$, $\alpha(t_1)$, $\alpha(t_2)$ non-collinear.

Finally we show that the cuspidal edge of C^* intersects C^* transversely provided condition 6 holds. The first part of condition 6 is equivalent to saying that no cuspidal edge of C^* intersects another cuspidal edge. The second part is equivalent to saying that triple points of C^* do not occur at a cuspidal edge.

Assume that $\alpha(t_0) = (0, 0, 0)$, $b(t_0) = (0, 0, 1)$, $F_C(t_1, b(t_0)) = F_C(t_0, b(t_0))$, $\tau(t_0) \neq 0$, $b(t_0)\cdot\alpha'(t_1) = 0$, but $b(t_0) \neq b(t_1)$. Since F_C has a simple cusp at $(t_0, b(t_0))$, we know that $F_C(t, b(t))$ is an immersion for values of t close to t_0. The normal to the image of DF_C at $F_C(t_1, b(t_0))$ is $(1, -a\cdot\alpha(t_1), -b\cdot\alpha(t_1))$ where a is the tangent vector to C at $\alpha(t_0)$ and b

$= n(t_0)$, the normal vector. The condition that the tangent to the cuspidal edge lies outside the tangent plane to C^* at $F_C(t_1, b(t_0))$ is that $(1, -a\cdot\alpha(t_1), -b\cdot\alpha(t_1))\cdot(\alpha(t_0)\cdot b'(t_0), 0, \tau(t_0)) \neq 0$. Since $\alpha(t_0) = (0, 0, 0)$ and $\tau(t_0) \neq 0$, the condition becomes $b\cdot\alpha(t_1) \neq 0$, or $\alpha(t_1)$ does not lie on the tangent line to C at $\alpha(t_0)$.

The condition that C^* has no quadruple points is equivalent to the first part of condition 5, while the condition that C^* does not have a self-intersection at a swallowtail point is equivalent to condition 7.

Thus α is in A if and only if F_C is stable at each (t, l) and $F_C(\Sigma(F_C))$ is in general position. This implies that $\alpha \in A$ if and only if F_C is stable.

COROLLARY. *The set A is open in $C^\infty(S^1, \mathbb{R}^3)$ as well as dense.*

Proof. Consider the mapping $G\colon C^\infty(S^1, \mathbb{R}^3) \to C^\infty(S^1 \times P^2, \mathbb{R}^3 \times \mathbb{P}^2)$ defined by $G(\alpha)(t, l) = (\alpha(t), l)$ and the mapping $F\colon C^\infty(S^1 \times \mathbb{P}^2, \mathbb{R}^3 \times \mathbb{P}^2) \to C^\infty(S^1 \times \mathbb{P}^2, L(\mathbb{P}^2))$ defined by $F(h) = f \circ h$ where $f(v, l) = (\Pi_l(v), l)$. The mapping G is clearly continuous in the Whitney topology, while the mapping F is also, for F is just the composition of maps in $C^\infty(S^1 \times \mathbb{P}^2, \mathbb{R}^3 \times \mathbb{P}^2)$ with a fixed map of $\mathbb{R}^3 \times \mathbb{P}^2 \to L(\mathbb{P}^2)$, and this is known to be continuous since $S^1 \times \mathbb{P}^2$ is compact. (See [6] p. 49, prop. 3.9).

Hence $F \circ G$ is continuous in the Whitney topology. The set U of C^∞ stable maps in $C^\infty(S^1 \times \mathbb{P}^2, L(\mathbb{P}^2))$ is open, hence $(F \circ G)^{-1}U$ is open in $C^\infty(S^1, \mathbb{R}^3)$. It follows from the proposition that $(F \circ G)^{-1}(U) = A$.

§4. SINGULARITIES OF THE DUAL SURFACE

If $C = \alpha(S^1)$, $\alpha \in A$, we have the following correspondence between the singularities of the dual surface C^* and the contact of planes in $\mathbb{R}^3$ with the curve C:

Double points of C^*	$\leftrightarrow$ Bitangent planes of C
Triple points of C^*	$\leftrightarrow$ Tritangent planes of C
Cuspidal points of C^*	$\leftrightarrow$ Osculating planes of C
Swallowtail points of C^*	$\leftrightarrow$ Osculating planes of stalls of C
Cuspidal/double points of C^*	$\leftrightarrow$ Osculating/bitangent planes of C

(Fig. 1.)

§5. THE INDEX OF A SWALLOWTAIL OF C^*.

