On the Existence of Bases of Class C^p of the Kernel and the Image of a Matrix Function

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ABSTRACT

This paper deals with the existence of bases of class C^p of the kernel and the image of a rectangular matrix function of q real variables. Fundamental results on this subject, extended from matrix functions to operator functions, have already been established by I. C. Gohberg and J. Leiterer, by means of general properties of cocycles in algebras of operator functions. This paper was prepared independently, and new elementary methods were found, using three tools of general interest: The first is a condition which guarantees the existence of global solutions defined on $\mathbb{R}^q$ of problems possessing local solutions. The second is the property that every subset of $\mathbb{C}^{m\times n}$ consisting of all the matrices of the same rank is analytically arcwise connected. The third is the smoothness of the Moore-Penrose inverse of a matrix function of class C^p. Applications to equivalences of class C^p, to linear equations on matrix functions of class C^p, to QR decomposition of class C^p, and to simultaneous unitary diagonalization of class C^p of a basis family of projector functions are expounded. Several results are established in the more general case of matrix functions defined on an open subset of a Banach space. This paper suggests problems on the generalization of its results from matrix functions to operator functions.

1. INTRODUCTION

The subject of this paper originates from the study of the nonlinear matrix differential equation

$$X(t)X'(t)=X'(t)X(t), \qquad t\in\Omega\subset\mathbb{R},$$

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and the results established here are likely to justify substitutions in matrix differential equations of order p. For example, the above equation was solved by I. J. Epstein [4] by substituting into it

$$X(t) = P(t)J(t)P(t)^{-1},$$

where $J(t)$ is in Jordan form and $P(t)$ is invertible. This substitution raises the problem of the differentiability of P and J. This differentiability has been established by J.-M. Gracia [8, Theorem 1] in the case where X is a matrix function of class C^1 and of constant Segre characteristic, defined on an open interval in $\mathbb{R}$. The proof of the existence of P of class C^p amounts to that of the existence of bases of class C^p of the spectral subspaces of X, which have the form $\operatorname{Ker}(X - \xi I_n)^m$, where ξ is an eigenvalue of class C^p of X.

The existence of bases of class C^p of the kernel and the image of a square matrix function of one real variable of constant rank and of class C^p was established by V. Doležal [3], and this result has been improved in [16–18]. A huge extension of this result to the case where A is an operator function satisfying very usual hypotheses, and defined on a contractible compact subset Ω of $\mathbb{R}^q$, has been achieved by I. C. Gohberg and J. Leiterer [7], by means of general properties of cocycles in algebras of operator functions. Moreover, these authors have shown that the hypothesis that Ω is contractible is necessary if and only if $q \geqslant 3$. In the more general case where Ω is compact, but not necessarily contractible, they have reduced the problem from the differentiable case to the continuous case. The present paper was prepared independently, and its main result, on the existence of bases of class C^p of the kernel and the image of a rectangular matrix function defined on a not necessarily bounded domain $\Omega \subset \mathbb{R}^q$, has been established by means of new elementary methods.

The main result of this paper is established in Section 8, by means of several preliminary results of general interest. Section 2 presents a generalization of a result of Y. Hirasawa [11] on the smoothness of the coefficients of a linear combination of vector functions. Section 3 is devoted to the QR decomposition of class C^p of a matrix function. Section 4 deals with the smoothness of the Moore-Penrose inverse of a matrix function and its immediate applications. Section 5 furnishes an example of a 2×2 hermitian matrix function of rank 1 and of class C^∞ defined on $\mathbb{R}^3 \setminus \{0\}$, whose image does not possess any continuous basis, even when it is restricted to the unit sphere. Section 6 establishes, under usual hypotheses, the existence of global solutions defined on $\mathbb{R}^q$ of problems possessing local solutions. Section 7 proves that every subset of $\mathbb{C}^{m\times n}$ consisting of all the matrices of the same rank is analytically arcwise connected. Section 8 contains the main result of this paper, namely Theorem 8.2, which establishes the existence of orthonormal bases of class C^p of $\operatorname{Ker} A$, $\operatorname{Im} A$, $(\operatorname{Ker} A)^{\perp}$, and $(\operatorname{Im} A)^{\perp}$, when A is a rectangular matrix function of constant rank and of class C^p, defined on a domain C^p-diffeomorphic to $\mathbb{R}^q$. As the results of [7], this theorem generalizes that of V. Doležal [3], by a quite different proof, and it answers the question raised by T. Kato [12, footnote, p. 136]. Section 9 expounds applications of this theorem, notably, on the general form of the solutions of class C^p of the linear matrix equation

$$A(t)X(t)=B(t);$$

on equivalences of class C^p between matrix functions:

$$A(t)=U(t)\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P(t), \qquad U(t)^*U(t)=U(t)U(t)^*=I_m;$$

on the rank decomposition of class C^p of matrix functions of rank r:

$$A(t)=U(t)P(t), \qquad U(t)\in\mathbb{C}^{m\times r}, \quad P(t)\in\mathbb{C}^{r\times n}, \quad U(t)^*U(t)=I_r;$$

and on the simultaneous unitary diagonalization of class C^p of a basis family $(P_1,\dots,P_s)$ of projector functions:

$$P_i(t)=U(t)\operatorname{diag}\left[0,I_{r_i},0\right]U(t)^{-1}.$$

This paper deals with rectangular matrix functions of class C^p ($p=0,1,2,\dots,\infty$) defined on a domain Ω. In Sections 2, 3, 4, the domain Ω is an open subset of a Banach space and the proofs are also valid in the analytic case. In Sections 8 and 9, the domain Ω is C^p-diffeomorphic to $\mathbb{R}^q$, and the proofs are not valid in the analytic case, because they are based on the existence of partitions of unity of class C^p, and the analytic continuation theorem implies that analytic partitions of unity do not exist. However, results on the analytic case have been established in [6], [12], [13], [15], [16], and [17].

Through this paper, the standard notation and terminology of [13] are used, with the following adjunctions and modifications:

$$\mathbb{R}_+^*=\{t\in\mathbb{R}|t>0\}, \qquad \mathbb{C}_r^{m\times n}=\{M\in\mathbb{C}^{m\times n}|\operatorname{rank} M=r\},$$

I_n and 0_n respectively denote the identity matrix and the null matrix of $\mathbb{C}^{n\times n}$, and $\langle x|y\rangle$ denotes the scalar product of any x,y in $\mathbb{C}^n$. If S is a vector subspace of $\mathbb{C}^n$, then $P_S \in \mathbb{C}^{n\times n}$ denotes the orthogonal projector on S. Let Ω be a set. If A is a map from Ω into $\mathbb{C}^{m\times n}$, then A^* denotes the map $t \mapsto A(t)^*$ from Ω into $\mathbb{C}^{n\times m}$. If A is a map from Ω into $\mathbb{C}_n^{n\times n}$, then A^{-1} does not denote any inverse map of A from $\mathbb{C}_n^{n\times n}$ into Ω, but denotes the map $t \mapsto A(t)^{-1}$ from Ω into $\mathbb{C}_n^{n\times n}$. If A is a map from Ω into $\mathbb{C}^{m\times n}$, then $\operatorname{Ker} A$ and $\operatorname{Im} A$ do not denote the sets $\{t\in\Omega | A(t)=0\}$ and $A(\Omega)$, but denote the maps $t\mapsto \operatorname{Ker} A(t)$ and $t \mapsto \operatorname{Im} A(t)$ respectively. If for every $t\in\Omega$, $S(t)$ is a vector subspace of $\mathbb{C}^n$, then $S^\perp$ and P_S denote the maps $t\mapsto S(t)^\perp$ and $t\mapsto P_{S(t)}$ respectively. If Ω_1 and Ω_2 are topological spaces, then $C^0(\Omega_1,\Omega_2)$ denotes the set of continuous maps from Ω_1 into Ω_2. If $p\in\{1,2,\dots\}$, $\mathbb{E}$, $\mathbb{F}$ are $\mathbb{R}$-normed spaces, and Ω is an open subset of $\mathbb{E}$, then $C^p(\Omega,\mathbb{F})$ denotes the set of maps from Ω into $\mathbb{F}$ of class C^p, that is to say, maps which are p times continuously Fréchet differentiable (when $\mathbb{E}$ is an $\mathbb{R}$-normed space and $\mathbb{F}$ is a $\mathbb{C}$-normed space, then, relative to the Fréchet differentiation, $\mathbb{F}$ is considered as an $\mathbb{R}$-Banach space), and finally,

$$C^\infty(\Omega,\mathbb{F}) = \bigcap_{p=0}^{\infty} C^p(\Omega,\mathbb{F}).$$

Through this paper, $p\in\{0,1,2,\dots\}\cup\{\infty\}$.

2. SMOOTHNESS OF THE COEFFICIENTS OF A LINEAR COMBINATION OF VECTOR FUNCTIONS

Let us recall the two following well-known lemmas, which will be extensively used throughout this paper.

LEMMA 2.1. *Let* $A\in\mathbb{C}^{m\times n}$. *Then*

(a) $\operatorname{Ker} A = (\operatorname{Im} A^*)^\perp$ *and* $\operatorname{Im} A = (\operatorname{Ker} A^*)^\perp$;
(b) $\operatorname{Ker}(A^*A) = \operatorname{Ker} A$ *and* $\operatorname{Im}(AA^*) = \operatorname{Im} A$;
(c) $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank} A = \operatorname{rank} A^*$.

Proof. See [2, Propositions 0.2.1 and 0.2.2, p. 31]. ∎

LEMMA 2.2 (Expression of the unique solution of the linear matrix equations $AX = B$ and $XA = B$ in the full rank case).

(a) *Let* $A \in \mathbb{C}_n^{m \times n}$, $B \in \mathbb{C}^{m \times s}$, *and* $X \in \mathbb{C}^{n \times s}$. *Then*

$$AX = B \quad \Leftrightarrow \quad X = (A^*A)^{-1}A^*B \text{ and } \operatorname{Im} B \subset \operatorname{Im} A.$$

(b) *Let* $A \in \mathbb{C}_m^{m \times n}$, $B \in \mathbb{C}^{s \times n}$, *and* $X \in \mathbb{C}^{s \times m}$. *Then*

$$XA = B \quad \Leftrightarrow \quad X = BA^*(AA^*)^{-1} \text{ and } \operatorname{Ker} A \subset \operatorname{Ker} B.$$

Proof. (a): By Lemma 2.1(c), (A^*A) is invertible. On the other hand, it is well known that $\operatorname{Im} B \subset \operatorname{Im} A$ if and only if there exists $Y \in \mathbb{C}^{n \times s}$ such that $B = AY$, which implies (a).

(b): By (a) applied to A^*, B^*, and X^*,

$$A^*X^* = B^* \quad \Leftrightarrow \quad X^* = (AA^*)^{-1}AB^* \text{ and } \operatorname{Im} B^* \subset \operatorname{Im} A^*.$$

On the other hand, by Lemma 2.1(a), $\operatorname{Im} B^* \subset \operatorname{Im} A^*$ if and only if $\operatorname{Ker} A \subset \operatorname{Ker} B$. Hence, the conclusion follows. ■

The following theorem generalizes the theorem of [11], with a simpler proof.

THEOREM 2.3 (Smoothness of the coefficients of a linear combination of vector functions). *Let* Ω *be an open subset of an* $\mathbb{R}$*-Banach space. Let* $b, a_1, \dots, a_n \in C^p(\Omega, \mathbb{C}^m)$, *and for every* $t \in \Omega$, *let* $x_1(t), \dots, x_n(t) \in \mathbb{C}$ *be such that* $b = x_1 a_1 + \cdots + x_n a_n$ *and* $a_1(t), \dots, a_n(t)$ *are linearly independent, for every* $t \in \Omega$. *Then* $x_1, \dots, x_n \in C^p(\Omega, \mathbb{C})$.

