Radical Relations in Orthogonal Groups

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ABSTRACT

Any relation between simple isometries is a consequence of relations of lengths $\leqslant 4$. This extends earlier results which deal with relations between reflections.

1. INTRODUCTION

If (V,Q) is a metric vector space with nontrivial radical (or singular if $\operatorname{char} K=2$), then the orthogonal group of (V,Q) contains elements that do not fix every element in the radical. The orthogonal group is larger than the group generated by reflections (cf. [3, p. 104]), even if the dimension of V is finite. But the reflections together with the simple radical isometries form a system of generators (cf. [3]). For the characterization of transformation groups it is important to find a set T of short relations between generators such that every relation is a consequence of these relations. This task is known as the relation problem [2].

In [5], the relation problem has been solved for unitary groups and for orthogonal groups whose quadratic form has an index at most 1. Using recent

*Research supported in part by the National Research Council of Canada, grant A7251.

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results [6] and results from [1], we can now prove a similar result regardless of the index of the quadratic form.

In Sec. 2 we collect some formulas which are needed in the proof of the main theorem. The proof of the preparatory Lemma 3 makes use of results in [5], while the proof of Theorem 1 combines results established in [1], [3], and [6].

Theorem 2 finally solves the relation problem for subgroups of orthogonal groups (cf. [7]).

2. FACTORIZATION OF SIMPLE RADICAL ISOMETRIES

Let V be a vector space over a commutative field K, and V^* its dual space. The dimension of V is arbitrary (it may be infinite). We shall consider a metric vector space (V,Q) where Q is a quadratic form. Let f be the bilinear form associated with Q. Then (V,f) also is a metric vector space. Its radical will be denoted by R. The singular radical will be denoted by SR.

LEMMA 1. *Let (V,Q) be a metric vector space, and assume $|K|>2$. If $s\in V\backslash R$ and $Q(s)=0$, then there are $a,b\in V\backslash R$ such that $Q(a),Q(b)\neq 0$ and*

$$s=\frac{a}{Q(a)}-\frac{b}{Q(b)}.$$

Proof. Since $s\notin R$, there is some $t\in V$ such that $Q(t)=0$ and $f(s,t)\neq 0$, i.e. $H=\langle s,t\rangle$ is a hyperbolic plane. Let $a\in H$ and $Q(a)\neq 0$; then $Q(a-\alpha t)=Q(a)-\alpha f(a,t)\neq Q(a),0$ for some $\alpha\in K\backslash\{0\}$, since $|K|\geqslant 3$. Let us denote αt simply by t again. We put $b=a-t$. Then $Q(a-b)=0$ and hence

$$Q\left(\frac{a}{Q(a)}-\frac{b}{Q(b)}\right)=0,$$

as an easy calculation shows. Since $Q(a)\neq Q(b)$, and since a and b are linearly independent, $a/Q(a)-b/Q(b)\notin K(a-b)=Kt$. Thus $a/Q(a)-b/Q(b)\in Ks\backslash\{0\}$. Therefore we can assume $a/Q(a)-b/Q(b)=s$. ■

A short calculation shows

LEMMA 2. *Let (V,Q) be a metric vector space, $a,b\in V\backslash R$ such that $Q(a),Q(b)\neq 0$, and $r\in$ SR. Then*

(i) $\sigma_a\sigma_{a+r}: x\to x-\dfrac{f(x,a)}{Q(a)}\,r$, *and*

(ii) $\sigma_a\sigma_{a+r}\sigma_b\sigma_{b+r}: x\to x-f\left(x,\dfrac{a}{Q(a)}-\dfrac{b}{Q(b)}\right)r.$