If $C = \alpha(S^1)$, $\alpha \in A$, then C^* is the boundary of a region R in $(\mathbb{P}^3)^*$ defined as follows. If $P \notin C^*$, then P is transverse to C, so $P \cap C$ has an even number of points. If $P \notin C^*$ let $n(P, C^*)$ be one-half this number of intersection points, and let $R = \text{Closure}\ \{P \in (\mathbb{P}^3)^* \mid P \notin C \text{ and } n(P, C^*) \equiv 1 \pmod 2\}$.

Now suppose P is a swallowtail point of C^*. We let $n(P, C^*) = \chi(S_\varepsilon \cap R)$, where S_ε is a small sphere in $(\mathbb{P}^3)^*$ containing P and transverse to C^*. (Here χ is the Euler characteristic.) A neighbourhood of P in C^* is homeomorphic to the cone on a figure eight, and $n(P, C^*) = 1$ if R is inside the figure eight, $n(P, C^*) = 0$ if R is outside the figure eight (Fig. 2).

PROPOSITION 3. *Let $C = \alpha(S^1)$, $\alpha \in A$, and let x be a stall of C with osculating plane P. Then $n(x, C) \equiv n(P, C^*)$ (mod 2).*

Proof. By definition $n(x, P)$ is congruent mod 2 to $n(Q, C^*)$, where $Q \notin C^*$ is a plane close to P such that Q does not intersect C near x. But for any $Q \notin C^*$, Q close to P, we have that Q intersects C near x in either, 0, 2 or 4 points, with 0 or 4 corresponding to Q inside the figure eight cone and 2 corresponding to Q outside the figure eight cone. This follows from the local

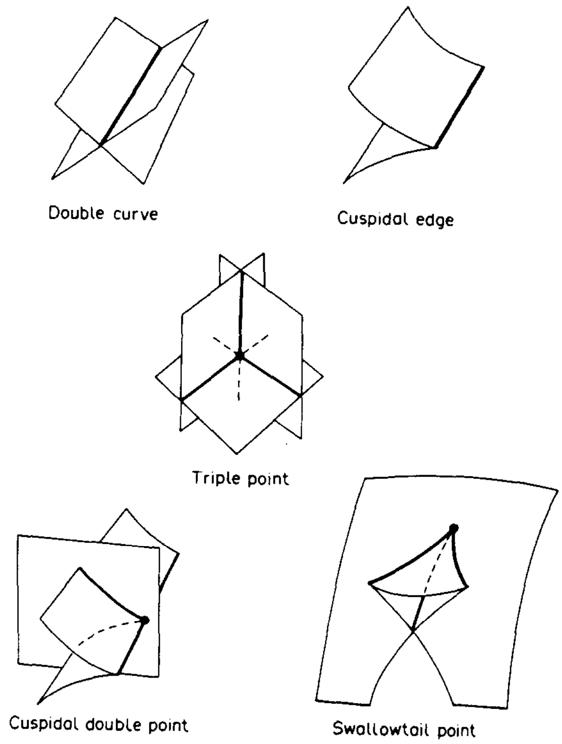

Fig. 1.

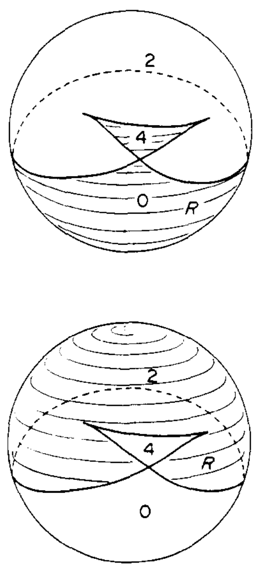

Fig. 2.

topology of the map $F_C\colon C \times \mathbb{P}^2 \to L(\mathbb{P}^2)$ at a swallowtail point x. For $(u, k) \in L(\mathbb{P}^2)\backslash F_C(\Sigma)$ near $F_C(x)$, $F_C^{-1}(u, k)$ has either 0, 2, or 4 points near x, as illustrated in Fig. 2. (This can be checked using the local equation of a swallowtail singularity.)

Now take $Q \notin C^*$, close to P and from inside the figure eight cone, such that Q does not intersect C near x. Then, if the points inside the figure eight cone are in R we have that

$$n(x, C) \equiv n(Q, C^*) \equiv 1 = n(P, C^*).$$

If the points inside the figure eight cone are not in R we have that

$$n(x, C) \equiv n(Q, C^*) \equiv 0 = n(P, C^*).$$

§6. COUNTING TRIPLE POINTS OF C^*.

Our theorem has been reduced to the following result.