Proof. Let

$$A = [a_1 \quad \cdots \quad a_n], \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then A is of class C^p, of rank n, and $Ax = b$. By virtue of Lemma 2.2(a),

$$x = (A^*A)^{-1}A^*b \in C^p(\Omega, \mathbb{C}^n).$$ ■

3. *QR* DECOMPOSITION OF CLASS C^p OF A RECTANGULAR MATRIX FUNCTION

PROPOSITION 3.1 (Gram-Schmidt orthonormalization of class C^p). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $a_1,\dots,a_n \in C^p(\Omega,\mathbb{C}^m)$ be such that at every point $t\in\Omega$, $a_1(t),\dots,a_n(t)$ are linearly independent. Then there exists a unique family of vector functions $u_1,\dots,u_n \in C^p(\Omega,\mathbb{C}^m)$ such that for every $t\in\Omega$, $i,j\in\{1,\dots,n\}$,*

$$\operatorname{span}\{a_1(t),\dots,a_j(t)\} = \operatorname{span}\{u_1(t),\dots,u_j(t)\},$$

$$\langle u_i(t)|u_j(t)\rangle = \delta_{ij}, \qquad \langle a_i(t)|u_i(t)\rangle \in \mathbb{R}_+^*.$$

Proof. It is easy to check by induction that a family $(u_1,\dots,u_n)$ possesses the above properties if and only if for every $t\in\Omega$, $j\in\{1,\dots,n\}$,

$$u_j(t) = \frac{v_j(t)}{\|v_j(t)\|},$$

where $\|v\|$ denotes $\langle v|v\rangle^{1/2}$ for every $v\in\mathbb{C}^m$, and

$$v_1(t) = a_1(t) \neq 0, \qquad j>1 \;\Rightarrow\; v_j(t) = a_j(t) - \sum_{i=1}^{j-1} \langle a_j(t)|u_i(t)\rangle u_i(t) \neq 0. \qquad \blacksquare$$

THEOREM 3.2 (*QR* decomposition of class C^p of a rectangular matrix function with constant partial ranks). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $r\in\{1,2,\dots\}$. Let*

$$A = [\,a_1 \quad \cdots \quad a_n\,] \in C^p(\Omega, \mathbb{C}_r^{m\times n})$$

be such that for each $j\in\{1,\dots,n\}$, $\operatorname{rank}[a_1 \dots a_j]$ is constant. For each $k\in\{1,\dots,r\}$, let

$$j_k = \min\{j\in\{1,\dots,n\} \,|\, \operatorname{rank}[\,a_1 \quad \cdots \quad a_j\,] = k\}.$$

Then there exists a unique pair $(Q, R=(r_{ij})) \in C^p(\Omega,\mathbb{C}_r^{m\times r})\times C^p(\Omega,\mathbb{C}_r^{r\times n})$

such that

$$A = QR, \qquad Q^*Q = I_r,$$

and for every $t \in \Omega$, $j \in \{1, \dots, n\}$, $k \in \{1, \dots, r\}$,

$$r_{kj_k}(t) \in \mathbb{R}_+^*, \qquad j < j_k \ \Rightarrow \ r_{kj} = 0.$$

Proof. It is easy to check by induction that $a_{j_1}(t), \dots, a_{j_r}(t)$ are linearly independent, and consequently constitute a basis of $\operatorname{Im} A(t)$, for every $t \in \Omega$. Therefore, by Proposition 3.1, there exists a unique family of vector functions $q_1, \dots, q_r \in C^p(\Omega, \mathbb{C}^m)$ such that for every $t \in \Omega$, $i, k \in \{1, \dots, r\}$,

$$\operatorname{span}\{q_1(t), \dots, q_k(t)\} = \operatorname{span}\{a_{j_1}(t), \dots, a_{j_k}(t)\}, \tag{1}$$

$$\langle q_i(t) | q_k(t) \rangle = \delta_{ik}, \tag{2}$$

$$\langle a_{j_k}(t) | q_k(t) \rangle \in \mathbb{R}_+^*. \tag{3}$$

Let $Q = [q_1 \ \cdots \ q_r]$. By (2), $Q^*Q = I_r$, hence Q is of rank r, and by (1), $\operatorname{Im} A = \operatorname{Im} Q$. Therefore, by Lemma 2.2(a), there exists $R = (r_{ij}) \in C^p(\Omega, \mathbb{C}^{r \times n})$ such that $QR = A$. Let $t \in \Omega$. The equality $R = Q^*A$ implies that

$$r_{ij}(t) = q_i(t)^* a_j(t) = \langle a_j(t) | q_i(t) \rangle \qquad \forall i \in \{1, \dots, r\},\ j \in \{1, \dots, n\}. \tag{4}$$

In particular, by (3),

$$r_{kj_k}(t) = \langle a_{j_k}(t) | q_k(t) \rangle \in \mathbb{R}_+^* \qquad \forall k \in \{1, \dots, r\}.$$

Let $k \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$ be such that $j < j_k$. If $k = 1$, then, by the definition of j_1, $a_j = 0$, which implies by (4) that $r_{kj} = 0$. Let us suppose that $k > 1$. By the definition of j_k, and by (1) and (2),

$$a_j(t) \in \operatorname{span}\{a_{j_1}(t), \dots, a_{j_{k-1}}(t)\} = \operatorname{span}\{q_1(t), \dots, q_{k-1}(t)\} \perp q_k(t);$$

hence by (4), $r_{kj}(t) = 0$.

Let $V = [v_1 \ \cdots \ v_r] \in C^p(\Omega, \mathbb{C}_r^{m\times r})$ and $S = (s_{ij}) \in C^p(\Omega, \mathbb{C}_r^{r\times n})$ be such that $A = VS$, $V^*V = I_r$, and

$$s_{kj_k}(t) \in \mathbb{R}_+^*, \ j < j_k \Rightarrow s_{kj} = 0 \qquad \forall t \in \Omega, \, k \in \{1,\ldots,r\}, \, j \in \{1,\ldots,n\}.$$

It proceeds from the quasitriangular form of S that

$$\operatorname{span}\{a_{j_1}(t),\ldots,a_{j_k}(t)\} = \operatorname{span}\{v_1(t),\ldots,v_k(t)\} \ \forall t \in \Omega, \, k \in \{1,\ldots,r\}.$$

On the other hand, $S = V^*A$; hence $\langle a_{j_k}(t) | v_k(t) \rangle = s_{kj_k}(t) \in \mathbb{R}_+^*$ for every $t \in \Omega$ and $k \in \{1,\ldots,r\}$. By the uniqueness assertion of Proposition 3.1, it follows that $v_1 = q_1, \ldots, v_r = q_r$, that is to say $V = Q$. Therefore, $S = V^*A = Q^*A = R$. ∎

COROLLARY 3.3 (QR decomposition of class C^p of an $m \times n$ matrix function of rank n). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $A \in C^p(\Omega, \mathbb{C}_n^{m\times n})$. Then there exists a unique pair (Q, R) such that $Q \in C^p(\Omega, \mathbb{C}_n^{m\times n})$, $R = (r_{ij}) \in C^p(\Omega, \mathbb{C}_n^{n\times n})$ is upper triangular, and*

$$A = QR, \qquad Q^*Q = I_n, \qquad r_{ii}(t) \in \mathbb{R}_+^* \quad \forall t \in \Omega, \, i \in \{1,\ldots,n\}.$$

Proof. For every $j \in \{1,\ldots,n\}$, let a_j denote the jth column of A. By hypothesis, for every $t \in \Omega$, $a_1(t),\ldots,a_n(t)$ are linearly independent. Therefore,

$$\operatorname{rank}[a_1,\ldots,a_j] = j \qquad \forall j \in \{1,\ldots,n\},$$

and the conclusion follows by Theorem 3.2. ∎

The following corollary deals with the local rank decomposition of a matrix function defined on an open subset Ω of a Banach space. A global version of this decomposition will be furnished by Corollary 9.4, but only in the particular case where Ω is C^p-diffeomorphic to $\mathbb{R}^q$.

COROLLARY 3.4 (Local rank decomposition of class C^p). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $r \in \{1,2,\ldots\}$, $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$, and $t_0 \in \Omega$. Then there exists an open neighborhood $\Omega_0 \subset \Omega$ of t_0, and there exist $U, C \in C^p(\Omega_0, \mathbb{C}_r^{m\times r})$, $V, B \in C^p(\Omega_0, \mathbb{C}_r^{r\times n})$, such that*

$$A(t) = U(t)B(t) = C(t)V(t) \quad \forall t \in \Omega_0, \qquad U^*U = I_r, \quad VV^* = I_r.$$

Proof. For each $j \in \{1,\dots,n\}$, let a_j denote the jth column of A. As A is of rank r, there exist $j_1,\dots,j_r \in \{1,\dots,n\}$ such that $(a_{j_1}(t_0),\dots,a_{j_r}(t_0))$ is a basis of $\operatorname{Im} A(t_0)$. By [5, Lemma 5.4], there exists an open neighborhood $\Omega_1 \subset \Omega$ of t_0 such that for every $t \in \Omega_1$, $(a_{j_1}(t),\dots,a_{j_r}(t))$ is a basis of $\operatorname{Im} A(t)$. By Proposition 3.1, there exists $U = [u_1 \ \dots \ u_r] \in C^p(\Omega_1, \mathbb{C}_r^{m\times r})$ such that $U^*U = I_r$ and $\operatorname{Im} U(t) = \operatorname{span}\{a_{j_1}(t),\dots,a_{j_r}(t)\} = \operatorname{Im} A(t)$ for every $t \in \Omega_1$. Let $t \in \Omega_1$. Let $B(t) = U(t)^*A(t)$. By Lemma 2.2(a), $U(t)B(t) = A(t)$, which implies that $B(t)$ is of rank r. Thus $B \in C^p(\Omega_1, \mathbb{C}_r^{r\times n})$.

By the part of Corollary 3.4 proved so far applied to A^*, there exists an open neighborhood $\Omega_0 \subset \Omega_1$ of t_0 and there exist $\tilde{U} \in C^p(\Omega_0, \mathbb{C}_r^{n\times r})$, $\tilde{B} \in C^p(\Omega_0, \mathbb{C}_r^{r\times m})$ such that

$$A(t)^* = \tilde{U}(t)\tilde{B}(t), \quad \tilde{U}(t)^*\tilde{U}(t) = I_r \qquad \forall t \in \Omega_0.$$

The remainder of the conclusion is obtained with $V = \tilde{U}^*$ and $C = \tilde{B}^*$. ■

4. SMOOTHNESS OF THE MOORE-PENROSE INVERSE OF A RECTANGULAR MATRIX FUNCTION AND APPLICATIONS

Let us recall the following theorem, due to J. Z. Hearon and J. W. Evans, on the smoothness of the Moore-Penrose inverse A^+ of a matrix function A. As it is very useful, we prove it by a new short proof.

THEOREM 4.1 (Smoothness of the Moore-Penrose inverse A^+ of a matrix function A: J. Z. Hearon and J. W. Evans [10, Theorem 1]). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $r \in \{0,1,\dots\}$. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then $A^+ \in C^p(\Omega, \mathbb{C}_r^{n\times m})$.*

Proof. If $r = 0$, then $A = 0$, and $A^+ = 0$ is of class C^p. Let us suppose that $r > 0$. It is sufficient to prove that A^+ is of class C^p on a neighborhood of every point of Ω. Let $t_0 \in \Omega$. By Corollary 3.4, there exists a neighborhood $\Omega_0 \subset \Omega$ of t_0, and there exist $U \in C^p(\Omega_0, \mathbb{C}_r^{m\times r})$, $B \in C^p(\Omega_0, \mathbb{C}_r^{r\times n})$, such that $U^*U = I_r$ and $A(t) = U(t)B(t)$ for every $t \in \Omega_0$. By virtue of [1, Theorem 5, p. 23],

$$A^+|_{\Omega_0} = B^*(BB^*)^{-1}U^* \in C^p\big(\Omega_0, \mathbb{C}_r^{n\times m}\big). \qquad ■$$

The following corollary deals with the existence of solutions of class C^p of the linear equation $AX = B$ on matrix functions defined on an open subset Ω of a Banach space. The general form of the solutions of class C^p of this equation will be furnished by Corollary 9.1, but only in the particular case where Ω is C^p-diffeomorphic to $\mathbb{R}^q$. Another complement of information lies in Lemma 2.2.

