Journal of Computer and System Sciences 78 (2012) 293–304

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

On restricted context-free grammars ☆

Jürgen Dassow [a], Tomáš Masopust [b,c,∗,1]

[a] Otto-von-Guericke-Universität Magdeburg, Fakultät für Informatik, PSF 4120, D-39016 Magdeburg, Germany
[b] Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, 616 62 Brno, Czech Republic
[c] CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Received 31 August 2010
Received in revised form 21 April 2011
Accepted 10 May 2011
Available online 13 May 2011

Keywords:
Context-free grammars
Derivation restriction
Normal forms
Generative power

ABSTRACT

Context-free grammars are widely used for the simple form of their rules. A derivation step consists of the choice of a nonterminal of the sentential form and of an application of a rule rewriting it. Several regulations of the derivation process have been studied to increase the power of context-free grammars. In the resulting grammars, however, not only the symbols to be rewritten are restricted, but also the rules to be applied. In this paper, we study context-free grammars with a simpler restriction where only symbols to be rewritten are restricted, not the rules, in the sense that any rule rewriting the chosen nonterminal can be applied. We prove that these grammars have the same power as random context, matrix, or programmed grammars. We also present two improved normal forms and discuss the characterization of context-sensitive languages by a variant using strings of length at most two instead of symbols.

1. Introduction and definitions

Context-free grammars are one of the most investigated families of grammars in formal language theory. We can see that each derivation step can be characterized so that a nonterminal of the current sentential form is chosen, and any of the rules rewriting this nonterminal is applied. On the other hand, however, it is well known that these grammars are not able to cover all aspects of natural and programming languages. Therefore, many types of grammars with context-free rules and with some additional mechanisms controlling the application of rules were defined. Such grammars can describe some of the phenomena of natural and programming languages. For instance, in 1971, van der Walt [14] introduced random context grammars as a type of regulated grammars which, e.g., can include the aspect that only declared variables are used in programming languages. The basic idea is that a (context-free) rule can only be applied if certain nonterminals are present or absent in the current sentential form. Formally, we have the following concept.

A *random context grammar* is a quadruple $G = (N, T, P, S)$ where N is an alphabet of nonterminals, T is an alphabet of terminals such that $N \cap T = \emptyset$, $S \in N$ is the start symbol, and P is a finite set of rules of the form $(A \to w, Q, R)$ with $A \in N$, $w \in (N \cup T)^*$, and $Q, R \subseteq N$. For a rule $p = (A \to w, Q, R) \in P$, $A \to w$ is called the *core rule* of p, Q is called the *permitting context* of p (or of $A \to w$, for short) and R is the *forbidding context* of p (or of $A \to w$). If for all rules $(A \to w, Q, R) \in P$, $Q = \emptyset$, then G is said to be a *forbidding* random context grammar. Analogously, if for all rules $(A \to w, Q, R) \in P$, $R = \emptyset$, then G is said to be a *permitting* random context grammar. If $(A \to w, Q, R) \in P$ implies $w \in (N \cup T)^+$ is a non-empty

☆ A poster of this work was presented at the Developments in Language Theory – DLT 2010 conference.

∗ Corresponding author at: Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, 616 62 Brno, Czech Republic.
E-mail addresses: dassow@iws.cs.uni-magdeburg.de (J. Dassow), masopust@math.cas.cz (T. Masopust).

1 The author was supported by the Czech Academy of Sciences, Institutional Research Plan No. AV0Z10190503.

0022-0000/$ – see front matter
doi:10.1016/j.jcss.2011.05.008

string, then G is *non-erasing*. A sentential form $x \in (N \cup T)^+$ can directly derive a word $y \in (N \cup T)^*$, written as $x \Longrightarrow y$, if and only if there is a rule $(A \to w, Q, R) \in P$ such that

1. $x = x_1 A x_2$ for certain words $x_1, x_2 \in (N \cup T)^*$,
2. $x_1 A x_2$ contains each letter of Q and no letter of R, and
3. $y = x_1 w x_2$.

The language generated by G is defined as $L(G) = \{z \in T^* \mid S \Longrightarrow^* z\}$, where $\Longrightarrow^*$ is the reflexive and transitive closure of the relation $\Longrightarrow$.