PROPOSITION 4. *Let $C = \alpha(S^1), \alpha \in A$. Let $T^*(C)$ be the number of triple points of C^*, and let $P_1, \ldots, P_k$ be the swallowtail points of C^*. Then*

$$T^*(C) \equiv \sum_{i=1}^{k} n(P_i, C^*) \qquad (\text{mod } 2).$$

Proof. Since $\alpha \in A$, the double locus of C^* is a collection of closed curves and curve segments with endpoints at swallowtail points of C^*. The above congruence will be obtained by traversing each of these curves and keeping track of the number of triple points of C^* encountered.

We first sketch the proof then proceed to provide the formal details which justify the construction.

A closed curve in the double locus must pass through triple points of the surface an even number of times. Any such curve has a neighborhood consisting of the region inside a collection of spheres around triple points and cuspidal double points, and tubes joining these spheres, with all these elements chosen so that they intersect as nicely as possible (a "controlled system of tubular neighborhoods", Fig. 3). Because the surface is orientable and the boundary of this neighborhood is orientable we can show that a curve on this boundary which follows the double curve will end up at its starting point after passing an even number of triple points. We then show that a curve in the double locus beginning at a swallowtail P and ending at another swallowtail Q will pass an even or an odd number of triple points depending on whether the stall indices at the endpoints are the same or different. The system of neighborhoods is chosen so that around every swallowtail we have a sphere meeting the surface in a figure eight. The orientability of the surface assures us that a curve on the neighborhood boundary which starts outside the figure eight at P will end up outside the figure eight at Q. The result follows from the fact that the index of points on this nearby curve will change precisely when the curve passes a triple point. Combining the results for all parts of the double locus we obtain the desired congruence.

We now provide the details for this argument.

The stable map $i \circ F_C: C \times \mathbb{P}^2 \to (\mathbb{P}^3)^*$ admits a Thom stratification [5], with the following stratification $\mathcal{S}$ in the target $(\mathbb{P}^3)^*$. The 0-strata of $\mathcal{S}$ are the triple points, the cuspidal double points, and the swallowtail points of C^*. The 1-strata of $\mathcal{S}$ are the open arcs of ordinary double points and ordinary cusp points of C^*. The 2-strata comprise the smooth points of C^*, and the 3-strata the rest of $(P^3)^*$. Let $\{T_U\}_{U \in \mathcal{S}}$ be a (controlled) system of tubular neighborhoods for $\mathcal{S}$ [5] [7]. By shrinking the tubes if necessary, we may assume that for each stratum U, the set $S_U = \bar{T}_U \setminus (\bar{U} \setminus U)$ is a smooth manifold, and S_U is transverse to S_V for all strata $V \subset \bar{U}$.

Recall that the singular locus Σ of $i \circ F_C$ is isomorphic to $C \times \mathbb{P}^1$, with $i \circ F_C(\Sigma) = C^*$, and that $f_C: \Sigma \to (\mathbb{P}^3)^*$ is the restriction of $i \circ F_C$. Let $D \subset C^*$ be the closure of the set of double points of f_C. Then D is the 1-skeleton of $\mathcal{S}$, and D can be uniquely expressed as a finite union of distinct connected C^∞ curves D_i with the following properties: Each curve D_i has as singular points only cusps, double points, triple points, and endpoints. The cusps of D_i occur at cuspidal double points of C^*. The double points and triple points of D_i, as well as the intersection points of D_i with D_j for $i \neq j$, occur at the triple points of C^*. The endpoints of D_i are the swallowtail points of C^*.

For each i, let T_i be the tubular neighborhood of D_i in $(\mathbb{P}^3)^*$, i.e. the union of the given tubular neighborhoods of the strata comprising D_i. Let S_i be the boundary of T_i. Then S_i is a smooth surface with corners. (A corner of S_i occurs where the boundary of a tubular

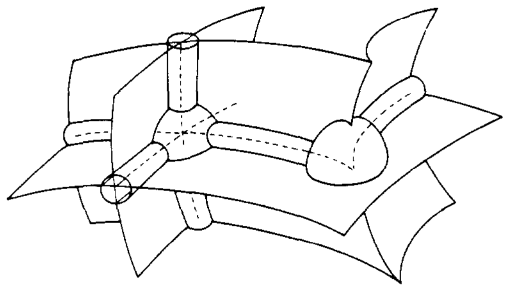

Fig. 3.

neighborhood of a 1-stratum intersects the boundary of a tubular neighborhood of an incident 0-stratum.) The surface S_i is orientable, since $(\mathbb{P}^3)^*$ and hence T_i are orientable. The surfaces C^* and S_i intersect transversely. Let C_i be the curve $C^* \cap S_i$. The region $R \subset (\mathbb{P}^3)^*$ (defined in section 5) intersects S_i in a surface R_i with boundary C_i.