3. RELATIONS IN THE ORTHOGONAL GROUP O

The group of all isometries of (V,Q) will be denoted by O. Let π be an isometry in O; then $B(\pi)=V^{\pi-1}$ is the path and $F(\pi)=\{x\in V;\ x^{\pi}=x\}$ is the fix of π. An isometry π will be called simple if $\dim B(\pi)=\operatorname{codim} F(\pi)=1$. Let T be the set of all simple isometries, $T_1=\{\tau\in T;\ B(\tau)\subset R\}$, and $T_2=T\setminus T_1$. The sets T, T_1, T_2 are normal in O. Also, $T^{-1}\subset T$, $T_1^{-1}\subset T_1$, and $T_2^{-1}\subset T_2$. Let $\mathfrak{F}$ be the free group generated by the set T. A word $(\pi)=(\sigma_1^{\varepsilon_1},\dots,\sigma_n^{\varepsilon_n})$ in $\mathfrak{F}$, where $\sigma_i\in T$, $\varepsilon_i=\pm I$ ($\sigma_i^I=\sigma_i$, and σ_i^{-I} is the inverse of σ_i in $\mathfrak{F}$), is a relation if $\sigma_1^{\eta_1},\dots,\sigma_n^{\eta_n}=1$, where $\eta_i=1$ if $\varepsilon_i=I$ and $\varepsilon_i=-1$ if ${}_i=-I$. If a word (π) of length n is a relation, we call π an n-relation. For our purposes, it is enough to consider words (π) where all $\varepsilon_i=I$ (namely, $T^{-1}\subset T$), and we shall calculate modulo the normal subgroup S which is generated by all n-relations with $n\leqslant 4$. Thus S clearly contains the 2-relations (σ,σ^{-I}) for all $\sigma\in T$.

The word $(\sigma_1',\dots,\sigma_m')=(\pi')$ in $\mathfrak{F}$ is derived from the word $(\sigma_1,\dots,\sigma_n)=(\pi)$ in $\mathfrak{F}$ if $(\pi')\equiv(\pi)$ mod S and if $\bigcap_{i=1}^{m}F(\sigma_i')\supset\bigcap_{i=1}^{n}F(\sigma_i)$. A word $(\sigma_1,\dots,\sigma_n)$ in $\mathfrak{F}$ is contracted if $\dim B(\sigma_i\sigma_{i+1})=2$ for $i=1,\dots,n-1$. It is v-ordered for some $v\in V$ if $v\in F(\sigma_i)$ and $v\notin F(\sigma_j)$ implies $i<j$. It has v-defect k if $v\notin F(\sigma_i)$ for exactly k of the elements σ_i, $i=1,\dots,n$.

Lemma 3. *If $(\pi)=(\tau_1,\dots,\tau_n)$ where $\tau_i\in T_1$ is a relation, then the empty word $\varnothing$ is derived from (π).*

Proof. Since T_1 is normal, we can assume by [5, Lemma 5] that (π) is v-ordered. It is easy to see that we can also assume that (π) is contracted. Neither v-ordering nor contraction increases the v-defect. If the v-defect is not zero, it can be reduced. Clearly, the v-defect cannot be 1. Now we assume the v-defect is greater than 1. Let $\tau,\omega\in T_1$ such that $B(\tau\omega)=2$ and $v\in V\setminus(F(\tau)\cup F(\omega))$. Then $v^{\tau\omega}-v=r\in B(\tau\omega)\setminus\{0\}\subset \mathrm{S}R$. We define $\omega':x\to x+x^{\psi}r$, where $\psi\in V^*$ such that $F(\tau\omega)^{\psi}=0$, $v^{\psi}=1$, and $D^{\psi}=0$ for some complement D of $F(\tau\omega)+Kv$ in V. Then $\omega'\in T_1$ is an isometry and $F(\tau\omega\omega'^{-1})=F(\tau\omega)+Kv$. We put $\tau\omega\omega'^{-1}=\tau'$. Then $\tau'\in T_1$ is also an isometry. We have $F(\tau)\cap F(\omega)=F(\tau')\cap F(\omega')$. Since $\operatorname{codim}\bigcap_{i=1}^{n}F(\tau_i)=k$ is finite, we can now use induction on k to finish the proof. ∎

Theorem 1. *Let $|K|>2$. If (π) is a relation in $\mathfrak{F}$, then* $(\pi)\equiv\varnothing$ mod S.

Proof. Let $(\pi)=(\sigma_1,\dots,\sigma_n)$ be a relation in $\mathfrak{F}$, and assume $\sigma_1,\dots,\sigma_n\in T$. Since T_i is normal in $\mathfrak{F}$, there are $\rho_i\in T_2$ and $\tau_i\in T_1$ such that $(\pi)\equiv(\rho_1,\dots,\rho_k,\tau_{k+1},\dots,\tau_n)$ mod S. Then $\rho_1\cdots\rho_k\tau_{k+1}\cdots\tau_n=1$ and $\kappa=\tau_{k+1}\cdots\tau_n=\rho_k\cdots\rho_1\in 0$. Clearly, $B(\kappa)\subset R$ and thus $B(\kappa)\subset$ SR by [3, Lemma