Denote the families of languages generated by random context grammars, non-erasing random context grammars, permitting random context grammars, non-erasing permitting random context grammars, forbidding random context grammars, and non-erasing forbidding random context grammars by $\mathcal{L}(RC)$, $\mathcal{L}(RC - \lambda)$, $\mathcal{L}(P)$, $\mathcal{L}(P - \lambda)$, $\mathcal{L}(F)$, and $\mathcal{L}(F - \lambda)$, respectively.

A random context grammar G is a *modified random context grammar* if it satisfies the following requirement instead of condition 2.

2′. $x_1 x_2$ contains each letter of Q and no letter of R.

Later we will see that these two conditions are equivalent.

It has been shown that the family $\mathcal{L}(RC)$ coincides with the family $\mathcal{L}(RE)$ of recursively enumerable languages whereas the family $\mathcal{L}(RC - \lambda)$ is equal to the family of languages generated by matrix or programmed grammars with appearance checking and without erasing rules and is a proper subfamily of the family $\mathcal{L}(CS)$ of context-sensitive or, equivalently, monotone languages.

In the sequel, further variants of random context grammars have been defined. For instance, in 1985, Păun [13] discussed *semi-conditional grammars*, where permitting and forbidding contexts are replaced with permitting and forbidding strings. According to the length of these strings, semi-conditional grammars of degree (i, j), for $i, j \geqslant 0$, were defined. It was shown that non-erasing semi-conditional grammars of degree (i, j), for $1 \leqslant i, j \leqslant 2$, $i \neq j$, characterize the family $\mathcal{L}(CS)$. Furthermore, it is shown in [7] that degree $(1, 1)$ is sufficient for these grammars to characterize the family $\mathcal{L}(RC)$, or $\mathcal{L}(RC - \lambda)$ if they are non-erasing (see also [10]). In addition, some further modifications are studied in [6,9]. The reader is also referred to the monographs [2] and [11].

Consider a derivation step of the above discussed regulated context-free grammars. This derivation step can be characterized so that a set of applicable rules is determined according to symbols appearing in the sentential form, a set of nonterminals that can be rewritten is determined according to the set of applicable rules, one of these nonterminals is chosen and rewritten by an applicable rule rewriting this nonterminal. Obviously, there are two types of rules rewriting the chosen nonterminal. The one consists of rules that cannot be applied to the current sentential form, while the other consists of those that can be applied. For instance, for a sentential form ABC and rules $(B \to \alpha, \{A\}, \emptyset)$ and $(B \to \beta, \emptyset, \{C\})$, the former rule is applicable, while the latter is not. Note that this is not how context-free grammars behave. If context-free grammars can rewrite a nonterminal, then this nonterminal can be rewritten by an arbitrary rule rewriting it.

Motivated by these observations, we define a new, simpler type of regulated context-free grammars.

Definition 1. A *restricted context-free grammar* is a quintuple $G = (N, T, P, S, f)$ where N is an alphabet of nonterminals, T is an alphabet of terminals such that $N \cap T = \emptyset$, $S \in N$ is the start symbol, P is a finite set of context-free rules (i.e., rules of the form $A \to w$ with $A \in N$ and $w \in (N \cup T)^*$), and $f : N \to \{+, -\} \times N$ is a function which maps every nonterminal to a signed nonterminal. The grammar G is non-erasing if all its rules are non-erasing.

We say that x directly derives y in G, written as $x \Longrightarrow y$, if the following two conditions are satisfied:

1. $x = x_1 A x_2$, $y = x_1 w x_2$, $A \to w \in P$,
2. $f(A) = (+, B)$ implies that x contains B, and $f(A) = (-, B)$ implies that x does not contain B.

The language generated by G consists of all words z over T with $S \Longrightarrow^* z$, where $\Longrightarrow^*$ is the reflexive and transitive closure of the relation $\Longrightarrow$, i.e., $L(G) = \{z \in T^* \mid S \Longrightarrow^* z\}$.

By this definition, a rule can only be applied if the condition given by the function f is satisfied. We note the differences to random context grammars mentioned above.

(i) The condition is associated with the nonterminal which is replaced by the rule under application, not with the rule. Thus, the context is the same for all rules with the same left-hand side.
(ii) For every nonterminal, there is only a forbidding context or only a permitting context. Moreover, the context is a singleton.