Now we shall count the triple points of C^* by constructing for each i a curve δ_i' on S_i which "follows" D_i, and which crosses C_i near triple points of C^*.

First we consider the case when D_i is a closed curve, i.e. no swallowtail points occur on D_i. Let $D_i' = f_C^{-1}(D_i) \subset \Sigma$. Let $d_i: S^1 \to D_i$ be a C^∞ parametrization of D_i with only cusps, double points, and triple points as singularities. Consider the pull-back

$$\begin{tikzcd} S \arrow[r, "d_i'"] \arrow[d, "f"'] & D_i' \arrow[d, "f_C"] \\ S^1 \arrow[r, "d_i"] & D_i \end{tikzcd}$$

The map f is a double cover; we will show below that it must be a trivial cover, so S is the disjoint union of two circles. Let $e_i: S^1 \to E_i'$ be the restriction of d_i' to one of these circles, with $E_i' = e_i(S^1)$. We wish to show that the number of $t \in S^1$ such that $d_i(t)$ is a triple point of C^* is *even.*

Fix an orientation of Σ, and choose a normal vector field n to e_i in Σ so that the tangent vector to e_i followed by n gives the orientation of Σ. We will say that a curve e_i' obtained by pushing e_i off itself along n lies to the *left* of e_i in Σ. (The vector field n is not defined at cusps of e_i, but this creates no problem in defining e_i'.) Using the stratification of f_C, we can construct such a curve e_i' with the property that $\mathrm{Im}\,(f_C \circ e_i') \subset \overline{T}_i$, and $f_C \circ e_i'(s) \in C_i$ except on the closures of the tubular neighborhoods of the triple points of C^*. If P is a triple point of C^*, let S_P^0 be the interior of $S_P \cap S_i$ in S_i. We can modify $f_C \circ e_i'$ in $\overline{T}_P$ to obtain a curve $\delta_i(t)$ in S_i such that δ_i and C_i are transverse in S_P, and $\delta_i(t) \cap C_i \cap S_P^0 \neq \emptyset$ if and only if $d_i(t)$ is a triple point of C^*. Fix an orientation of S_i, and push δ_i slightly to the left on S_i to obtain the desired curve δ_i'. We can choose $\delta_i'(t)$ so that it intersects C_i precisely when $d_i(t)$ is a triple point of C^*, and these intersections are transverse (*Fig.* 4a, b). Now since δ_i' and C_i are closed curves on the oriented surface S_i, the number of intersection points of δ_i' and C_i is indeed even.

Suppose, however, that the double cover $f: S \to S^1$ constructed is nontrivial, so that $S = S^1$, and $d_i': S^1 \to D_i'$ with $f_C \circ d_i' = d_i \circ f$. Let δ_i be constructed from d_i' as above, and let δ_i' be obtained by pushing δ_i to the left on S_i. Choose $t_0 \in S^1$ such that $d_i(t_0)$ is an ordinary double point of C^*. Let F_0 be the circle fiber of S_i over $d_i(t_0)$. Then $F_0 \cap C_i$ has four points, two of which, say x and y, are on the image of the curve δ_i. Now the curve δ_i bounds a region in S_i, so δ_i' either lies inside this region or outside it. But two points of δ_i', say x' and y', lie in F_0, and they are separated by the points x and y, a contradiction. We conclude that f must be trivial.

Next we consider the case when D_i is an arc, with endpoints at swallowtails of C^*. Then there are parametrizations d_i: $[0, 1] \to D_i$ and d_i': $[-1, 1] \to D_i'$ with $d_i'(-1) = d_i'(1)$, such that $f_C \circ d_i'(s) = d_i(|s|)$. We wish to show that the number of $t \in [0, 1]$ such that $d_i(t)$ is a triple point of C^* has the same parity as $n(P, C^*) + n(Q, C^*)$, where $P = d_i(0)$ and $Q = d_i(1)$ are the