COROLLARY 4.2 (Existence of solutions of class C^p of the matrix linear equations $AX = B$ and $XA = B$). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then the following two assertions are true:*

(a) *Let $B \in C^p(\Omega, \mathbb{C}^{m\times s})$ be such that* $\operatorname{Im} B(t) \subset \operatorname{Im} A(t)$ *for every $t \in \Omega$. Then there exists $X \in C^p(\Omega, \mathbb{C}^{n\times s})$ such that $AX = B$.*

(b) *Let $B \in C^p(\Omega, \mathbb{C}^{s\times n})$ be such that* $\operatorname{Ker} A(t) \subset \operatorname{Ker} B(t)$ *for every $t \in \Omega$. Then there exists $X \in C^p(\Omega, \mathbb{C}^{s\times m})$ such that $XA = B$.*

Proof. (a): Let $X = A^+B$. By Theorem 4.1, X is of class C^p. Let $t \in \Omega$. By [2, Theorem 1.1.1], $A(t)X(t) = P_{\operatorname{Im} A(t)}B(t)$, and since $\operatorname{Im} B(t) \subset \operatorname{Im} A(t)$, $P_{\operatorname{Im} A(t)}B(t) = B(t)$. Thus $AX = B$.

(b): By Lemma 2.1(a), $\operatorname{Im} B(t)^* \subset \operatorname{Im} A(t)^*$ for every $t \in \Omega$. Therefore, by (a), there exists $Y \in C^p(\Omega, \mathbb{C}^{m\times s})$ such that $A^*Y = B^*$. Hence $XA = B$, where $X = Y^* \in C^p(\Omega, \mathbb{C}^{s\times m})$. ■

COROLLARY 4.3 [Smoothness of the orthogonal projectors on $\operatorname{Ker} A(t)$ and $\operatorname{Im} A(t)$]. *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then $P_{\operatorname{Im} A}$, $P_{(\operatorname{Im} A)^\perp}$, $P_{\operatorname{Ker} A}$, and $P_{(\operatorname{Ker} A)^\perp}$ are of class C^p.*

Proof. By [2, Theorem 1.1.1], $P_{\operatorname{Im} A} = AA^+$ and $P_{\operatorname{Im} A^*} = A^+A$. Hence by Lemma 2.1(a),

$$P_{(\operatorname{Ker} A)^\perp} = P_{\operatorname{Im} A^*} = A^+A, \qquad P_{\operatorname{Ker} A} = I_n - A^+A, \qquad P_{(\operatorname{Im} A)^\perp} = I_m - AA^+.$$

By Theorem 4.1, A^+ is of class C^p, and therefore the conclusion is a direct consequence of the above relations. ■

COROLLARY 4.4 (Smoothness of the orthogonal projector on the intersection of two vector subspaces). *Let Ω be an open subset of an $\mathbb{R}$-Banach space. Let $P_1, P_2 \in C^p(\Omega, \mathbb{C}^{n\times n})$ be orthogonal projector valued functions* *(that is to say, $P_i(t)^2 = P_i(t) = P_i(t)^*$ for every $i \in \{1,2\}$, $t \in \Omega$) such that* $\operatorname{rank}(P_1 + P_2)$ *is constant. Then*

$$P_{(\operatorname{Im} P_1)\cap(\operatorname{Im} P_2)} \in C^p(\Omega, \mathbb{C}^{n\times n}).$$

Proof. By Theorem 4.1, $(P_1 + P_2)^+$ is of class C^p. On the other hand, by virtue of [1, Theorem 3, p. 199] (theorem due to W. N. Anderson and R. J. Duffin),

$$P_{(\operatorname{Im} P_1)\cap(\operatorname{Im} P_2)} = 2P_1(P_1 + P_2)^+ P_2,$$

and hence the conclusion follows. ■

5. EXAMPLE OF A 2×2 HERMITIAN MATRIX FUNCTION OF RANK 1 AND OF CLASS C^∞ DEFINED ON $\mathbb{R}^3 \setminus \{0\}$ WHOSE IMAGE DOES NOT POSSESS ANY CONTINUOUS BASIS

This section is devoted to an example of a hermitian matrix function A of class C^∞ and of rank 1 defined on $\mathbb{R}^3 \setminus \{0\}$ whose image does not possess any continuous basis, not even when it is restricted to the unit sphere. A very similar (but nonhermitian) example, defined on $\mathbb{C} \cup \{\infty\}$, was published in 1976 by I. C. Gohberg and J. Leiterer [7, §5, Counterexample 1]. This example shows that in [6, Corollary 13.6.5], it is necessary to suppose that K is contractible (except when $K \subset \mathbb{R}^2$ [7, Corollary 4.2]).

In this section, $\Omega = \mathbb{R}^3 \setminus \{0\}$ and the following notation will be used. Let

$$\|t\| = \sqrt{t_1^2 + t_2^2 + t_3^2} \quad \forall t = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}^T \in \mathbb{R}^3, \qquad S^2 = \left\{ t \in \mathbb{R}^3 \,\middle|\, \|t\| = 1 \right\}.$$

The matrix $A(t)$ that we will define will depend only on the direction of $t \in \mathbb{R}^3$. In order to show easily that A is of class C^∞, we will first define a matrix function $\tilde{A}(\theta, \varphi)$ of the angular coordinates of $t \in \mathbb{R}^3$. In order to make the bases of $\operatorname{Im} A$ discontinuous around the poles of the unit sphere without destroying the smoothness of A, we will make A constant on a neighborhood of the poles by means of a function $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\eta(\theta) = 1 \quad \forall \theta \in \left]-\infty, \tfrac{1}{3}\pi\right], \qquad \eta(\theta) = 0 \quad \forall \theta \in \left[\tfrac{2}{3}\pi, \infty\right[,$$

$$0 < \eta(\theta) < 1 \qquad \forall \theta \in \left]\tfrac{1}{3}\pi, \tfrac{2}{3}\pi\right[,$$

whose existence is well known. Let

$$v(\theta,\varphi)=\begin{bmatrix}\eta(\theta)\\ [1-\eta(\theta)]e^{-i\varphi}\end{bmatrix},$$

where $i^2=-1$, and let $\tilde{A}\in C^\infty(\mathbb{R}^2,\mathbb{C}_1^{2\times 2})$ be defined by

$$\begin{aligned}\tilde{A}(\theta,\varphi)&=\begin{bmatrix}\eta(\theta)v(\theta,\varphi) & [1-\eta(\theta)]e^{i\varphi}v(\theta,\varphi)\end{bmatrix}\\ &=\begin{bmatrix}\eta(\theta)^2 & \eta(\theta)[1-\eta(\theta)]e^{i\varphi}\\ \eta(\theta)[1-\eta(\theta)]e^{-i\varphi} & [1-\eta(\theta)]^2\end{bmatrix} \qquad \forall(\theta,\varphi)\in\mathbb{R}^2.\end{aligned}$$

Let

$$\Omega_1=\left\{\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\mathbb{R}^3 \middle| t_2\neq 0 \text{ or } t_1>0\right\},$$

$$\Omega_2=\left\{\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\mathbb{R}^3 \middle| t_2\neq 0 \text{ or } t_1<0\right\},$$

$$\Omega_3=\Omega_1\cup\Omega_2=\left\{\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\mathbb{R}^3 \middle| (t_1,t_2)\neq(0,0)\right\}.$$

It is easy to check that there exist $\Theta\in C^0(\mathbb{R}^3\setminus\{0\},[0,\pi])$, $\Phi_1\in C^\infty(\Omega_1,]-\pi,\pi[)$, and $\Phi_2\in C^\infty(\Omega_2,]0,2\pi[)$ such that $\Theta|_{\Omega_3}\in C^\infty(\Omega_3,]0,\pi[)$ and

$$t_1=\|t\|\sin\Theta(t)\cos\Phi_i(t) \qquad \forall i\in\{1,2\},\ t=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\Omega_i,$$

$$t_2=\|t\|\sin\Theta(t)\sin\Phi_i(t) \qquad \forall i\in\{1,2\},\ t=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\Omega_i,$$

$$t_3=\|t\|\cos\Theta(t) \qquad \forall t=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T\in\mathbb{R}^3\setminus\{0\}.$$

For every $t=[t_1\ t_2\ t_3]^T\in\Omega$, let

$$A(t)=\begin{cases}\tilde{A}(\Theta(t),\Phi_1(t)) & \text{if } t_2\neq 0 \text{ or } t_1>0,\\ \tilde{A}(\Theta(t),\Phi_2(t)) & \text{if } t_2=0 \text{ and } t_1<0,\\ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \text{if } t_1=t_2=0 \text{ and } t_3>0,\\ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} & \text{if } t_1=t_2=0 \text{ and } t_3<0,\end{cases}$$

and let

$$\Omega_n = \{t \in \Omega | 0 \leqslant \Theta(t) < \tfrac{1}{3}\pi\}, \qquad \Omega_s = \{t \in \Omega | \tfrac{2}{3}\pi < \Theta(t) \leqslant \pi\}.$$

As $\tilde{A}(\theta, \varphi + 2\pi) = \tilde{A}(\theta, \varphi)$ for every $\theta, \varphi \in \mathbb{R}$, it follows directly from the definitions that

$$A(t) = \tilde{A}(\Theta(t), \Phi_1(t)) \quad \forall t \in \Omega_1, \qquad A(t) = \tilde{A}(\Theta(t), \Phi_2(t)) \quad \forall t \in \Omega_2,$$

$$A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \forall t \in \Omega_n, \qquad A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \forall t \in \Omega_s;$$

hence $A \in C^\infty(\Omega, \mathbb{C}_1^{2\times 2})$, because $(\Omega_1, \Omega_2, \Omega_n, \Omega_s)$ is an open covering of Ω.

Now let us show *ad absurdum* that the image of A does not possess any continuous basis, even when it is restricted to $S^2 \subset \Omega$. Let $v = [v_1 \ v_2]^T \in C^0(S^2, \mathbb{C}^2 \setminus \{0\})$ be such that

$$\operatorname{Im} A(t) = \operatorname{span}\{v(t)\} \qquad \forall t \in S^2.$$

Let us show that if such a function v existed, the unit circle $S^1 = \{z \in \mathbb{C} | |z| = 1\}$ in $\mathbb{C}$ would be contractible, in particular simply connected, by means of a homotopy $H(z, \lambda)$ that we will define below. Let

$$\Sigma_1 = \{t \in S^2 | 0 \leqslant \Theta(t) < \tfrac{2}{3}\pi\}, \qquad \Sigma_2 = \{t \in S^2 | \tfrac{1}{3}\pi < \Theta(t) \leqslant \pi\}.$$

Let $j \in \{1,2\}$. Let $a_j = [a_{1j} \ a_{2j}]^T$ be the jth column of A. For every $t \in S^2$, as $a_j(t) \in \operatorname{Im} A(t) = \operatorname{span}\{v(t)\}$, there exists $\alpha_j(t) \in \mathbb{C}$ such that $a_j(t) = \alpha_j(t) v(t)$. Then

$$\alpha_1(t) v_1(t) = a_{11}(t) = [\eta(\Theta(t))]^2 \neq 0, \qquad v_1(t) \neq 0 \quad \forall t \in \Sigma_1,$$

$$\alpha_2(t) v_2(t) = a_{22}(t) = [1 - \eta(\Theta(t))]^2 \neq 0, \quad v_2(t) \neq 0 \quad \forall t \in \Sigma_2,$$

which allows us to define, for every $i \in \{1,2\}$, $u_i \in C^0(\Sigma_i, S^1)$ by

$$u_i(t) = \frac{v_i(t)}{|v_i(t)|} \qquad \forall t \in \Sigma_i.$$

Let

$$z_1 = u_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \in S^1, \qquad z_2 = u_2\left(\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right) \in S^1,$$

$$r(z,\theta) = \begin{bmatrix} \mathcal{R}e\, z \sin\theta \\ \mathcal{I}m\, z \sin\theta \\ \cos\theta \end{bmatrix} \qquad \forall z \in S^1,\ \theta \in \mathbb{R}.$$

It is easy to see that $r \in C^0(S^1 \times \mathbb{R}, S^2)$ and that for every $(z,\theta) \in S^1 \times [0,\pi]$, $\Theta(r(z,\theta)) = \theta$,

$$\theta \in [0, \tfrac{2}{3}\pi[\ \Rightarrow\ r(z,\theta) \in \Sigma_1, \qquad \theta \in]\tfrac{1}{3}\pi, \pi] \ \Rightarrow\ r(z,\theta) \in \Sigma_2,$$

which grounds the following definitions:

$$H_1(z,\lambda) = \frac{u_1(r(z,\lambda\pi))}{z_2} \qquad \forall z \in S^1,\quad \lambda \in [0, \tfrac{2}{3}[,$$

$$H_2(z,\lambda) = \frac{z u_2(r(z,\lambda\pi))}{z_2} \qquad \forall z \in S^1,\quad \lambda \in]\tfrac{1}{3}, 1],$$

$$H(z,\lambda) = H_1(z,\lambda) \qquad \forall z \in S^1,\quad \lambda \in [0, \tfrac{2}{3}[,$$

$$H(z,\lambda) = H_2(z,\lambda) \qquad \forall z \in S^1,\quad \lambda \in [\tfrac{2}{3}, 1].$$

It is immediately seen that $H_1 \in C^0(S^1 \times [0, \tfrac{2}{3}[, S^1)$ and $H_2 \in C^0(S^1 \times]\tfrac{1}{3}, 1], S^1)$.

