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A certain class of rapidly convergent series representations for $\zeta(2n+1)$

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Abstract

For a natural number n, the authors propose and develop three new series representations for the Riemann Zeta function $\zeta(2n+1)$. The infinite series occurring in each of these three representations for $\zeta(2n+1)$ converges remarkably faster than that in Wilton's result. Furthermore, one of the three series representations for $\zeta(2n+1)$ involves the most rapidly convergent series among all the hitherto known members of the family of series representations considered here. Relevant connections of the results presented in this paper with many other known series representations for $\zeta(2n+1)$ are also briefly indicated. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the usual notations, let $\zeta(s)$ and $\zeta(s,a)$ denote the Riemann and the Hurwitz Zeta functions defined (for $\Re(s)>1$) by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad (\Re(s)>1) \tag{1.1}$$

and

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s)>1;\ a \neq 0,-1,-2,\ldots) \tag{1.2}$$

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and (for $\Re(s) \leqslant 1;\ s \neq 1$) by their analytic continuations (see, for details, [8]). Then, in terms of the familiar Bernoulli numbers B_n defined by means of the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi), \tag{1.3}$$

it is fairly well known that

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.4}$$

but no such simple (and useful) representation exists for $\zeta(2n+1)$ $(n \in \mathbb{N})$. Indeed there are many known series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which converge much more rapidly than those given by the defining series in (1.1). For example, we have the series representation:

$$\zeta(2n+1) = (-1)^{n-1}\, \pi^{2n} \left[\frac{H_{2n+1} - \log \pi}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}), \tag{1.5}$$

which was given, over seven decades ago, by Wilton [10, p. 92] (see also Hansen [3, p. 357, Entry (54.6.9)]), and the following result given recently by Srivastava [6, p. 10, Eq. (42)] (see also [5, p. 5, Eq. (3.4)]):

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \left(\frac{\pi}{2}\right)^{2n} \Bigg[& \frac{H_{2n+1} - \log(\pi/2)}{(2n+1)!} + \frac{2(4^n - 1)}{(2n+2)!} B_{2n+2} \log 2 \\ & - \frac{2^{2n+1} - 1}{(2n+1)!} \zeta'(-2n-1) - \frac{2^{4n+3}}{(2n+1)!} \zeta'\left(-2n-1, \frac{1}{4}\right) \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{(\pi/2)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} \Bigg] \quad (n \in \mathbb{N}), \end{aligned} \tag{1.6}$$

where (*and in what follows*) a prime denotes the derivative of $\zeta(s)$ or $\zeta(s, a)$ with respect to s, an empty sum is to be interpreted as nil, and H_n denotes the familiar harmonic numbers defined by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}). \tag{1.7}$$

Of the two seemingly analogous representations (1.5) and (1.6), the infinite series in (1.6) would obviously converge more rapidly, with their general terms having the order estimates:

$$O(k^{-2n-2} \cdot m^{-2k}) \quad (k \to \infty;\ n \in \mathbb{N};\ m = 2 \text{ and } 4). \tag{1.8}$$

The main object of this paper is to derive three (presumably new) members of the class of the series representations (1.5) and (1.6). The general terms of the infinite series occurring in these three members (given by Theorem 4 below) have the order estimates:

$$O(k^{-2n-2} \cdot m^{-2k}) \quad (k \to \infty;\ n \in \mathbb{N};\ m = 3, 4, 6), \tag{1.9}$$

which exhibit the fact that *each* of the three series representations derived here for $\zeta(2n+1)$ converges more rapidly than Wilton's result (1.5) and two of them at least as rapidly as Srivastava's result (1.6).

2. A set of lemmas

We begin by defining the sequence $\{\beta_n(x)\}_{n=0}^{\infty}$ by means of the generating function [cf. Eq. (1.3)]:

$$F(x,t) := \frac{t-\log x}{\mathrm{e}^t - x} = \sum_{n=0}^{\infty} \beta_n(x) \frac{t^n}{n!} \quad (1 \leqslant x \leqslant 1+c;\ c>0), \tag{2.1}$$

so that, clearly,

$$\beta_n(1) = B_n \quad (n \in N_0 := \mathbb{N} \cup \{0\}). \tag{2.2}$$

Since the zeros of $\mathrm{e}^t - x$ are given by

$$t = 2n\pi i + \log x \quad (n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}), \tag{2.3}$$

the radius of convergence of the series in (2.1) is at least 2π. Hence, by the Cauchy–Hadamard theorem for absolute convergence (cf., e.g., [11, p. 30]), we have