3]. Since $R \subset F(\rho_i)$, we get $R \subset F(\kappa)$, and therefore by [3, Lemma 2], there are $\tau_i': x \to x + x^{\psi_i} r_i$, where $r_i \in B(\kappa)$, $\psi_i \in V^*$, and $F(\kappa) \subset F(\tau_i')$, such that $\kappa = \tau_{k+1}' \cdots \tau_{n'}'$. Since $\kappa = \rho_k \cdots \rho_1$, we get by [3, Lemma 18] that $\psi_i = f_{a_i}$ for $a_i \in V \backslash R$. Since $r_i \in \mathrm{SR}$, we get that $\tau_i': x \to x + f(x, a_i) r_i$ are isometries. By Lemmas 1 and 2, there are $\sigma_i' \in T_2$ such that $(\tau_{k+1}', \ldots, \tau_{n'}') \equiv (\sigma_1', \ldots, \sigma_m')$ mod S and $\kappa = \tau_{k+1}' \cdots \tau_{n'}' = \sigma_1' \cdots \sigma_m'$. Since $\kappa = \rho_k \cdots \rho_1$, we obtain $(\sigma_1', \ldots, \sigma_m') \equiv (\rho_k, \ldots, \rho_1)$ by [1, Satz 2] and [6, Theorem 6.1]. Thus we get $(\pi) \equiv (\rho_1, \ldots, \rho_k, \tau_{k+1}, \ldots, \tau_n) \equiv (\tau_{k+1}', \ldots, \tau_{n'}', \tau_{k+1}, \ldots, \tau_n)$ mod S. We use Lemma 3 to finish the proof. ■

The results in Theorem 1 can be extended to a class of subgroups of the orthogonal groups. Let $\bar{V}$ be a subspace of V, $\bar{T} = \{\sigma \in T;\ B(\sigma) \subset \bar{V}\}$, $\bar{T}_\perp = \{\sigma \in T;\ B(\sigma) \subset \bar{V} \cap R\}$, $\bar{T}_2 = \bar{T} \backslash \bar{T}_1$, $\bar{O}$ the subgroup of 0 generated by $\bar{T}$, $\bar{\mathfrak{F}}$ the free group generated by $\bar{T}$, and $\bar{S}$ the normal subgroup of $\bar{\mathfrak{F}}$ generated by all n-relations with $n \leqslant 4$.

Theorem 2. *Let* $|K| > 3$. *If* (π) *is a relation in* $\bar{\mathfrak{F}}$, *then* $(\pi) \equiv \varnothing$ mod $\bar{S}$.

We note that in case $|K| = 3$, the result is also true if $\bar{V} \neq H + \mathrm{rad}\,\bar{V}$, or $\mathrm{rad}\,\bar{V} \subset R$, or $\dim \bar{V} < 4$. Here, H is a hyperbolic plane and $\mathrm{rad}\,\bar{V}$ is the radical of $\bar{V}$.

The proof is similar to that of Theorem 1. We shall list the necessary modifications. We replace T, T_i, and S by $\bar{T}, \bar{T}_i$, and $\bar{S}$, respectively. If $Q(x) = 0$ for all $x \in \bar{V} \backslash R$, then $(\pi) = (\tau_{k+1}, \ldots, \tau_n)$ and we can apply Lemma 3. Now we assume there is some $x \in \bar{V} \backslash R$ such that $Q(x) \neq 0$. Clearly, $B(\kappa) \subset \bar{V} \cap \mathrm{SR}$, $B(\tau_i') \subset B(\kappa)$, and $a_i \in [\Sigma_i B(\rho_i)] \backslash R$ by the last sentence in the proof of Lemma 18 in [3]. The existence of $\sigma_i' \in T_2$ follows again with the help of Lemmas 1 and 2. This is clear if either $Q(a_i) \neq 0$ or if $Q(a_i) = 0$ and $a_i \in \bar{V} \backslash \mathrm{rad}\,\bar{V}$. If $Q(a_i) = 0$ and $a_i \in \mathrm{rad}\,\bar{V} \backslash R$, then we take any $a' \in \bar{V} \backslash R$ such that $Q(a') \neq 0$ (such an a' exists by our assumption) and put $b' = a' - a$. Then $Q(b') = Q(a')$. Therefore, we can apply Lemma 2(ii). We use [7, Theorem] and [6, Theorem 8.3] instead of [1, Satz 2] and [5, Theorem 6.1].

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Received 27 May 1980