Let us show that

$$H_1(z,\lambda) = H_2(z,\lambda) \qquad \forall z \in S^1,\ \lambda \in]\tfrac{1}{3}, \tfrac{2}{3}[.$$

Let $z \in S^1$ and $\lambda \in]\tfrac{1}{3}, \tfrac{2}{3}[$. Let $\theta = \lambda\pi$ and $t = r(z,\theta)$. Since $z \in S^1$, there exists $\varphi \in [0, 2\pi[$ such that $z = e^{i\varphi}$. Then

$$\tfrac{1}{3}\pi < \theta < \tfrac{2}{3}\pi, \qquad 0 < \eta(\theta) < 1, \qquad \Theta(t) = \theta, \quad t \in \Sigma_1 \cap \Sigma_2 \subset \Omega_3.$$

If $z \neq 1$, then $t \in \Omega_2$ and $\Phi_2(t) = \varphi$. If $z = 1$, then $t \in \Omega_1$ and $\Phi_1(t) = 0 = \varphi$.

In both cases, $A(t)=\tilde{A}(\theta,\varphi)$, and

$$\alpha_1(t)v_1(t)=a_{11}(t)=[\eta(\theta)]^2\neq 0,$$

$$\alpha_1(t)v_2(t)=a_{21}(t)=\eta(\theta)[1-\eta(\theta)]e^{-i\varphi}\neq 0,$$

which validates the following computation:

$$\frac{u_1(t)}{u_2(t)}=\frac{\alpha_1(t)v_1(t)}{|\alpha_1(t)||v_1(t)|}\frac{|\alpha_1(t)||v_2(t)|}{\alpha_1(t)v_2(t)}$$

$$=\frac{[\eta(\theta)]^2}{|[\eta(\theta)]^2|}\frac{|\eta(\theta)[1-\eta(\theta)]e^{-i\varphi}|}{\eta(\theta)[1-\eta(\theta)]e^{-i\varphi}}=e^{i\varphi}=z.$$

Therefore,

$$H_1(z,\lambda)=\frac{u_1(t)}{z_2}=\frac{zu_2(t)}{z_2}=H_2(z,\lambda).$$

Thus H restricted to $S^1\times[0,\frac{2}{3}[$ coincides with H_1, and H restricted to $S^1\times\,]\frac{1}{3},1]$ coincides with H_2; hence $H\in C^0(S^1\times[0,1],S^1)$. On the other hand,

$$H(z,0)=\frac{z_1}{z_2}\in S^1,\quad H(z,1)=z\qquad \forall z\in S^1,$$

that is to say, S^1 is contractible. This implies that S^1 is simply connected [9, Proposition (3.2), p. 12]. But S^1 is not simply connected; therefore v does not exist.

REMARKS. For every $t\in\Omega$, as $A(t)$ is a 2×2 hermitian matrix of rank 1, $\operatorname{Ker}A(t)$ and $\operatorname{Im}A(t)$ are two orthogonal vector subspaces of $\mathbb{C}^2$ of dimension 1. Therefore, the nonexistence of a continuous basis of $\operatorname{Im}A$ implies the nonexistence of a continuous basis of $\operatorname{Ker}A$. On the other hand, for every $t\in\Omega$, $A(t)$ is diagonalizable, and $\operatorname{Ker}A(t)$ and $\operatorname{Im}A(t)$ are the two eigensubspaces of $A(t)$. It follows that there does not exist any continuous eigenvector of A, in spite of the fact that the eigenvalues of A, namely 0 and $2\eta^2-2\eta+1$, are of class C^∞. In other words, there exists a diagonal matrix function D of class C^∞ such that $A(t)$ is similar to $D(t)$ for every $t \in \Omega$, but there does not exist any continuous matrix function P of rank 2 such that $A = PDP^{-1}$.

6. CONNECTION OF LOCAL SOLUTIONS INTO GLOBAL SOLUTIONS

This section deals with the connection of local solutions into global solutions of a problem (P) (in most cases the solutions will be required to be functions of class C^p). This problem is supposed to possess at every point $t \in \mathbb{R}^q$ a solution f_t defined on an open neighborhood V_t of t, and to have the following property: If f_1 and f_2 are solutions of (P) defined on open subsets Ω_1 and Ω_2 of $\mathbb{R}^q$ respectively, and if $\Omega_1 \cap \Omega_2$ is convex, then f_1 and f_2 may be connected into a solution of (P) defined on $\Omega_1 \cup \Omega_2$. In fact, we will only use the weaker hypothesis of the existence of this connection in the particular case when Ω_1 and Ω_2 are themselves convex.

The objective of this section is to show that these local solutions may be connected into global solutions. A first difficulty lies in the fact that the diameter of V_t may be arbitrarily small. A second one lies in the fact that $C_1 \cup C_2$ is not necessarily convex, even when C_1 and C_2 are. The first difficulty is surmounted by Lemma 6.1 and the second one by Lemma 6.2 below.

The statements of this section are formulated in an axiomatic form, which is to be understood in the following way: for every open subset Ω of $\mathbb{R}^q$, $\mathscr{S}(\Omega)$ will represent the set of all the solutions of (P) defined on Ω, and $\mathscr{R}(\Omega)$ will represent the assertion "there exists at least one solution of (P) defined on Ω". The empty map defined on $\varnothing$ will be considered as a solution of (P) on $\Omega = \varnothing$, that is to say, $\mathscr{R}(\varnothing)$ will be considered as true.

LEMMA 6.1 (Condition for a locally true property to be true on $]-r, r[^q$). *Let $r \in \mathbb{R}_+^*$. Let $\mathscr{R}(\Omega)$ be a relation defined (either true or false) for every open subset Ω of $\mathbb{R}^q$, and satisfying the following conditions*:

(a) *For every $t \in [-r, r]^q$, there exists an open neighborhood $\Omega_t \subset \mathbb{R}^q$ of t such that $\mathscr{R}(\Omega_t)$ is true.*

(b) *For all convex open subsets C_1 and C_2 of $\mathbb{R}^q$,*

$$\mathscr{R}(C_1) \text{ and } \mathscr{R}(C_2) \quad \Rightarrow \quad \mathscr{R}(C_1 \cup C_2).$$

(c) *For all open subsets* Ω_1 *and* Ω_2 *of* $\mathbb{R}^q$,

$$\Omega_1 \subset \Omega_2 \text{ and } \mathscr{R}(\Omega_2) \quad \Rightarrow \quad \mathscr{R}(\Omega_1).$$

Then, under these conditions, $\mathscr{R}(]-r,r[^q)$ *is true.*

REMARK. Conditions (a) and (c) imply that $\mathscr{R}(\varnothing)$ is true.

Proof. This lemma will be proved by induction on the dimension q of $\mathbb{R}^q$, and for this purpose, it will be denoted by $\mathscr{L}(q)$. Let us first prove the lemma when $q=1$. Since $\mathscr{R}$ satisfies hypotheses (a) and (c) of $\mathscr{L}(1)$, for every $t \in [-r,r]$ there exists an open interval $I_t \subset \mathbb{R}$ such that $t \in I_t$ and $\mathscr{R}(I_t)$ is true. Since $[-r,r]$ is compact and connected, it follows that there exists a finite family $(s_1,\dots,s_m)$ of points of $[-r,r]$ (numbered in such a way that for every $k \in \{1,\dots,m\}$, $I_{s_1} \cup \cdots \cup I_{s_k}$ is an interval) such that

$$[-r,r] \subset I_{s_1} \cup \cdots \cup I_{s_m}.$$

Then by induction and by (b), $\mathscr{R}(I_{s_1} \cup \cdots \cup I_{s_k})$ is true for every $k \in \{1,\dots,m\}$. By (c), the validity of $\mathscr{R}(I_{s_1} \cup \cdots \cup I_{s_m})$ implies that of $\mathscr{R}(]-r,r[)$. Thus $\mathscr{L}(1)$ is true.

Let $q_0 \in \{2,3,\dots\}$ be such that $\mathscr{L}(1),\dots,\mathscr{L}(q_0-1)$ are true. Let us prove the lemma when $q=q_0$. Since $q_0 \geqslant 2$, there exist $q_1, q_2 \in \{1,\dots, q_0-1\}$ such that $q_0 = q_1+q_2$. For every open subset Ω of $\mathbb{R}^{q_1}$, let

$$\mathscr{R}_1(\Omega) = \mathscr{R}\big(\Omega \times]-r,r[^{q_2}\big).$$

Let us show that $\mathscr{R}_1$ satisfies the hypotheses of $\mathscr{L}(q_1)$. Let $t_1 \in [-r,r]^{q_1}$. Since $\mathscr{R}$ satisfies hypotheses (a) and (c) of $\mathscr{L}(q_0)$, for every $t \in [-r,r]^{q_0}$ there exist open intervals $I_{1t},\dots,I_{q_0 t} \subset \mathbb{R}$ such that $t \in I_{1t} \times \cdots \times I_{q_0 t}$ and $\mathscr{R}(I_{1t} \times \cdots \times I_{q_0 t})$ is true. Let

$$C_{1t} = I_{1t} \times \cdots \times I_{q_1 t}, \quad C_{2t} = I_{q_1+1t} \times \cdots \times I_{q_0 t} \qquad \forall t \in [-r,r]^{q_0}.$$

Since the set $\{t_1\} \times [-r,r]^{q_2}$ is compact, it possesses a finite number m of points $s_1,\dots,s_m$ such that

$$\{t_1\} \times [-r,r]^{q_2} \subset \bigcup_{i=1}^{m} C_{1s_i} \times C_{2s_i}. \qquad (*)$$

Let $C_1 = C_{1s_1} \cap \cdots \cap C_{1s_m}$. Let us show that $\mathscr{R}_1(C_1)$ is true, by means of the following relation $\mathscr{R}_2$. For every open subset Ω of $\mathbb{R}^{q_2}$, let

$$\mathscr{R}_2(\Omega) = \mathscr{R}(C_1 \times \Omega).$$

Let us show that $\mathscr{R}_2$ satisfies the hypotheses of $\mathscr{L}(q_2)$. Let $t_2 \in [-r, r]^{q_2}$. By the inclusion $(*)$, there exists $k \in \{1, \dots, m\}$ such that $t_2 \in C_{2s_k}$. Since $\mathscr{R}$ satisfies hypothesis (c) of $\mathscr{L}(q_0)$, the validity of $\mathscr{R}(C_{1s_k} \times C_{2s_k})$ implies that of $\mathscr{R}_2(C_{2s_k})$. Thus $\mathscr{R}_2$ satisfies hypothesis (a) of $\mathscr{L}(q_2)$. Since $\mathscr{R}$ satisfies hypotheses (b) and (c) of $\mathscr{L}(q_0)$ and C_1 is convex, $\mathscr{R}_2$ satisfies hypotheses (b) and (c) of $\mathscr{L}(q_2)$. As $q_2 < q_0$, $\mathscr{L}(q_2)$ is true, and therefore $\mathscr{R}_2(]-r, r[^{q_2}) = \mathscr{R}_1(C_1)$ is true. Thus $\mathscr{R}_1$ satisfies hypothesis (a) of $\mathscr{L}(q_1)$. It is easy to check that $\mathscr{R}_1$ satisfies the other hypotheses of $\mathscr{L}(q_1)$. As $q_1 < q_0$, $\mathscr{L}(q_1)$ is true, and therefore, $\mathscr{R}_1(]-r, r[^{q_1}) = \mathscr{R}(]-r, r[^{q_0})$ is true. Thus $\mathscr{L}(q_0)$ is true. It follows that $\mathscr{L}(q)$ is true for all $q \in \{1, 2, \dots\}$. ■