Lemma 1. *Let the sequence* $\{\beta_n(x)\}_{n=0}^{\infty}$ *be defined by* (2.1). *Then there exists some nonnegative real number* κ *such that*

$$\liminf_{n\to\infty} \left(\frac{|\beta_n(x)|}{n!} \right)^{-1/n} = 2\pi + \kappa \quad (\kappa \geqslant 0). \tag{2.4}$$

We now consider the following Dirichlet series [cf. Eq. (1.1)]:

$$\omega(s,x) := \sum_{n=1}^{\infty} \frac{x^{-n-1}}{n^s} + \frac{\log x}{s-1} \sum_{n=1}^{\infty} \frac{x^{-n-1}}{n^{s-1}} \quad (1 \leqslant x \leqslant 1+c;\ c>0), \tag{2.5}$$

so that, obviously,

$$\omega(s,1) = \zeta(s). \tag{2.6}$$

In case $1 < x \leqslant 1+c$ $(c>0)$, we can see that the function $\omega(s,x)$ is meromorphic, that is, holomorphic on the whole complex s-plane except for a simple pole at $s=1$ with residue

$$\frac{\log x}{x-1}.$$

Lemma 2. *Let* $\beta_n(x)$ *and* $\omega(s,x)$ *be defined by* (2.1) *and* (2.5), *respectively. Then*

$$\omega(1-n,x) = -\frac{\beta_n(x)}{n} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{2.7}$$

Proof. Relationship (2.7) is well known when $x=1$. So we assume that $1 < x \leqslant 1+c$ $(c>0)$.

For $t \in \mathbb{C}$ with $|t| < \log x$, we find from the generating function (2.1) that

$$\begin{aligned} F(x,t) &= -\left(\frac{t-\log x}{x} \right) \sum_{j=0}^{\infty} x^{-j}\, \mathrm{e}^{jt} \\ &= -\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} x^{-j-1} j^n \right) \frac{t^{n+1}}{n!} \end{aligned}$$

$$
\begin{aligned}
&+(\log x)\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}x^{-j-1}j^{n}\right)\frac{t^n}{n!}\\
&=\frac{\log x}{x-1}+\frac{1-x+\log x}{(x-1)^2}t\\
&+\sum_{n=2}^{\infty}\left[n\left(-\sum_{j=1}^{\infty}x^{-j-1}j^{n-1}\right)+(\log x)\sum_{j=1}^{\infty}x^{-j-1}j^{n}\right]\frac{t^n}{n!}. \qquad (2.8)
\end{aligned}
$$

Assertion (2.7) of Lemma 2 would now follow from (2.8) if we apply the definition (2.5) and compare the coefficients of t^n in (2.1) and (2.8).

Next, we prove

Lemma 3. *For* $n\in\mathbb{N}$ *and* $|\theta|<2\pi$ $(\theta\in\mathbb{R})$,

$$
\begin{aligned}
&(2n+1)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n+2}}\sin(k\theta)+\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n+1}}[\sin(k\theta)\log x-\theta\cos(k\theta)]\\
&=\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\frac{\log x}{x(x-1)}+\sum_{k=0}^{n-1}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}(2n-2k)\,\omega(2n-2k+1,x)\\
&+\sum_{k=n+1}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\beta_{2k-2n}(x) \quad (1<x\leqslant 1+c;\ c>0). \qquad (2.9)
\end{aligned}
$$

Proof. Denote, for convenience, the left-hand side of (2.9) by $\Omega_n(x,\theta)$. Then it is easily seen that

$$
\begin{aligned}
\Omega_n(x,\theta)&=(2n+1)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n+2}}\sum_{j=0}^{\infty}\frac{(-1)^j(k\theta)^{2j+1}}{(2j+1)!}\\
&+\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n+1}}\left[(\log x)\sum_{j=0}^{\infty}\frac{(-1)^j(k\theta)^{2j+1}}{(2j+1)!}-\theta\sum_{j=0}^{\infty}\frac{(-1)^j(k\theta)^{2j}}{(2j)!}\right]\\
&=\sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}(2n+1)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n-2j+1}}\\
&+\sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\sum_{k=1}^{\infty}x^{-k-1}\left(\frac{\log x}{k^{2n-2j}}-\frac{2j+1}{k^{2n-2j+1}}\right)\\
&=\sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\left[(2n-2j)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n-2j+1}}+(\log x)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n-2j}}\right], \qquad (2.10)
\end{aligned}
$$

where the various interchanges of the order of summation are justified by absolute convergence of the series involved under the conditions stated already with Lemma 3.