Thus, each derivation step of restricted context-free grammars can be characterized so that a set of applicable nonterminals is determined according to symbols appearing in the sentential form, an applicable nonterminal is chosen and rewritten by an arbitrary rule rewriting this nonterminal. Note that except for the context checking, these grammars behave like context-free grammars. Therefore, we believe that restricted context-free grammars are the simplest type of grammars where the derivation is controlled by the structure of the sentential form.

In [8], it is shown that every recursively enumerable language is generated by a restricted context-free grammar with erasing rules, but the power of the non-erasing variant was left open. In this paper, we prove that this variant has the same power as random context (matrix, programmed) grammars. In addition, we improve some normal forms proved in [8].

Analogously to the case of random context grammars, we define two special types of restricted context-free grammars – *permitting* and *forbidding* restricted context-free grammars – so that we define $f : N \to \{+\} \times N$ or $f : N \to \{-\} \times N$, respectively. We denote the families of languages generated by restricted context-free grammars, non-erasing restricted context-free grammars, permitting restricted context-free grammars, permitting non-erasing restricted context-free grammars, forbidding restricted context-free grammars, and forbidding non-erasing restricted context-free grammars by $\mathcal{L}(rCF)$, $\mathcal{L}(rCF - \lambda)$, $\mathcal{L}(rCF_+)$, $\mathcal{L}(rCF_+ - \lambda)$, $\mathcal{L}(rCF_-)$, and $\mathcal{L}(rCF_- - \lambda)$, respectively.

In [4], Gazdag proved that $\mathcal{L}(rCF_+ - \lambda)$ and $\mathcal{L}(rCF_-)$ properly contain the family $\mathcal{L}(CF)$ of context-free languages. It is easy to modify Example 1 of [4] to show the proper containment $\mathcal{L}(CF) \subset \mathcal{L}(rCF_- - \lambda)$.

2. Hierarchy results

The aim of this section is to characterize the generative power of restricted context-free grammars. First, however, we prove the following lemma.

Lemma 2. *For every (permitting, forbidding) random context grammar G, there is a (permitting, forbidding) modified random context grammar G' such that $L(G) = L(G')$, and vice versa. Moreover, G is non-erasing if and only if G' is non-erasing.*

Proof. Let $G = (N, T, P, S)$ be a (permitting, forbidding) random context grammar. Construct the modified random context grammar $G' = (N, T, P', S)$ of the same type so that

$$P' = \{(A \to x, Q \setminus \{A\}, R) \mid (A \to x, Q, R) \in P \text{ and } A \notin R\}.$$

The derivation step $x_1 A x_2 \Longrightarrow x_1 w x_2$ made by an application of $(A \to w, Q, R) \in P$ in G means that $A \notin R$. Thus, $x_1 A x_2 \Longrightarrow x_1 w x_2$ can also be generated in G' by an application of the rule $(A \to w, Q \setminus \{A\}, R) \in P'$. Conversely, given a sentential form over $N \cup T$ in G', every derivation step is of the form $x_1 A x_2 \Longrightarrow x_1 w x_2$ made by an application of $(A \to w, Q \setminus \{A\}, R) \in P'$. Then, $x_1 A x_2 \Longrightarrow x_1 w x_2$ is generated by an application of the rule $(A \to w, Q, R) \in P$ in G because $A \notin R$. Thus, $L(G) = L(G')$.

On the other hand, let $G = (N, T, P, S)$ be a modified (permitting, forbidding) random context grammar. Construct the random context grammar $G' = (N \cup N', T, P', S)$ of the same type, where $N' = \{A' \mid A \in N\}$, $N \cap N' = \emptyset$, and P' is defined as follows:

1. $P' = \{(A \to A', \emptyset, N') \mid A \in N\} \cup \{(A' \to x, Q, R) \mid (A \to x, Q, R) \in P\}$ for G being a modified (forbidding) random context grammar, or
2. $P' = \{(A \to A', \emptyset, \emptyset) \mid A \in N\} \cup \{(A' \to x, Q, \emptyset) \mid (A \to x, Q, \emptyset) \in P\}$ for G being a modified permitting random context grammar.

The derivation step $x_1 A x_2 \Longrightarrow x_1 w x_2$ in G is simulated by the two step derivation $x_1 A x_2 \Longrightarrow x_1 A' x_2 \Longrightarrow x_1 w x_2$ in G'. Thus, $L(G) \subseteq L(G')$ is satisfied.