LEMMA 6.2 (Connection of solutions defined on domains of the form $\mathbb{R}^q \times B \times C_0$ into solutions defined on $\mathbb{R}^q \times \mathbb{R} \times C_0$). *Let $q_1, q_2 \in \{0, 1, \dots\}$. Let $q = q_1 + 1 + q_2$. If $q_2 \neq 0$, let C_0 be a convex open subset of $\mathbb{R}^{q_2}$. For every subset S of $\mathbb{R}$, let*

$$\varphi(S) = \begin{cases} \mathbb{R}^{q_1} \times S \times C_0 & \text{if } q_1, q_2 > 0, \\ \mathbb{R}^{q_1} \times S & \text{if } q_1 > 0 = q_2, \\ S \times C_0 & \text{if } q_1 = 0 < q_2, \\ S & \text{if } q_1 = 0 = q_2. \end{cases}$$

Let F be a set. Let $\mathscr{S}$ be a map which associates every open subset Ω of $\mathbb{R}^q$ with a set $\mathscr{S}(\Omega)$ of maps from Ω into F and satisfies the following conditions:

(a) *For every bounded open subset B of $\mathbb{R}$, $\mathscr{S}(\varphi(B))$ is not empty.*

(b) *For all convex open subsets C_1, C_2 of $\mathbb{R}^q$, and for all $f_1 \in \mathscr{S}(C_1)$, $f_2 \in \mathscr{S}(C_2)$, there exists $f \in \mathscr{S}(C_1 \cup C_2)$ such that*

$$f(t) = f_1(t) \quad \forall t \in C_1 \setminus C_2, \qquad f(t) = f_2(t) \quad \forall t \in C_2 \setminus C_1.$$

(c) *For all open subsets Ω_1, Ω_2 of $\mathbb{R}^q$, and for all f,*

$$\Omega_1 \subset \Omega_2 \text{ and } f \in \mathscr{S}(\Omega_2) \quad \Rightarrow \quad f|_{\Omega_1} \in \mathscr{S}(\Omega_1).$$

(d) *For every open subset* Ω *of* $\mathbb{R}^q$, *for every map* f *from* Ω *into* F, *and for every open covering* $(\Omega_i)_{i\in I}$ *of* Ω,

$$f|_{\Omega i} \in \mathscr{S}(\Omega_i) \quad \forall i \in I \quad \Rightarrow \quad f \in \mathscr{S}(\Omega).$$

Then, under these conditions, $\mathscr{S}(\varphi(\mathbb{R}))$ *is not empty.*

REMARK. Condition (a) implies that $\mathscr{S}(\varnothing)$ is not empty.

Proof. Let us define the following convex open subsets of $\mathbb{R}^q$: Let $D_0 = \varnothing$, and let

$$D_{m,n} = \varphi(]m,n[) \qquad \forall m \in \mathbb{Z},\ n \in \{m+1, m+2, \ldots\},$$

$$D_n = D_{-n,n} \qquad \forall n \in \{1,2,\ldots\}.$$

Let us show by induction that there exists a sequence $(f_k)_{k\in\mathbb{N}}$ such that for every $k \in \{0,1,\ldots\}$,

$$f_k \in \mathscr{S}(D_{k+1}), \qquad f_{k+1}(t) = f_k(t), \quad \forall t \in D_k. \tag{1}$$

By (a), for all $m \in \mathbb{Z}$, $n \in \{m+1, m+2, \ldots\}$, there exists $g_{m,n} \in \mathscr{S}(D_{m,n})$. Let $f_0 = g_{-1,1}$. Let $n \in \{0,1,\ldots\}$ be such that the existence of $f_0, \ldots, f_n$ satisfying (1) is established. By (b) applied to f_n and $g_{n,n+2}$, there exists $h \in \mathscr{S}(D_{-n-1,n+2})$ such that

$$h(t) = f_n(t) \qquad \forall t \in D_{n+1} \setminus D_{n,n+2}.$$

By (b) applied to h and $g_{-n-2,-n}$, there exists $f_{n+1} \in \mathscr{S}(D_{n+2})$ such that

$$f_{n+1}(t) = h(t) \qquad \forall t \in D_{-n-1,n+2} \setminus D_{-n-2,-n}.$$

These relations imply that

$$f_{n+1}(t) = f_n(t) \qquad \forall t \in D_n.$$

It follows directly from the definitions that $(D_k \setminus D_{k-1})_{k=1}^{\infty}$ is a partition of $\varphi(\mathbb{R})$, which allows us to define the following map f from $\varphi(\mathbb{R})$ into F by

$$f(t) = f_k(t) \qquad \forall k \in \{1,2,\ldots\},\ t \in D_k \setminus D_{k-1}.$$

Let us show that

$$f(t) = f_n(t) \qquad \forall n \in \{1,2,\ldots\},\ t \in D_n. \tag{2}$$

Let $n \in \{1,2,\ldots\}$ and $t \in D_n$. Since $(D_k \setminus D_{k-1})_{k=1}^{n}$ is a partition of D_n, there exists one $i \in \{1,\ldots,n\}$ such that $t \in D_i \setminus D_{i-1}$, and by the definition of f, $f(t) = f_i(t)$. If $i < n$, then by (1), $f_i(t) = f_{i+1}(t) = \cdots = f_n(t)$. Thus $f(t) = f_n(t)$. By (2) and (c),

$$f|_{D_k} = f_k|_{D_k} \in \mathscr{S}(D_k) \qquad \forall k \in \{1,2,\ldots\},$$

and since $(D_k)_{k=1}^{\infty}$ is an open covering of $\varphi(\mathbb{R})$, it follows by (d) that $f \in \mathscr{S}(\varphi(\mathbb{R}))$. ■

THEOREM 6.3 (Connection of local solutions into global solutions defined on $\mathbb{R}^q$). *Let F be a set, and $\mathscr{S}$ be a map which associates every open subset Ω of $\mathbb{R}^q$ with a set $\mathscr{S}(\Omega)$ of maps from Ω into F and satisfies the following conditions:*

(a) *For every $t \in \mathbb{R}^q$, there exists an open neighborhood $\Omega_t \subset \mathbb{R}^q$ of t such that $\mathscr{S}(\Omega_t)$ is not empty.*

(b) *For all convex open subsets C_1, C_2 of $\mathbb{R}^q$, and for all $f_1 \in \mathscr{S}(C_1)$, $f_2 \in \mathscr{S}(C_2)$, there exists $f \in \mathscr{S}(C_1 \cup C_2)$ such that*

$$f(t) = f_1(t) \quad \forall t \in C_1 \setminus C_2, \qquad f(t) = f_2(t) \quad \forall t \in C_2 \setminus C_1.$$

(c) *For all open subsets Ω_1, Ω_2 of $\mathbb{R}^q$, and for all f,*

$$\Omega_1 \subset \Omega_2 \text{ and } f \in \mathscr{S}(\Omega_2) \quad \Rightarrow \quad f|_{\Omega_1} \in \mathscr{S}(\Omega_1).$$

(d) *For every open subset Ω of $\mathbb{R}^q$, for every map f from Ω into F, and for every open covering $(\Omega_i)_{i \in I}$ of Ω,*

$$f|_{\Omega_i} \in \mathscr{S}(\Omega_i) \quad \forall i \in I \quad \Rightarrow \quad f \in \mathscr{S}(\Omega).$$

Then, under these conditions, $\mathscr{S}(\mathbb{R}^q)$ is not empty.

REMARK. Conditions (a) and (c) imply that $\mathscr{S}(\varnothing)$ is not empty.

Proof. For every open subset Ω of $\mathbb{R}^q$, let $\mathscr{R}(\Omega)$ denote the assertion "$\mathscr{S}(\Omega)$ is not empty." Let $\mathscr{A}(0)$ denote the assertion "for every bounded open subset B of $\mathbb{R}^q$, $\mathscr{R}(B)$ is true," and if $q > 1$, for every $m \in \{1, \dots, q-1\}$, let $\mathscr{A}(m)$ denote the assertion "for every bounded open subset B of $\mathbb{R}^{q-m}$, $\mathscr{R}(\mathbb{R}^m \times B)$ is true."

Let us show by induction that $\mathscr{A}(0), \dots, \mathscr{A}(q-1)$ are true. Hypotheses (a), (b), and (c) imply that $\mathscr{R}$ satisfies all the hypotheses of Lemma 6.1, with any $r \in \mathbb{R}_+^*$. Therefore, by virtue of this lemma, $\mathscr{R}(]-r, r[^q)$ is true for every $r \in \mathbb{R}_+^*$, which implies, by (c), that $\mathscr{A}(0)$ is true. If $q = 1$, then the proof by induction is finished. Let us suppose that $q > 1$. Let $m \in \{1, \dots, q-1\}$ be such that $\mathscr{A}(m-1)$ is true, and let us show that $\mathscr{A}(m)$ is true. Let B_0 be a bounded open subset of $\mathbb{R}^{q-m}$. Then there exists $r_0 \in \mathbb{R}_+^*$ such that $B_0 \subset]-r_0, r_0[^{q-m}$. Let $C_0 =]-r_0, r_0[^{q-m}$. For every subset S of $\mathbb{R}$, let

$$\varphi(S) = \begin{cases} \mathbb{R}^{m-1} \times S \times C_0 & \text{if} \quad m > 1, \\ S \times C_0 & \text{if} \quad m = 1. \end{cases}$$

Since $\mathscr{A}(m-1)$ is true and C_0 is bounded, $\mathscr{R}(\varphi(B))$ is true for every bounded open subset B of $\mathbb{R}$, that is to say, $(\varphi, \mathscr{S})$ satisfies hypothesis (a) of Lemma 6.2, with $q_1 = m-1$ and $q_2 = q-m$. As hypotheses (b), (c), (d) of Lemma 6.2 and Theorem 6.3 are the same, and C_0 is convex and open, $(\varphi, \mathscr{S})$ satisfies all the hypotheses of Lemma 6.2. Therefore, by virtue of this lemma, $\mathscr{R}(\varphi(\mathbb{R})) = \mathscr{R}(\mathbb{R}^m \times C_0)$ is true. Hence, by (c), $\mathscr{R}(\mathbb{R}^m \times B_0)$ is true. So $\mathscr{A}(m)$ is true. Thus $\mathscr{A}(0), \dots, \mathscr{A}(q-1)$ are true.