Upon separating the j-sum in (2.10) into three parts, we have

$$\Omega_n(x,\theta)=\sum_{j=0}^{n-1}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\left[(2n-2j)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n-2j+1}}+(\log x)\sum_{k=1}^{\infty}\frac{x^{-k-1}}{k^{2n-2j}}\right]$$
$$+\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}(\log x)\sum_{k=1}^{\infty}x^{-k-1}+\sum_{j=n+1}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\sum_{k=1}^{\infty}x^{-k-1}\left(\frac{2n-2j}{k^{2n-2j+1}}+\frac{\log x}{k^{2n-2j}}\right). \tag{2.11}$$

Finally, in view of the definition (2.5) and Lemma 2, we find from (2.11) that

$$\Omega_n(x,\theta)=\sum_{j=0}^{n-1}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}(2n-2j)\,\omega(2n-2j+1,x)$$
$$+\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\,\frac{\log x}{x(x-1)}+\sum_{j=n+1}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\beta_{2j-2n}(x), \tag{2.12}$$

which evidently completes the proof of Lemma 3.

3. Preliminary results

By applying Lemma 3, we first prove

Theorem 1. *For $n\in\mathbb{N}$ and $|\theta|<2\pi$ ($\theta\in\mathbb{R}$),*

$$(2n+1)\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^{2n+2}}-\theta\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^{2n+1}}-2n\theta\,\zeta(2n+1)$$
$$=2(-1)^n\theta^{2n+1}\left(\sum_{k=1}^{n-1}\frac{(-1)^k\cdot k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\theta^{2k}}-\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{(2\pi/\theta)^{2k}}\right). \tag{3.1}$$

Proof. If $n\in\mathbb{N}$ and $\theta\in\mathbb{R}$, it is easily observed that each series on the left-hand side of (2.9) is uniformly convergent with respect to x on the closed interval $[1,1+c]$ ($c>0$). On the other hand, it follows from Lemma 1 that the series on the right-hand side of (2.9) is also uniformly convergent with respect to x on $[1,1+c]$, provided that $|\theta|<2\pi$. Thus, by letting $x\to 1+$ in Lemma 3, we obtain

$$(2n+1)\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^{2n+2}}-\theta\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^{2n+1}}$$
$$=2n\theta\,\zeta(2n+1)+\sum_{k=1}^{n-1}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}(2n-2k)\,\zeta(2n-2k+1)$$
$$+\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}+\sum_{k=n+1}^{\infty}\frac{(-1)^k\,\theta^{2k+1}}{(2k+1)!}B_{2k-2n}, \tag{3.2}$$

where we have also used the relationships (2.2) and (2.6).

Now, in view of the familiar relationship (1.4), we can write the last term on the right-hand side of (3.2) in the form:

$$\sum_{k=n+1}^{\infty}\frac{(-1)^k\,\theta^{2k+1}}{(2k+1)!}\,\frac{(-1)^{k-n-1}(2k-2n)!}{2^{2k-2n-1}\,\pi^{2k-2n}}\zeta(2k-2n)$$

$$=2(-1)^{n-1}\theta^{2n+1}\sum_{k=1}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{(2\pi/\theta)^{2k}}$$

$$=-\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}+2(-1)^{n-1}\theta^{2n+1}\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{(2\pi/\theta)^{2k}}, \tag{3.3}$$

since $\zeta(0)=-\frac{1}{2}$. Assertion (3.1) of Theorem 4 follows from (3.2) upon substituting the expression given by (3.3) for the last term on the right-hand side of (3.2).