Conversely, given a sentential form over $N \cup T$ in G', the only possible two step derivation in a (forbidding) random context grammar G' is of the form $x_1 A x_2 \Longrightarrow x_1 A' x_2 \Longrightarrow x_1 w x_2$, where $x_1 A x_2 \Longrightarrow x_1 w x_2$ is a derivation in G. Therefore, we also get $L(G') \subseteq L(G)$.

In the case of permitting random context grammars, we can obtain sentential forms with some occurrences of primed letters, since we can change any nonterminal B to B' at any time. Let $g : (N \cup N')^* \to N^*$ be a homomorphism defined by $g(B) = g(B') = B$, for $B \in N$. Then, $x_1 A' x_2 \Longrightarrow x_1 w x_2$ with $x_1, x_2 \in (N \cup N')^*$ holds in G' according to $(A' \to w, Q, \emptyset)$ only if $g(x_1) A g(x_2) \Longrightarrow g(x_1) w g(x_2)$ holds in G. Hence, $L(G') \subseteq L(G)$ is shown.

Summarized, we have shown that $L(G) = L(G')$. □

The following lemma is the key statement to prove the main result of this paper.

Lemma 3. *For every (non-erasing) modified random context grammar $G = (N, T, P, S)$ and every symbol $d \in T$, there is a (non-erasing) restricted context-free grammar G' such that $L(G') = \{d\}L(G)$.*

Proof. Let $G = (N, T, P, S)$ be a modified random context grammar, and let $N = \{A_1, A_2, \dots, A_n\}$, for some $n \geqslant 1$. For $A \in N$, let m_A be the number of rules of the form $(A \to w, Q, R)$ in P. We set

$$m = m_{A_1} + m_{A_2} + \cdots + m_{A_n}.$$

Moreover, for $1 \leqslant i \leqslant m_A$, let

$$\bigl(A \to w_{i,A}, \{B_{i,A,1}, B_{i,A,2}, \dots, B_{i,A,r_{i,A}}\}, \{C_{i,A,1}, C_{i,A,2}, \dots, C_{i,A,s_{i,A}}\}\bigr) \tag{1}$$

be the ith rule with a core rule with left-hand side A. We construct the restricted context-free grammar $G' = (N', T, P', S', f)$ where

$$N' = N \cup \{S', Y\} \cup \{(A, i) \mid A \in N,\ 1 \leqslant i \leqslant m_A\} \cup \{(A, i)' \mid A \in N,\ 1 \leqslant i \leqslant m_A\}$$
$$\cup \{(A, i, j) \mid A \in N,\ 1 \leqslant i \leqslant m_A,\ 1 \leqslant j \leqslant 2m + r_{i,A} + s_{i,A} + 1\},$$

P' consists of all rules given in the following enumeration, and f is defined as in the following enumeration:

1. for S', we have the rule $S' \to YS$ and $f(S') = (+, S')$,
2. for $A \in N$, we have the rules $A \to (A, i)$ with $1 \leqslant i \leqslant m_A$ and $f(A) = (+, Y)$,
3. for (A, i) with $A \in N$, $1 \leqslant i \leqslant m_A$, we have the rule $(A, i) \to (A, i)'$ and $f((A, i)) = (-, (A, i)')$,
4. for $(A, i)'$ with $A \in N$, $1 \leqslant i \leqslant m_A$, we have the rule
$$(A, i)' \to w_{i,A} \quad \text{and} \quad f\bigl((A, i)'\bigr) = \bigl(+, (A, i, 2m + r_{i,A} + s_{i,A} + 1)\bigr),$$
5. for Y, we have the rules $Y \to (A, i, 1)$ with $A \in N$, $1 \leqslant i \leqslant m_A$, $Y \to d$ and $f(Y) = (+, Y)$,
6. for (A, i, j) with $A \in N$, $1 \leqslant i \leqslant m_A$, $\sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant \sum_{t=1}^{k} m_{A_t}$, $1 \leqslant k \leqslant n$, we have the rule
$$(A, i, j) \to (A, i, j+1) \quad \text{and} \quad f\bigl((A, i, j)\bigr) = \left(-, \left(A_k, j - \sum_{t=1}^{k-1} m_{A_t}\right)\right),$$
7. for (A_h, i, j) with $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant m + \sum_{t=1}^{k} m_{A_t}$, $1 \leqslant k \leqslant h-1$, we have the rule
$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\bigl((A_h, i, j)\bigr) = \left(-, \left(A_k, j - m - \sum_{t=1}^{k-1} m_{A_t}\right)'\right),$$
8. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{h-1} m_{A_t} + 1 \leqslant j \leqslant m + i - 1 + \sum_{t=1}^{h-1} m_{A_t}$, we have the rule
$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\bigl((A_h, i, j)\bigr) = \left(-, \left(A_h, j - m - \sum_{t=1}^{h-1} m_{A_t}\right)'\right),$$
9. for $(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t})$ with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, we have the rule
$$\left(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t}\right) \to \left(A_h, i, m + i + 1 + \sum_{t=1}^{h-1} m_{A_t}\right) \quad \text{and}$$
$$f\left(\left(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t}\right)\right) = \bigl(+, (A_h, i)'\bigr),$$
10. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{h-1} m_{A_t} + 1 + i \leqslant j \leqslant m + \sum_{t=1}^{h} m_{A_t}$, we have the rule
$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\bigl((A_h, i, j)\bigr) = \left(-, \left(A_h, j - m - \sum_{t=1}^{h-1} m_{A_t}\right)'\right),$$
11. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant m + \sum_{t=1}^{k} m_{A_t}$, $h+1 \leqslant k \leqslant n$, we have the rule
$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\bigl((A_h, i, j)\bigr) = \left(-, \left(A_k, j - m - \sum_{t=1}^{k-1} m_{A_t}\right)'\right),$$
12. for (A, i, j) with $1 \leqslant i \leqslant m_A$, $2m + 1 \leqslant j \leqslant 2m + r_{i,A}$, we have the rule
$$(A, i, j) \to (A, i, j+1) \quad \text{and} \quad f\bigl((A, i, j)\bigr) = (+, B_{i,A,j-2m}),$$

13. for (A,i,j) with $1 \leqslant i \leqslant m_A$, $2m + r_{i,A} + 1 \leqslant j \leqslant 2m + r_{i,A} + s_{i,A}$, we have the rule

$$(A,i,j) \to (A,i,j+1) \quad \text{and} \quad f\big((A,i,j)\big) = (-, C_{i,A,j-2m-r_{i,A}}),$$

14. for $(A,i,2m+r_{i,A}+s_{i,A}+1)$ with $1 \leqslant i \leqslant m_A$, we have the rule

$$(A,i,2m+r_{i,A}+s_{i,A}+1) \to Y \quad \text{and} \quad f\big((A,i,2m+r_{i,A}+s_{i,A}+1)\big) = \big(-, (A,i)'\big).$$

The rule of group 1 generates YS from S', S derives words of $L(G)$ and Y derives d. Assume that we have a sentential form Yx with some sentential form x of G. As long as Y is present we can apply the rules of group 2, i.e., we replace some nonterminals A by (A,i) with $1 \leqslant i \leqslant m_A$. Moreover, we can replace exactly one occurrence of (A,i) by $(A,i)'$, see the rules of group 3. Thus, the sentential form can contain at most one occurrence of $(A,i)'$ for each $A \in N$ and $1 \leqslant i \leqslant m_A$. Now assume that we change Y according to the rules of group 5. Then, we obtain a sentential form which starts with $(A,i,1)$, for some $A \in N$ and $1 \leqslant i \leqslant m_A$, or with d. Assume that we applied $Y \to (A_h,i,1)$, i.e., we choose a nonterminal A_h and the ith rule with a core rule with left-hand side A_h. By our above settings, this ith rule has the form given in (1). Now we essentially have to apply the rules given in groups 6–13. Rules of group 6 check the absence of nonterminals

$$(A_1,1),(A_1,2),\ldots,(A_1,m_{A_1}),(A_2,1),(A_2,2),\ldots,(A_2,m_{A_2}),\ldots,(A_n,1),(A_n,2),\ldots,(A_n,m_{A_n}).$$

We mention that these checks have not been done in succession because we can replace some (A,i) by $(A,i)'$ as long as the absence of (A,i) was not checked. Rules of group 7 check the absence of nonterminals

$$(A_1,1)',(A_1,2)',\ldots,(A_1,m_{A_1})',(A_2,1)',\ldots,(A_2,m_{A_2})',\ldots,(A_{h-1},1)',\ldots,(A_{h-1},m_{A_{h-1}})'.$$