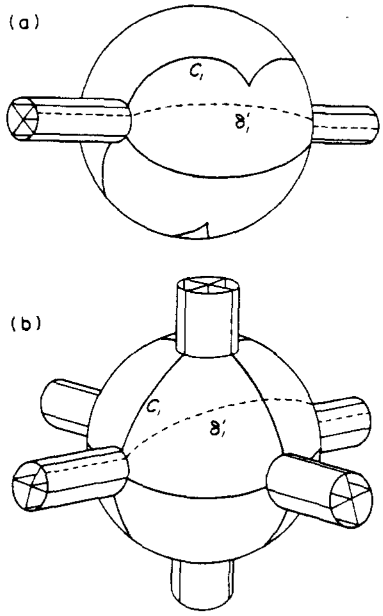

Fig. 4.

endpoints of D_i. First we recall the definition of $n(P, C^*)$. Let S_P be the sphere boundary of the tubular neighborhood of P. The curve $C^* \cap S_P$ is homeomorphic to a figure eight, and $n(P, C^*) = 1$ if and only if the points inside this figure eight are in the region R.

By the same method as above, we construct a closed curve δ_i: $[-1, 1] \to S_i$ with $\delta_i(0) \in S_P$ and $\delta_i(-1) = \delta_i(1) \in S_Q$, such that $\delta_i(s) \in C_i$ except on the boundaries of the tubular neighborhoods of the triple points and cuspidal double points on D_i. (In particular, δ_i coincides with C_i on S_P and on S_Q.) By pushing δ_i to the *right* in Σ instead of to the left, we obtain another closed curve ε_i on S_i disjoint from δ_i. Let A_i be the region of S_i such that $\partial A_i = \mathrm{Im}\,\delta_i$ and $A_i \not\supset \mathrm{Im}\,\varepsilon_i$. Let B_i be the region of S_i such that $\partial B_i = \mathrm{Im}\,\varepsilon_i$ and $B_i \not\supset \mathrm{Im}\,\delta_i$. Let $\Gamma_i = A_i \cup B_i$. Then the points of $\Gamma_i \cap S_P$ lie inside $C^* \cap S_P$ and the points of $\Gamma_i \cap S_Q$ lie inside $C^* \cap S_Q$ (Fig. 5).

We next construct as above an arc δ_i: $[0, 1] \to S_i$ by pushing $\delta_i | [0, 1]$ to the left, with $\delta'_i(0) \in S_P, \delta'_i(1) \in S_Q$, and such that $\delta'_i(t)$ intersects C_i (transversely) precisely when $d_i(t)$ is a triple point of C^*. Since δ'_i does not cross δ_i or ε_i, Im δ'_i must lie either in Γ_i or in S_i/Γ_i. Therefore $\delta'_i(0)$ lies inside $C^* \cap S_P$ if and only if $\delta'_i(1)$ lies inside $C^* \cap S_Q$. So δ'_i crosses C_i an even number of times if and only if $n(P, C^*) = n(Q, C^*)$, as desired.

Putting together the cases when D_i has no endpoints and when D_i has endpoints, we conclude that, since each triple point lies on 3 branches of D, $3T(C^*) \equiv \sum_{i=1}^{k} n(P_i, C^*)$, which gives the stated result.

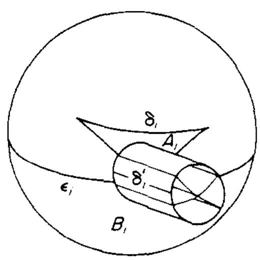

Fig. 5.

§7. RELATED RESULTS

We can also show that the number of tritangent planes of a surface in 3-space is congruent mod 2 to the number of cusps of the Gauss map, appropriately indexed. (The local geometry of a surface near a cusp of the Gauss map is described in [2].)

The same topological technique gives a relation between the number of triple points and Whitney umbrella points (pinch points) of a stable map of a surface to a 3-manifold.

It is interesting to compare our results for real space curves with R. Piene's formulas for complex space curves [8, §4]. For a generic curve in projective 3-space, she obtains the formula [8, p. 116]

$$T = t + d(1,2) + \gamma + k_0 + 2k_1,$$

where t is the number of tritangent planes, $d(1,2)$ is the number of osculating bitangents, γ is the number of flex bitangents, k_0 is the number of stalls, and k_1 is the number of flexes. On the other hand [8, p. 113],

$$T = \tfrac{1}{6}[(r_1-4)(r_1-3)(r_1-2) - 6g],$$

where r_1 is the rank of the curve (or the degree of its dual surface) and g is the genus of the curve. For a generic real curve in $\mathbb{R}^3$ there are no flexes, $d(1,2)$ is even, and the degree of the dual surface is even, so Piene's formulas suggest that the number of tritangent planes should be congruent mod 2 to the number of stalls (ignoring the genus). However, the appearance of the nonlocal index $n(x, C)$ of a stall is not predicted by her formulas.

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