Since each series in (3.1) is uniformly convergent with respect to θ on the open interval $(-2\pi,2\pi)$, by executing termwise differentiation in (3.1) with respect to θ, we have

Theorem 2. *For* $n\in\mathbb{N}$ *and* $|\theta|<2\pi$ $(\theta\in\mathbb{R})$,

$$2n\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^{2n+1}}+\theta\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^{2n}}-2n\,\zeta(2n+1)$$

$$=2(-1)^n\theta^{2n}\left(\sum_{k=1}^{n-1}\frac{(-1)^k\cdot k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\theta^{2k}}-\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\,\frac{\zeta(2k)}{(2\pi/\theta)^{2k}}\right). \tag{3.4}$$

In their special cases when $\theta=\pi$, Theorems 1 and 2 readily yield

Corollary 1. *For* $n\in\mathbb{N}$,

$$\zeta(2n+1)=(-1)^n\frac{2(2\pi)^{2n}}{(2n-1)2^{2n}+1}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\pi^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{2^{2k}}\right] \tag{3.5}$$

and

$$\zeta(2n+1)=(-1)^n\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{2k}}\right]. \tag{3.6}$$

Remark 1. The series representation (3.5) was given by Srivastava [5, p. 4, Eq. (2.5)] (see also [7, p. 393, Eq. (3.20)]). Indeed, as already observed by Srivastava [op. cit.], (3.5) provides a significantly simpler (and much more rapidly convergent) version of one of the two *main* results of Cvijović and Klinowski [1, p. 1265, Theorem B].

Remark 2. The series representation (3.6) is the other *main* result of Cvijović and Klinowski [1, p. 1265, Theorem A], whose companion was referred to in Remark 1.

Remark 3. Since

$$\frac{(2k-1)!}{(2n+2k+1)!} = \frac{1}{2n+1}\left[\frac{(2k-1)!}{(2n+2k)!} - \frac{(2k)!}{(2n+2k+1)!}\right] \quad (n,k \in \mathbb{N}), \tag{3.7}$$

it is not difficult to obtain Wilton's result (1.5) by combining the series representation (3.5) with another result of Srivastava [5, p. 1, Eq. (1.3)] (see also [7, p. 389, Eq. (2.9)]), which we shall recall here as Eq. (4.1) below.

With a view to applying Theorems 1 and 2 in their other special cases when

$$\theta = \tfrac{2}{3}\pi, \quad \tfrac{1}{2}\pi, \quad \text{and} \quad \tfrac{1}{3}\pi, \tag{3.8}$$

we now evaluate several trigonometric sums given by

Lemma 4. *For* $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{n^s} = \frac{3^{1-s}-1}{2}\zeta(s), \tag{3.9}$$

$$\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n^s} = \sqrt{3}\left\{\frac{3^{-s}-1}{2}\,\zeta(s) + 3^{-s}\,\zeta\left(s, \frac{1}{3}\right)\right\}, \tag{3.10}$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/2)}{n^s} = 2^{-s}(2^{1-s}-1)\,\zeta(s), \tag{3.11}$$

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^s} = (2^{-s}-1)\,\zeta(s) + 2^{1-2s}\,\zeta\left(s, \frac{1}{4}\right), \tag{3.12}$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^s} = \frac{1}{2}(6^{1-s} - 3^{1-s} - 2^{1-s} + 1)\,\zeta(s) \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^s} = \sqrt{3}\left[\frac{3^{-s}-1}{2}\,\zeta(s) + 6^{-s}\left\{\zeta\left(s, \frac{1}{6}\right) + \zeta\left(s, \frac{1}{3}\right)\right\}\right]. \tag{3.14}$$

Proof. Although a *direct* proof of each of the trigonometric sums (3.9) to (3.14) may seem to be fairly elementary, we choose to derive these sums by suitably specializing one or the other of the following known results recorded (for example) by Hansen [3, p. 223, Entry (14.4.3); p. 244, Entry (17.4.3)]:

$$\sum_{n=1}^{\infty} \frac{\sin(nx+y)}{n^s} = \frac{(2\pi)^s}{2\Gamma(s)}\csc(\pi s)\left[\cos\left(y - \frac{1}{2}\pi s\right)\,\zeta\left(1-s, \frac{x}{2\pi}\right) - \cos\left(y + \frac{1}{2}\pi s\right)\,\zeta\left(1-s, 1-\frac{x}{2\pi}\right)\right] \quad (\Re(s) > 1;\ 0 < x < 2\pi) \tag{3.15}$$

and

$$\sum_{n=1}^{\infty}\frac{\cos(nx+y)}{n^s}=\frac{(2\pi)^s}{2\Gamma(s)}\csc(\pi s)\left[\sin\left(y+\frac{1}{2}\pi s\right)\zeta\left(1-s,1-\frac{x}{2\pi}\right)\right.$$
$$\left.-\sin\left(y-\frac{1}{2}\pi s\right)\zeta\left(1-s,\frac{x}{2\pi}\right)\right]\quad(\Re(s)>1;\ 0<x<2\pi). \tag{3.16}$$