For every subset S of $\mathbb{R}$, let

$$\tilde{\varphi}(S) = \begin{cases} \mathbb{R}^{q-1} \times S & \text{if} \quad q > 1, \\ S & \text{if} \quad q = 1. \end{cases}$$

Since $\mathscr{A}(q-1)$ is true, $\mathscr{R}(\tilde{\varphi}(B))$ is true for every bounded open subset B of $\mathbb{R}$. Hence, like $(\varphi, \mathscr{S})$ above, $(\tilde{\varphi}, \mathscr{S})$ satisfies all the hypotheses of Lemma 6.2, with $q_1 = q-1$ and $q_2 = 0$. Therefore, by virtue of this lemma, $\mathscr{R}(\tilde{\varphi}(\mathbb{R})) = \mathscr{R}(\mathbb{R}^q)$ is true. ∎

7. EVERY SUBSET OF $\mathbb{C}^{m \times n}$ CONSISTING OF ALL THE MATRICES OF THE SAME RANK IS ANALYTICALLY ARCWISE CONNECTED

LEMMA 7.1. *The subset $\mathbb{C}_n^{n \times n}$ of $\mathbb{C}^{n \times n}$ is analytically arcwise connected, that is to say, for all $A_0, B_0 \in \mathbb{C}_n^{n \times n}$, there exists an analytic map Z from $\mathbb{R}$ into $\mathbb{C}_n^{n \times n}$ such that $Z(0) = A_0$ and $Z(1) = B_0$.*

Proof. Let $A_0, B_0 \in \mathbb{C}_n^{n \times n}$. Let $C_0 = B_0 A_0^{-1}$. Since every matrix of $\mathbb{C}^{n \times n}$ is triangularizable, there exist $P_0, T_0 \in \mathbb{C}^{n \times n}$ such that P_0 is invertible, T_0 is upper triangular, and

$$C_0 = P_0 T_0 P_0^{-1}.$$

Since T_0 is upper triangular, there exist $D_0, N_0 \in \mathbb{C}^{n \times n}$ such that D_0 is diagonal, N_0 is strictly upper triangular, and

$$T_0 = D_0 + N_0.$$

Since A_0 and B_0 are invertible, so are C_0, T_0, and D_0. Consequently, there exist

$$d_{01}, \dots, d_{0n} \in \mathbb{C} \setminus \{0\}, \qquad \rho_{01}, \dots, \rho_{0n} \in \mathbb{R}_+^*, \qquad \theta_{01}, \dots, \theta_{0n} \in [0, 2\pi[$$

such that

$$D_0 = \operatorname{diag}[d_{01}, \dots, d_{0n}], \qquad d_{01} = \rho_{01} e^{i\theta_{01}}, \dots, \qquad d_{0n} = \rho_{0n} e^{i\theta_{0n}},$$

where $i^2 = -1$. Let

$$\eta(t) = \frac{1 - \cos \pi t}{2} \qquad \forall t \in \mathbb{R}.$$

It is immediately seen that η is an analytic map from $\mathbb{R}$ into $\mathbb{R}$ such that

$$\eta(0) = 0, \quad \eta(1) = 1, \qquad 0 \leqslant \eta(t) \leqslant 1 \quad \forall t \in \mathbb{R}.$$

For every $t \in \mathbb{R}$, let

$$d_j(t) = \left[1 - \eta(t) + \eta(t)\rho_{0j}\right] e^{i\eta(t)\theta_{0j}} \quad \forall j \in \{1, \dots, n\},$$

$$D(t) = \operatorname{diag}[d_1(t), \dots, d_n(t)],$$

$$T(t) = D(t) + \eta(t) N_0, \qquad Y(t) = P_0 T(t) P_0^{-1}, \qquad Z(t) = Y(t) A_0.$$

Then

$$D(0)=I_n,\qquad T(0)=I_n,\qquad Y(0)=I_n,\qquad Z(0)=A_0,$$

$$D(1)=D_0,\qquad T(1)=T_0,\qquad Y(1)=C_0,\qquad Z(1)=B_0.$$

On the other hand, for every $t\in\mathbb{R}$, $j\in\{1,\dots,n\}$,

$$|d_j(t)|=[1-\eta(t)]+\eta(t)\rho_{0j}>0;$$

therefore,

$$\det Z(t)=d_1(t)\cdots d_n(t)\det A_0\neq 0\qquad \forall t\in\mathbb{R}.$$

Thus Z has the required properties. ■

In the proof of the next theorem, we will denote by $I_r^{m\times n}$ the following matrix of $\mathbb{C}_r^{m\times n}$:

$$I_r^{m\times n}=\begin{cases}\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix} & \text{if } 0<r<m,n,\\ \begin{bmatrix}I_r & 0\end{bmatrix} & \text{if } 0<r=m<n,\\ \begin{bmatrix}I_r\\ 0\end{bmatrix} & \text{if } 0<r=n<m,\\ I_r & \text{if } 0<r=m=n,\\ 0 & \text{if } r=0.\end{cases}$$

THEOREM 7.2. *The subset $\mathbb{C}_r^{m\times n}$ of $\mathbb{C}^{m\times n}$ is analytically arcwise connected, that is to say, for all $A_0,B_0\in\mathbb{C}_r^{m\times n}$, there exists an analytic map Z from $\mathbb{R}$ into $\mathbb{C}_r^{m\times n}$ such that $Z(0)=A_0$ and $Z(1)=B_0$.*

Proof. Let $A_0,B_0\in\mathbb{C}_r^{m\times n}$. By [13, 2.8, p. 53], there exist $A_1,B_1\in\mathbb{C}_m^{m\times m}$, $A_2,B_2\in\mathbb{C}_n^{n\times n}$ such that

$$A_0=A_1I_r^{m\times n}A_2,\qquad B_0=B_1I_r^{m\times n}B_2.$$

By Lemma 7.1, there exist analytic maps Z_1 from $\mathbb{R}$ into $\mathbb{C}_m^{m\times m}$ and Z_2 from

$\mathbb{R}$ into $\mathbb{C}_n^{n\times n}$ such that

$$Z_1(0)=A_1,\qquad Z_1(1)=B_1,\qquad Z_2(0)=A_2,\qquad Z_2(1)=B_2.$$

Then $Z=Z_1 I_r^{m\times n} Z_2$ has the required properties. ■

8. EXISTENCE OF BASES OF CLASS C^p

LEMMA 8.1 (Connection of two bases of class C^p defined on two convex sets). *Let C_1 and C_2 be convex open subsets of a separable Hilbert space. Let $A\in C^p(C_1,\mathbb{C}_r^{m\times r})$ and $B\in C^p(C_2,\mathbb{C}_r^{m\times r})$ be such that*

$$\operatorname{Im} A(t)=\operatorname{Im} B(t)\qquad \forall t\in C_1\cap C_2.$$

Then there exists $V\in C^p(C_1\cup C_2,\mathbb{C}_r^{m\times r})$ such that

$$V(t)=A(t)\quad \forall t\in C_1\setminus C_2,\qquad V(t)=B(t)\quad \forall t\in C_2\setminus C_1,$$

$$\operatorname{Im} V(t)=\operatorname{Im} A(t)=\operatorname{Im} B(t)\qquad \forall t\in C_1\cap C_2.$$

Proof. If $C_1\subset C_2$ or $C_2\subset C_1$ or $C_1\cap C_2=\varnothing$, then $V=B$ or $V=A$ or ($V|_{C_1}=A$ and $V|_{C_2}=B$) is suitable, respectively. Let us suppose that $C_1\setminus C_2$, $C_2\setminus C_1$, and $C_1\cap C_2$ are not empty, and let $t_0\in C_1\cap C_2$. By Corollary 4.2(a), there exists $X\in C^p(C_1\cap C_2,\mathbb{C}^{r\times r})$ such that

$$B(t)X(t)=A(t)\qquad \forall t\in C_1\cap C_2.$$

Since $\operatorname{rank} A=r$ and $\operatorname{rank} X\leqslant r$, this relation implies that $\operatorname{rank} X=r$. Consequently, by virtue of Theorem 7.2, there exists an analytic map Y from $\mathbb{R}$ into $\mathbb{C}_r^{r\times r}$ such that

$$Y(0)=X(t_0),\qquad Y(1)=I_r.$$

By [14, corollary of Theorem II.3.2, p. 36], there exists a partition of unity $(\eta,1-\eta)$ of class C^∞ on $C_1\cup C_2$, subordinated to the open covering (C_2,C_1) of $C_1\cup C_2$. That is to say, there exists $\eta\in C^\infty(C_1\cup C_2,[0,1])$ such that

$$\eta(t)=0\quad \forall t\in C_1\setminus C_2,\qquad \eta(t)=1\quad \forall t\in C_2\setminus C_1. \tag{1}$$

Let

$$\Omega_{ij}=\left\{t\in C_1\cup C_2\middle|\tfrac{1}{5}i<\eta(t)<\tfrac{1}{5}j\right\}\qquad \forall i,j\in\{-\infty,0,1,2,3,4,5,\infty\},$$

$$\tilde{\Omega}_{ij}=\left\{t\in C_1\cup C_2\middle|\tfrac{1}{5}i\leqslant\eta(t)\leqslant\tfrac{1}{5}j\right\}\qquad \forall i,j\in\{0,1,2,3,4,5\}$$

($\tilde{\Omega}_{ij}$ is not necessarily equal to the closure of Ω_{ij}). As above for η, there exist $\alpha,\beta\in C^{\infty}(C_1\cup C_2,[0,1])$ [subordinated to $(\Omega_{1\infty},\Omega_{-\infty 2})$ and $(\Omega_{3\infty},\Omega_{-\infty 4})$] such that

$$\alpha(t)=0\quad \forall t\in\tilde{\Omega}_{01},\qquad \alpha(t)=1\quad \forall t\in\tilde{\Omega}_{25},$$

$$\beta(t)=0\quad \forall t\in\tilde{\Omega}_{03},\qquad \beta(t)=1\quad \forall t\in\tilde{\Omega}_{45}.$$

The relations (1) imply that

$$\Omega_{-\infty 5}\subset C_1,\qquad \Omega_{0\infty}\subset C_2,\qquad \Omega_{05}\subset C_1\cap C_2,$$

and since $C_1\cap C_2$ is convex,

$$[1-\alpha(t)]t+\alpha(t)t_0\in C_1\cap C_2\qquad \forall t\in\Omega_{05}.$$

These observations allow us to define

$$V(t)=A(t)\qquad \forall t\in\tilde{\Omega}_{01},$$

$$V(t)=B(t)X([1-\alpha(t)]t+\alpha(t)t_0)\qquad \forall t\in\Omega_{12},$$

$$V(t)=B(t)X(t_0)\qquad \forall t\in\tilde{\Omega}_{23},$$

$$V(t)=B(t)Y(\beta(t))\qquad \forall t\in\Omega_{34},$$

$$V(t)=B(t)\qquad \forall t\in\tilde{\Omega}_{45}.$$

It follows directly from the definitions that

$$\begin{aligned} V(t) &= A(t) && \forall t \in \Omega_{-\infty 1}, \\ V(t) &= B(t)X([1-\alpha(t)]t + \alpha(t)t_0) && \forall t \in \Omega_{03}, \\ V(t) &= B(t)Y(\beta(t)) && \forall t \in \Omega_{2\infty}, \end{aligned} \tag{2}$$

which shows that V is of class C^p on $\Omega_{-\infty 1}, \Omega_{03}, \Omega_{2\infty}$, and therefore $V \in C^p(C_1 \cup C_2, \mathbb{C}^{m\times r})$. By (1), $C_1 \setminus C_2 \subset \tilde{\Omega}_{01}$ and $C_2 \setminus C_1 \subset \tilde{\Omega}_{45}$; consequently

$$V(t) = A(t) \quad \forall t \in C_1 \setminus C_2, \qquad V(t) = B(t) \quad \forall t \in C_2 \setminus C_1.$$

On the other hand, A, B, X, and Y are of rank r; consequently, V is of rank r, and by (2),

$$\operatorname{Im} V(t) = \operatorname{Im} A(t) \text{ or } \operatorname{Im} V(t) = \operatorname{Im} B(t) \qquad \forall t \in C_1 \cup C_2.$$

Hence,

$$\operatorname{Im} V(t) = \operatorname{Im} A(t) = \operatorname{Im} B(t) \qquad \forall t \in C_1 \cap C_2. \qquad \blacksquare$$

The following theorem (the main result of this paper) deals with the existence of bases of class C^p of the kernel and the image of a matrix function defined on a domain C^p-diffeomorphic to $\mathbb{R}^q$. A similar result on operator functions defined on a contractible compact subset of $\mathbb{R}^q$ has been established by I. C. Gohberg and J. Leiterer (direct consequence of [7, Theorem 3.6 and Proposition 4.1]). Both results furnish an extended version of [6, Corollary 13.6.5], where it is necessary to suppose the domain contractible, because of the counterexamples produced in [7, Counterexample 5.1] and in Section 5 of the present paper.

THEOREM 8.2 (Existence of orthonormal bases of class C^p of the kernel and the image of a rectangular matrix function of q real variables: generalization of the theorem [3] due to V. Doležal). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then there exist*

$$u_1, \dots, u_m \in C^p(\Omega, \mathbb{C}^m), \qquad v_1, \dots, v_n \in C^p(\Omega, \mathbb{C}^n)$$

such that for every $t \in \Omega$,

(a) *if* $r > 0$, *then* $(u_1(t), \ldots, u_r(t))$ *is an orthonormal basis of* $\operatorname{Im} A(t)$,
(b) *if* $r < m$, *then* $(u_{r+1}(t), \ldots, u_m(t))$ *is an orthonormal basis of* $[\operatorname{Im} A(t)]^{\perp}$,
(c) *if* $r > 0$, *then* $(v_1(t), \ldots, v_r(t))$ *is an orthonormal basis of* $[\ker A(t)]^{\perp}$,
(d) *if* $r < n$, *then* $(v_{r+1}(t), \ldots, v_n(t))$ *is an orthonormal basis of* $\operatorname{Ker} A(t)$.