For example, if we set

$$x=\tfrac{2}{3}\pi\quad\text{and}\quad y=0,$$

in (3.15), we find that

$$\sum_{n=1}^{\infty}\frac{\sin(2n\pi/3)}{n^s}=\frac{(2\pi)^s}{4\Gamma(s)}\csc\left(\frac{1}{2}\pi s\right)\left[\zeta\left(1-s,\frac{1}{3}\right)-\zeta\left(1-s,\frac{2}{3}\right)\right]. \tag{3.17}$$

Since

$$\zeta(s)=\frac{1}{m^s-1}\sum_{j=1}^{m-1}\zeta\left(s,\frac{j}{m}\right)\quad(m\in\mathbb{N}\setminus\{1\}), \tag{3.18}$$

which is an immediate consequence of the definitions (1.1) and (1.2), the trigonometric sum (3.10) would follow from (3.17) when we appropriately apply Rademacher's formula (cf., e.g., Magnus et al. [4, p. 23]):

$$\zeta\left(s,\frac{m}{n}\right)=2\Gamma(1-s)(2n\pi)^{s-1}\left[\sin\left(\frac{1}{2}\pi s\right)\sum_{j=1}^{n}\cos\left(\frac{2mj\pi}{n}\right)\zeta\left(1-s,\frac{j}{n}\right)\right.$$
$$\left.+\cos\left(\frac{1}{2}\pi s\right)\sum_{j=1}^{n}\sin\left(\frac{2mj\pi}{n}\right)\zeta\left(1-s,\frac{j}{n}\right)\right]\quad(m,n\in\mathbb{N}), \tag{3.19}$$

as well as its well-known special case when $m=n=1$:

$$\zeta(s)=2(2\pi)^{s-1}\sin(\tfrac{1}{2}\pi s)\,\Gamma(1-s)\,\zeta(1-s). \tag{3.20}$$

The other trigonometric sums asserted by Lemma 4 can be proven similarly.

Next, we prove

Theorem 3. *For $n\in\mathbb{N}$,*

$$\zeta(2n+1)=(-1)^n\frac{2(2\pi)^{2n}}{n(3^{2n+1}-1)}\left[\frac{(3^{2n}-1)\pi}{4\sqrt{3}\,(2n)!}B_{2n}+\frac{(-1)^n\pi}{\sqrt{3}\,(2\pi)^{2n}}\zeta\left(2n,\frac{1}{3}\right)\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{3^{2k}}\right], \tag{3.21}$$

$$\zeta(2n+1)=(-1)^n\frac{2(2\pi)^{2n}}{n(2^{4n+1}+2^{2n}-1)}\left[\frac{2^{2n-3}(2^{2n}-1)\pi}{(2n)!}B_{2n}+\frac{(-1)^n\pi}{2(2\pi)^{2n}}\zeta\left(2n,\frac{1}{4}\right)\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k)!}\frac{\zeta(2k+1)}{(\pi/2)^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{4^{2k}}\right] \tag{3.22}$$

and

$$\zeta(2n+1)=(-1)^n\frac{2(2\pi)^{2n}}{n(6^{2n}+3^{2n}+2^{2n}-1)}\left[\frac{2^{2n-3}(3^{2n}-1)\pi}{\sqrt{3}\,(2n)!}B_{2n}\right.$$
$$+\frac{(-1)^n\,\pi}{2\sqrt{3}\,(2\pi)^{2n}}\left\{\zeta\left(2n,\frac{1}{3}\right)+\zeta\left(2n,\frac{1}{6}\right)\right\}$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k)!}\frac{\zeta(2k+1)}{(\pi/3)^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right], \tag{3.23}$$

in terms of the Bernoulli numbers B_n *defined by* (1.3).

Proof. Upon specializing the parameter θ in Theorem 2 as in (3.8), if we make use of the corresponding assertions of Lemma 4 and relationship (1.4), we shall obtain the series representations (3.21) to (3.23) of Theorem 3. The details may be omitted.