Proof. The conclusion is obvious in the particular case where $r = 0$, that is to say, where $A = 0$. Let us suppose that $r > 0$. By hypothesis on Ω, there exists a C^p-diffeomorphism φ from $\mathbb{R}^q$ into Ω. Let

$$B = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} = A \circ \varphi \in C^p\left(\mathbb{R}^q, \mathbb{C}_r^{m \times n}\right).$$

For every open subset Λ of $\mathbb{R}^q$, let

$$\mathscr{S}(\Lambda) = \left\{ W \in C^p\left(\Lambda, \mathbb{C}_r^{m \times r}\right) \middle| \operatorname{Im} W(t) = \operatorname{Im} B(t)\ \forall t \in \Lambda \right\}.$$

Let us show that $\mathscr{S}$ satisfies the hypotheses of Theorem 6.3. Let $t_0 \in \mathbb{R}^q$. Since $B(t_0)$ is of rank r, there exist $j_1, \ldots, j_r \in \{1, \ldots, n\}$ such that $b_{j_1}(t_0), \ldots, b_{j_r}(t_0)$ are linearly independent. Let

$$\tilde{B} = \begin{bmatrix} b_{j_1} & \cdots & b_{j_r} \end{bmatrix}.$$

By [5, Lemma 5.4], there exists an open neighborhood $\Lambda_0 \subset \mathbb{R}^q$ of t_0 such that for every $t \in \Lambda_0$, $\tilde{B}(t)$ is of rank r. This implies that $\tilde{B}|_{\Lambda_0} \in \mathscr{S}(\Lambda_0)$. Thus $\mathscr{S}$ satisfies hypothesis (a) of Theorem 6.3. By Lemma 8.1, $\mathscr{S}$ satisfies hypothesis (b) of this theorem, and it is obvious that $\mathscr{S}$ satisfies hypotheses (c) and (d). Therefore, by virtue of Theorem 6.3, there exists $W \in \mathscr{S}(\mathbb{R}^q)$. Let

$$V = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} = W \circ \varphi^{-1} \in C^p\left(\Omega, \mathbb{C}_r^{m \times r}\right).$$

Let $t \in \Omega$. It follows directly from the definitions that

$$\operatorname{Im} V(t) = \operatorname{Im} W\left(\varphi^{-1}(t)\right) = \operatorname{Im} B\left(\varphi^{-1}(t)\right) = \operatorname{Im} A(t);$$

consequently, $(v_1(t),\dots,v_r(t))$ is a basis of $\operatorname{Im} A(t)$. Therefore, by Proposition 3.1, there exist $u_1,\dots,u_r \in C^p(\Omega,\mathbb{C}^m)$ such that, for every $t\in\Omega$, $(u_1(t),\dots,u_r(t))$ is an orthonormal basis of $\operatorname{Im} A(t)$.

By Corollary 4.3,

$$P_{(\operatorname{Im} A)^\perp} \in C^p(\Omega,\mathbb{C}^{m\times m}), \qquad P_{\operatorname{Ker} A}, P_{(\operatorname{Ker} A)^\perp} \in C^p(\Omega,\mathbb{C}^{n\times n}).$$

Since A is of constant rank, so are these three projector valued functions. By applying to them Theorem 8.2(a) just proved above, the conclusions (b), (c), and (d) are obtained. ■

Corollary 8.3 (Existence of bases of class C^p of a vector subspace valued function E and of $E^\perp$). *Let $\Omega\subset\mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $r\in\{1,2,\dots\}$. Let E be a map from Ω into the set of all the vector subspaces of dimension r of $\mathbb{C}^n$ possessing the following property: for every $t\in\Omega$, there exists an open neighborhood $\Omega_t\subset\Omega$ of t and there exist $v_{t1},\dots,v_{tr}\in C^p(\Omega_t,\mathbb{C}^n)$ such that for every $u\in\Omega_t$, $(v_{t1}(u),\dots,v_{tr}(u))$ is a basis of $E(u)$. Then there exist $u_1,\dots,u_n\in C^p(\Omega,\mathbb{C}^n)$ such that for every $t\in\Omega$, $(u_1(t),\dots,u_r(t))$ is an orthonormal basis of $E(t)$ and if $r<n$, $(u_{r+1}(t),\dots,u_n(t))$ is an orthonormal basis of $E(t)^\perp$.*

Proof. The corollary is obvious if $r=n$. Let us suppose that $r<n$. Let $t\in\Omega$. By hypothesis, there exists an open neighborhood $\Omega_t\subset\Omega$ of t and there exists $V_t\in C^p(\Omega_t,\mathbb{C}_r^{n\times r})$ such that $\operatorname{Im} V_t(u)=E(u)$ for every $u\in\Omega_t$. By Corollary 4.3,

$$P_E|_{\Omega_t} = P_{\operatorname{Im} V_t} \in C^p\left(\Omega_t,\mathbb{C}_r^{n\times n}\right).$$

Since $(\Omega_t)_{t\in\Omega}$ is an open covering of Ω, it follows that $P_E\in C^p(\Omega,\mathbb{C}_r^{n\times n})$, and the conclusion is obtained by applying Theorem 8.2(a) and (b) to P_E.

9. APPLICATIONS

The following corollary furnishes an extension of Corollary 4.2, as indicated just before it.

Corollary 9.1 (General form of the solutions of class C^p of the linear equation $AX=B$ on matrix functions). *Let $\Omega\subset\mathbb{R}^q$ be C^p-diffeomorphic to*

$\mathbb{R}^q$. Let $n \in \{2,3,\ldots\}$ and $r \in \{1,\ldots,n-1\}$ (see Lemma 2.2(a) for the case $r = n$). Let $d = n - r$. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$ and $B \in C^p(\Omega, \mathbb{C}^{m\times s})$ be such that $\operatorname{Im} B(t) \subset \operatorname{Im} A(t)$, for every $t \in \Omega$. Then there exist $\tilde{X} \in C^p(\Omega, \mathbb{C}^{n\times s})$ and $U \in C^p(\Omega, \mathbb{C}_d^{n\times d})$ such that

$$A\tilde{X} = B, \qquad AU = 0, \qquad U^*U = I_d,$$

and for all X,

$$X \in C^p(\Omega, \mathbb{C}^{n\times s}) \text{ and } AX = B \quad \Leftrightarrow \quad \exists Y \in C^p(\Omega, \mathbb{C}^{d\times s}), \quad X = \tilde{X} + UY.$$

Proof. By Corollary 4.2, there exists $\tilde{X} \in C^p(\Omega, \mathbb{C}^{n\times s})$ such that $A\tilde{X} = B$. By Theorem 8.2(d), there exist $u_1,\ldots,u_d \in C^p(\Omega, \mathbb{C}^n)$ such that for every $t \in \Omega$, $(u_1(t),\ldots,u_d(t))$ is an orthonormal basis of $\operatorname{Ker} A(t)$. Let $U = [u_1 \ \cdots \ u_d]$. It follows directly from the definition of U that $U \in C^p(\Omega, \mathbb{C}_d^{n\times d})$, $AU = 0$, and $U^*U = I_d$. Let $X \in C^p(\Omega, \mathbb{C}^{n\times s})$ be such that $AX = B$. Then $A(X - \tilde{X}) = 0$; hence

$$\operatorname{Im}\left[X(t) - \tilde{X}(t)\right] \subset \operatorname{Ker} A(t) = \operatorname{Im} U(t) \qquad \forall t \in \Omega.$$

By virtue of Corollary 4.2, there exists $Y \in C^p(\Omega, \mathbb{C}^{d\times s})$ such that $UY = X - \tilde{X}$. The converse is obvious. ∎

REMARK. The general form of the solutions of class C^p of the equation $XA = B$ may be obtained by applying Corollary 9.1 to the equation $A^*X^* = B^*$, as in Corollary 4.2.

COROLLARY 9.2 (Equivalences of class C^p to the rank canonical form). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then there exist*

$$P, V \in C^p\left(\Omega, \mathbb{C}_m^{m\times m}\right), \qquad Q, U \in C^p\left(\Omega, \mathbb{C}_n^{n\times n}\right),$$

and if $r > 0$, there exist $B, C \in C^p(\Omega, \mathbb{C}_r^{r\times r})$ such that (using the notation *defined just before Theorem* 7.2)

$$A = PI_r^{m\times n}U = VI_r^{m\times n}Q, \qquad U^*U = UU^* = I_n, \quad V^*V = VV^* = I_m,$$

$$P^*P = \begin{cases} I_m, & \text{if} \quad r = 0, \\ \operatorname{diag}[B, I_{m-r}], & \text{if} \quad 0 < r < m, \\ B, & \text{if} \quad r = m, \end{cases}$$

$$QQ^* = \begin{cases} I_n, & \text{if} \quad r = 0, \\ \operatorname{diag}[C, I_{n-r}], & \text{if} \quad 0 < r < n, \\ C, & \text{if} \quad r = n, \end{cases}$$

and if $r > 0$, $B(t), C(t)$ *are positive definite for every* $t \in \Omega$. *Moreover, if for every* $t \in \Omega$, $A(t)$ *is a partial isometry, then* P *and* Q *may be chosen such that* $P(t)$ *and* $Q(t)$ *are unitary for every* $t \in \Omega$.

Proof. If $r = 0$, then $A = 0 = I_m I_0^{m\times n} I_n$ and the lemma is trivial. Let us suppose that $r > 0$. By virtue of Theorem 8.2, there exist $w_1, \dots, w_n \in C^p(\Omega, \mathbb{C}^n)$ such that for every $t \in \Omega$, $(w_1(t), \dots, w_r(t))$ is an orthonormal basis of $[\operatorname{Ker} A(t)]^\perp$, and if $r < n$, $(w_{r+1}(t), \dots, w_n(t))$ is an orthonormal basis of $\operatorname{Ker} A(t)$; moreover, if $r < m$, there exist $p_{r+1}, \dots, p_m \in C^p(\Omega, \mathbb{C}^m)$ such that for every $t \in \Omega$, $(p_{r+1}(t), \dots, p_m(t))$ is an orthonormal basis of $[\operatorname{Im} A(t)]^\perp$. Let

$$W = [\,w_1 \quad \cdots \quad w_n\,], \qquad U = W^*, \qquad p_1 = Aw_1, \dots, \qquad p_r = Aw_r,$$

$$P = [\,p_1 \quad \cdots \quad p_m\,], \qquad P_1 = [\,p_1 \quad \cdots \quad p_r\,], \qquad B = P_1^* P_1,$$

and if $r < m$, $P_2 = [\,p_{r+1} \quad \cdots \quad p_m\,]$. Then

$$AW = \left[\,Aw_1 \quad \cdots \quad Aw_n\,\right] = PI_r^{m\times n}, \qquad W^*W = WW^* = I_n,$$

$$U^*U = UU^* = I_n;$$

hence

$$A = PI_r^{m\times n}W^* = PI_r^{m\times n}U.$$

If $r < m$,

$$P^*P = \begin{bmatrix} P_1^* \\ P_2^* \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} P_1^*P_1 & P_1^*P_2 \\ P_2^*P_1 & P_2^*P_2 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & I_{m-r} \end{bmatrix},$$

and if $r = m$, $P^*P = P_1^*P_1 = B$. Let $t \in \Omega$. Since $w_1(t), \dots, w_r(t)$ are linearly independent and belong to $[\operatorname{Ker} A(t)]^{\perp}$, $p_1(t), \dots, p_r(t)$ are linearly independent too. Consequently, $P_1(t)$ is of rank r, $P(t)$ is of rank m, and by Lemma 2.1(c), $B(t)$ is of rank r. It follows that $B(t)$ is positive definite. Moreover, if $A(t)$ is a partial isometry, then $p_1(t), \dots p_r(t)$ are orthonormal, and therefore $P(t)$ is unitary.