By comparing the series representation (3.22) with a known result due to Srivastava [6, p. 9, Eq. (41)] (see also [5, p. 5, Eq. (3.3)], we get an interesting identity involving the Zeta functions $\zeta(s)$ and $\zeta(s,a)$ (and their derivatives with respect to s), which is given by

Corollary 2. *For* $n\in\mathbb{N}$,

$$\frac{(-1)^n\pi}{2(2\pi)^{2n}}\zeta\left(2n,\frac{1}{4}\right)=\{(2^{2n-2}-1)\log 2-2^{2n-3}(2^{2n}-1)\pi\}\frac{B_{2n}}{(2n)!}$$
$$-\frac{2^{2n-1}-1}{2(2n-1)!}\zeta'(1-2n)-\frac{4^{2n-1}}{(2n-1)!}\zeta'\left(1-2n,\frac{1}{4}\right). \tag{3.24}$$

4. The main series representations

We first recall here the following series representations for $\zeta(2n+1)$, which were given earlier by Srivastava [5, pp. 1–2, Eqs. (1.3) to (1.6)] (see also [7, p. 389, Eqs. (2.9) to (2.12)]):

Lemma 5. *For* $n\in\mathbb{N}$,

$$\zeta(2n+1)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n+1}-1}\left[\frac{H_{2n}-\log\pi}{(2n)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}}\right.$$
$$\left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{2^{2k}}\right], \tag{4.1}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+1}-1}\left[\frac{H_{2n}-\log(2\pi/3)}{(2n)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}\right.$$
$$\left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{3^{2k}}\right], \tag{4.2}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{2^{4n+1}+2^{2n}-1}\left[\frac{H_{2n}-\log(\pi/2)}{(2n)!}\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{(\pi/2)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{4^{2k}}\right] \tag{4.3}$$

and

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1)+2^{2n}-1}\left[\frac{H_{2n}-\log(\pi/3)}{(2n)!}\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{(\pi/3)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right], \tag{4.4}$$

where (just as elsewhere in this paper) an empty sum is to be interpreted as nil.

As pointed out by Srivastava [7, p. 391], by suitably combining the series representations (4.1) and (4.3), one can easily obtain a slightly modified (and corrected) version of a result proven by Tsumura [9, p. 383, Theorem B] as well as a variant of Tsumura's result (cf. [7, p. 391, Eq. (3.8)], which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived by Ewell [2, p. 1010, Corollary] by employing some entirely different techniques from those used here (and by Tsumura [9] and Srivastava [7]). Furthermore, Srivastava's series representation (4.1) converges faster than each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [12, p. 1590, Eq. (3.13)] and by Cvijović and Klinowski [1, p. 1265, Theorem A] (already referred to in Remark 2 in the preceding section). See also Remark 3 for relevant connection of the series representation (4.1) with Wilton's result (1.5).

By applying Theorem 3 and Srivastava's series representations (4.2) to (4.4), we now prove our main results given by

Theorem 4. *For* $n\in\mathbb{N}$,

$$\zeta(2n+1)=(-1)^{n-1}\left(\frac{2\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log(2\pi/3)}{(2n+1)!}+\frac{(3^{2n+2}-1)\pi}{2\sqrt{3}(2n+2)!}B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{\sqrt{3}(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{3}\right)$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{3^{2k}}\right], \tag{4.5}$$

$$\zeta(2n+1)=(-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n}\left[\frac{H_{2n+1}-\log(\pi/2)}{(2n+1)!}+\frac{2^{2n}(2^{2n+2}-1)\pi}{(2n+2)!}B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{2(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{4}\right)$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(\pi/2)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{4^{2k}}\right] \tag{4.6}$$

and

$$\zeta(2n+1)=(-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log(\pi/3)}{(2n+1)!}+\frac{2^{2n}(3^{2n+2}-1)\pi}{\sqrt{3}(2n+2)!}B_{2n+2}\right.$$
$$+\frac{(-1)^{n-1}}{2\sqrt{3}(2\pi)^{2n+1}}\left\{\zeta\left(2n+2,\frac{1}{3}\right)+\zeta\left(2n+2,\frac{1}{6}\right)\right\}$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(\pi/3)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{6^{2k}}\right]. \tag{4.7}$$

Proof. In order to prove the first assertion (4.5) of Theorem 4, we apply the series representations (3.21) and (4.2) with n replaced by $n+1$ in *each* case. By equating the two expressions for $\zeta(2n+3)$ thus obtained, we find that

$$-\frac{1}{n+1}\left[\frac{(3^{2n+2}-1)\pi}{4\sqrt{3}(2n+2)!}B_{2n+2}+\frac{(-1)^{n-1}}{2\sqrt{3}\,(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{3}\right)\right.$$
$$+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k+2)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}+\frac{(-1)^{n-1}\cdot n}{2!}\frac{\zeta(2n+1)}{(2\pi/3)^{2n}}$$
$$\left.-\frac{1}{2(2n+2)!}+\sum_{k=1}^{\infty}\frac{(2k)!}{(2n+2k+2)!}\frac{\zeta(2k)}{3^{2k}}\right]$$
$$=\frac{H_{2n+2}-\log(2\pi/3)}{(2n+2)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+2)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}$$
$$+\frac{(-1)^n}{2!}\frac{\zeta(2n+1)}{(2\pi/3)^{2n}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+2)!}\frac{\zeta(2k)}{3^{2k}}, \tag{4.8}$$

since $\zeta(0)=-\frac{1}{2}$.

Now, in view of the identities:

$$\frac{(2k-1)!}{(2n+2k+1)!}=\frac{(2k)!}{(2n+2k+2)!}+2(n+1)\frac{(2k-1)!}{(2n+2k+2)!} \tag{4.9}$$

and

$$\frac{1}{2(2n-2k+1)!}=\frac{n+1}{(2n-2k+2)!}-\frac{k}{(2n-2k+2)!}, \tag{4.10}$$

if we multiply both sides of (4.8) by $n+1$ and combine similar terms, we have

$$(n+1)\frac{H_{2n+2}-\log(2\pi/3)}{(2n+2)!}-\frac{1}{2(2n+2)!}$$
$$+\frac{(3^{2n+2}-1)\pi}{4\sqrt{3}(2n+2)!}B_{2n+2}+\frac{(-1)^{n-1}}{2\sqrt{3}(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{3}\right)$$

$$+\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(2\pi/3)^{2k}}+\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{3^{2k}}$$
$$=\frac{(-1)^{n-1}}{2}\frac{\zeta(2n+1)}{(2\pi/3)^{2n}}\quad(n\in\mathbb{N}). \tag{4.11}$$

Since

$$(n+1)\frac{H_{2n+2}-\log(2\pi/3)}{(2n+2)!}-\frac{1}{2(2n+2)!}=\frac{H_{2n+1}-\log(2\pi/3)}{2(2n+1)!}, \tag{4.12}$$

the assertion (4.5) of Theorem 4 would follow readily from (4.11).

Our derivations of the remaining assertions (4.6) and (4.7) of Theorem 4 are much akin to that of (4.5). We similarly make use of the series representations (3.22) and (4.3) to derive the assertion (4.6), and the series representations (3.23) and (4.4) to derive the assertion (4.7).

Each of the main series representations (4.5) to (4.7), which we have derived for $\zeta(2n+1)$ $(n\in\mathbb{N})$ in this paper, belongs to the class containing Wilton's formula (1.5) and Srivastava's formula (1.6). If, for convenience, we denote the summands of the *infinite* series in the representations (1.5), (1.6), and (4.5) to (4.7) by $\mathscr{S}_k^{(j)}$ $(j=1,2,3,4,5)$, respectively, and apply Stirling's formula (cf., e.g. [4, p. 12]) and the fact that $\zeta(2k)\to 1$ as $k\to\infty$, we easily obtain the following order estimates [cf. Eqs. (1.8) and (1.9)]:

$$\mathscr{S}_k^{(j)}=\mathrm{O}(k^{-2n-2}\cdot m^{-2k})\quad(k\to\infty;\ n\in\mathbb{N}) \tag{4.13}$$

$$(m=2 \text{ when } j=1;\ m=3 \text{ when } j=3;\ m=4 \text{ when } j=2 \text{ and } j=4;\ m=6 \text{ when } j=5).$$

Clearly, therefore, of these five series representations for $\zeta(2n+1)$, our result (4.7) involves the most rapidly convergent series. On the other hand, the rate of convergence of the series involved in each of our results (4.5) to (4.7) is obviously much better than that of the series involved in Wilton's result (1.5). And, in particular, the rate of convergence of the series involved in our result (4.6) is as good as that of the series involved in Srivastava's result (1.6).

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