By the part of Corollary 9.2 proved so far applied to $A^* \in C^p(\Omega, \mathbb{C}_r^{n\times m})$ (still with $r > 0$), there exist

$$\tilde{P} \in C^p\left(\Omega, \mathbb{C}_n^{n\times n}\right), \qquad \tilde{U} \in C^p\left(\Omega, \mathbb{C}_m^{m\times m}\right), \qquad \tilde{B} \in C^p\left(\Omega, \mathbb{C}_r^{r\times r}\right)$$

such that

$$A^* = \tilde{P} I_r^{n\times m} \tilde{U}, \qquad \tilde{U}^*\tilde{U} = \tilde{U}\tilde{U}^* = I_m,$$

$$\tilde{P}^*\tilde{P} = \begin{cases} \operatorname{diag}\left[\tilde{B}, I_{n-r}\right] & \text{if} \quad r < n, \\ \tilde{B} & \text{if} \quad r = n, \end{cases}$$

and $\tilde{B}(t)$ is positive definite for every $t \in \Omega$. Moreover, if for every $t \in \Omega$, $A(t)$ is a partial isometry, then so is $A(t)^*$ [1, Theorem 6.3.3, p. 252], and $\tilde{P}^*\tilde{P} = \tilde{P}\tilde{P}^* = I_n$. The remainder of the conclusion is obtained with $Q = \tilde{P}^*$, $V = \tilde{U}^*$, and $C = \tilde{B}$. ■

The following corollary generalizes the main part of Corollary 9.2.

Corollary 9.3 (Equivalences of class C^p). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $A, B \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then there exist*

$$P, V \in C^p\left(\Omega, \mathbb{C}_m^{m\times m}\right), \qquad Q, U \in C^p\left(\Omega, \mathbb{C}_n^{n\times n}\right)$$

such that

$$A = PBU = VBQ, \qquad U^*U = UU^* = I_n, \quad V^*V = VV^* = I_m.$$

Moreover, if for every $t \in \Omega$, $A(t)$ and $B(t)$ are partial isometries, then P and Q may be chosen such that $P(t)$ and $Q(t)$ are unitary for every $t \in \Omega$.

Proof. The conclusion is obtained by applying Corollary 9.2 to A and to B, and by composing the equivalence between A and $I_r^{m\times n}$ with the equivalence between $I_r^{m\times n}$ and B. ∎

The following corollary furnishes a global version of Corollary 3.4, as indicated just before it.

COROLLARY 9.4 (Rank decomposition of class C^p). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $r \in \{1,2,\dots\}$. Let $A \in C^p(\Omega, \mathbb{C}_r^{m\times n})$. Then there exist $U, C \in C^p(\Omega, \mathbb{C}_r^{m\times r})$ and $V, B \in C^p(\Omega, \mathbb{C}_r^{r\times n})$ such that*

$$A = UB = CV, \qquad U^*U = I_r, \quad VV^* = I_r.$$

Proof. By Theorem 8.2, there exist $u_1, \dots, u_r \in C^p(\Omega, \mathbb{C}^m)$ such that for every $t \in \Omega$, $(u_1(t), \dots, u_r(t))$ is an orthonormal basis of $\operatorname{Im} A(t)$. Let $U = [u_1 \ \cdots \ u_r] \in C^p(\Omega, \mathbb{C}_r^{m\times r})$. By Corollary 4.2(a), there exists $B \in C^p(\Omega, \mathbb{C}^{r\times n})$ such that $UB = A$. Then $r = \operatorname{rank} A \leqslant \operatorname{rank} B \leqslant r$ implies that $\operatorname{rank} B = r$, and on the other hand, it is obvious that $U^*U = I_r$.

By the part of Corollary 9.4 proved so far applied to A^*, there exist $\tilde{U} \in C^p(\Omega, \mathbb{C}_r^{n\times r})$ and $\tilde{B} \in C^p(\Omega, \mathbb{C}_r^{r\times m})$ such that $A^* = \tilde{U}\tilde{B}$, $\tilde{U}^*\tilde{U} = I_r$, and the remainder of the conclusion is obtained with $V = \tilde{U}^*$ and $C = \tilde{B}^*$. ∎

REMARK. If Ω is a connected topological space and $P \in C^0(\Omega, \mathbb{C}^{n\times n})$ is a projector valued function, then $\operatorname{rank} P$ is constant, because $\operatorname{rank} P = \operatorname{tr} P$ is a continuous function with values in $\{0, 1, \dots\}$.

The following corollary furnishes an extension of [12, II.4.5].

COROLLARY 9.5 (Simultaneous diagonalization of class C^p of a basis family of projector valued functions). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $P_1, \dots, P_s \in C^p(\Omega, \mathbb{C}^{n\times n})$ be such that*

$$P_iP_j = \delta_{ij}P_i, \ \ P_i \neq 0, \ \ \forall i, j \in \{1, \dots, s\}, \qquad P_1 + \cdots + P_s = I_n.$$

For every $i \in \{1, \dots, s\}$, let $r_i = \operatorname{rank} P_i$ (cf. the remark above) and $D_i =$

$\operatorname{diag}[\delta_{i1}I_{r_1},\ldots,\delta_{is}I_{r_s}]$. *Then there exists* $U \in C^p(\Omega, \mathbb{C}_n^{n\times n})$ *such that*

$$P_i(t) = U(t)D_iU(t)^{-1} = U_{t_0}(t)P_i(t_0)U_{t_0}(t)^{-1}$$

$$\forall i \in \{1,\ldots,s\}, \quad t, t_0 \in \Omega,$$

where $U_{t_0}(t) = U(t)U(t_0)^{-1}$, *for every* $t, t_0 \in \Omega$. *Moreover, if for every* $i \in \{1,\ldots,s\}$ *and* $t \in \Omega$, $P_i(t)$ *is an orthogonal projector (that is to say,* $P_i(t)^* = P_i(t)$*), then* $U(t)$ *and* $U_{t_0}(t)$ *are unitary for every* $t, t_0 \in \Omega$.

Proof. It is well known that the hypothesis on $(P_i)_{i=1}^s$ implies that

$$\operatorname{Im} P_1(t) \dotplus \cdots \dotplus \operatorname{Im} P_s(t) = \mathbb{C}^n \qquad \forall t \in \Omega. \tag{1}$$

Consequently, $r_1 + \cdots + r_s = n$. By virtue of Theorem 8.2, for every $i \in \{1,\ldots,s\}$, there exist $u_{i1},\ldots,u_{ir_i} \in C^p(\Omega, \mathbb{C}^n)$ such that for every $t \in \Omega$, $(u_{i1}(t),\ldots,u_{ir_i}(t))$ is an orthonormal basis of $\operatorname{Im} P_i(t)$. Let

$$U_i = \begin{bmatrix} u_{i1} & \cdots & u_{ir_i} \end{bmatrix} \quad \forall i \in \{1,\ldots,s\}, \qquad U = \begin{bmatrix} U_1 & \cdots & U_s \end{bmatrix}.$$

Let $i \in \{1,\ldots,s\}$. For every $k \in \{1,\ldots,s\}$ we have $P_kU_k = U_k$, because $\operatorname{Im} P_k = \operatorname{Im} U_k$; consequently

$$\begin{aligned} P_iU = P_i\begin{bmatrix} U_1 & \cdots & U_s \end{bmatrix} &= P_i\begin{bmatrix} P_1U_1 & \cdots & P_sU_s \end{bmatrix} \\ &= \begin{bmatrix} P_iP_1U_1 & \cdots & P_iP_sU_s \end{bmatrix} \\ &= \begin{bmatrix} \delta_{i1}U_1 & \cdots & \delta_{is}U_s \end{bmatrix} = UD_i. \end{aligned}$$

The relation (1) implies that $\operatorname{rank} U = n$; consequently, $P_i = UD_iU^{-1}$, and

$$\begin{aligned} P_i(t) = U(t)D_iU(t)^{-1} &= U(t)U(t_0)^{-1}P_i(t_0)U(t_0)U(t)^{-1} \\ &= U_{t_0}(t)P_i(t_0)U_{t_0}(t)^{-1} \qquad \forall t, t_0 \in \Omega, \end{aligned}$$

where $U_{t_0}(t) = U(t)U(t_0)^{-1}$ for every $t, t_0 \in \Omega$.

If for every $t \in \Omega$, $P_1(t),\ldots,P_s(t)$ are orthogonal projectors, then the direct sum (1) is orthogonal, and therefore $U(t)$ is unitary for every $t \in \Omega$. ■

REMARK. Corollary 9.5 may be applied to the diagonalization of class C^p of a projector valued function P, by applying it to the basis family $(P, I_n - P)$.

COROLLARY 9.6 (Unitary rank block diagonalization of class C^p). *Let $\Omega \subset \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $n \in \{2,3,\dots\}$ and $r \in \{1,\dots,n-1\}$. Let $A \in C^p(\Omega, \mathbb{C}_r^{n\times n})$ be such that*

$$\operatorname{Im} A(t) = [\operatorname{Ker} A(t)]^{\perp} \qquad \forall t \in \Omega$$

(this condition is satisfied in particular when $A(t)$ is normal). Then there exist $U \in C^p(\Omega, \mathbb{C}_n^{n\times n})$ and $A_1 \in C^p(\Omega, \mathbb{C}_r^{r\times r})$ such that

$$A = U \operatorname{diag}[A_1, 0_{n-r}] U^*, \qquad U^*U = UU^* = I_n.$$

Proof. By Theorem 8.2, there exist $u_1, \dots, u_n \in C^p(\Omega, \mathbb{C}^n)$ such that for every $t \in \Omega$, $(u_1(t), \dots, u_r(t))$ is an orthonormal basis of $\operatorname{Im} A(t)$ and $(u_{r+1}(t), \dots, u_n(t))$ is an orthonormal basis of $[\operatorname{Im} A(t)]^{\perp}$. Let

$$U_1 = [u_1 \quad \cdots \quad u_r], \qquad U_2 = [u_{r+1} \quad \cdots \quad u_n], \qquad U = [U_1 \quad U_2].$$

Then for every $t \in \Omega$, $U(t)$ is unitary, and

$$\operatorname{Im}[A(t)U_1(t)] \subset \operatorname{Im} A(t) = \operatorname{Im} U_1(t), \qquad \operatorname{rank} U_1(t) = r;$$

therefore, by Corollary 4.2(a), there exists $A_1 \in C^p(\Omega, \mathbb{C}^{r\times r})$ such that $U_1 A_1 = AU_1$. On the other hand, $AU_2 = 0$, because, by hypothesis, $(\operatorname{Im} A)^{\perp} = \operatorname{Ker} A$. Thus,

$$AU = [AU_1 \quad AU_2] = [U_1 A_1 \quad 0] = U \operatorname{diag}[A_1, 0_{n-r}],$$

which implies the conclusion. ■

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REFERENCES

1 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, 1974.
2 S. L. Campbell and C. D. Meyer Jr., *Generalized Inverses of Linear Transformations*, Pitman, 1979.
3 V. Doležal, The existence of a continuous basis of a certain linear subspace of E_r which depends on a parameter, *Časopis Pěst. Mat.* 89:466–469 (1964).
4 I. J. Epstein, Conditions for a matrix to commute with its integral, *Proc. Amer. Math. Soc.* 14:266–270 (1963).
5 J.-Cl. Evard, On matrix functions which commute with their derivative. *Linear Algebra Appl.* 68:145–178 (1985).
6 I. C. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, 1986.
7 I. C. Gohberg and J. Leiterer, Über Algebren stetiger Operatorfunctionen, *Studia Math.* LVII:1–26 (1976).
8 J.-M. Gracia, Smooth jordanization of conservative matrices (generalized version in preparation).
9 M. J. Greenberg and J. R. Harper, *Algebraic Topology*, Benjamin-Cumings, 1981.
10 J. Z. Hearon and J. W. Evans, Differentiable generalized inverses, *J. Res. Nat. Bur. Standards* 72B(2):109–113 (1968).
11 Y. Hirasawa, A remark on homogeneous linear equations with variable coefficients, *Kodai Math. J.* 9:266–271 (1986).
12 T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer-Verlag, 1982.
13 P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic, 1985.
14 S. Lang, *Differential Manifolds*, Addison-Wesley, 1972.
15 P. Saphar, Contribution à l'étude des applications linéairs dans un espace de Banach, *Bull. Soc. Math. France* 92:363–384 (1964).
16 Y. Sibuya, Some global properties of matrices of functions of one variable, *Math. Ann.* 161:67–77 (1965).
17 L. M. Silverman and R. S. Bucy, Generalizations of a theorem of Doležal, *Math. Systems Theory* 4(4):334–339 (1970).
18 L. Weiss and P. L. Falb, Doležal's theorem, linear algebra with continuously parametrized elements, and time varying systems, *Math. Systems Theory* 3(1):67–75 (1969).

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