Rules in group 8 check the absence of $(A_h,1)'$, $(A_h,2)'$, …, $(A_h,i-1)'$. The rule in group 9 checks the presence of the nonterminal $(A_h,i)'$. Rules of group 10 check the absence of nonterminals $(A_h,i+1)',(A_h,i+2)',\ldots,(A_h,m_{A_h})'$, and rules of group 11 check the absence of nonterminals

$$(A_{h+1},1)',\ldots,(A_{h+1},m_{A_{h+1}})',(A_{h+2},1)',\ldots,(A_{h+2},m_{A_{h+2}})',\ldots,(A_n,1)',\ldots,(A_n,m_{A_n})'.$$

If all these checks are positive, then the derivation before the checks was of the form

$$Yw = Yx_1A_hx_2 \Longrightarrow Yx_1(A_h,i)x_2 \Longrightarrow Yx_1(A_h,i)'x_2$$

(where the last step can be performed until the absence of (A_h,i) is checked by a rule of group 6). Now we check by the rules of group 12 the presence of letters in the permitting context of the rule $A_h \to w_{i,A_h}$, and by the rules of group 13 the absence of nonterminals in the forbidding context (note that we here need the assumption that G is a modified random context grammar because we test only the occurrence/non-occurrence in x_1x_2). Now the only applicable rule is that of group 4 which gives the following sentential form $(A_h,i,2m+r_{i,A_h}+s_{i,A_h}+1)x_1w_{i,A_h}x_2$. Then, we have to apply the rule of group 14 which gives the sentential form $Yx_1w_{i,A_h}x_2$. Thus we have simulated one derivation step of G in G'.

Now, assume that we applied the rule $Y \to d$ of group 5, which gives the sentential form dw. If w contains a nonterminal, we can perform some replacements according to the rules of group 3, but the derivation cannot terminate (see the rules of group 4). Therefore, every terminating derivation in G' has the form

$$S' \Longrightarrow YS \Longrightarrow^* Yy_1 \Longrightarrow^* Yy_2 \Longrightarrow^* \cdots \Longrightarrow^* Yy_z \Longrightarrow dy_z$$

where $S \Longrightarrow^* y_1 \Longrightarrow^* y_2 \Longrightarrow^* \cdots \Longrightarrow^* y_z$ is a terminating derivation in G. Hence $L(G') = \{d\}L(G)$. □

The following theorem is the main result of this paper.

Theorem 4. *$\mathcal{L}(RC-\lambda) = \mathcal{L}(rCF-\lambda)$ and $\mathcal{L}(RC) = \mathcal{L}(rCF)$.*

Proof. (i) Let $G = (N,T,P,S,f)$ be a (non-erasing) restricted context-free grammar. We construct the (non-erasing) random context grammar $G' = (N,T,P',S)$ with

$$P' = \big\{\big(A \to w, \{B\}, \emptyset\big) \bigm| A \to w \in P,\ f(A) = (+,B)\big\} \cup \big\{\big(A \to w, \emptyset, \{B\}\big) \bigm| A \to w \in P,\ f(A) = (-,B)\big\}.$$

It is easy to see that $x_1Ax_2 \Longrightarrow x_1wx_2$ holds in G' if and only if it holds in G. Hence, $L(G') = L(G)$, which proves the inclusions $\mathcal{L}(rCF) \subseteq \mathcal{L}(RC)$ and $\mathcal{L}(rCF-\lambda) \subseteq \mathcal{L}(RC-\lambda)$.

(ii) Let G be a random context grammar. By Lemma 2, there is a modified random context grammar G_1 with $L(G) = L(G_1)$. According to Lemma 3, we construct the restricted context-free grammar G_1' where we use the rule $Y \to \lambda$ instead of $Y \to d$ in group 5. Then, we obtain that $L(G_1') = L(G)$. Thus, we have shown that $\mathcal{L}(RC) \subseteq \mathcal{L}(rCF)$.

(iii) Let $L \subseteq T^*$ be a language generated by a non-erasing random context grammar. For $a \in T$, let $L_a = \{w \mid aw \in L\}$. Obviously,

$$L = \bigcup_{a \in T} \{a\}L_a. \tag{2}$$

By Lemma 2 and [2, Corollary of Theorem 1.3.2], L_a is generated by a non-erasing modified random context grammar G_a. By Lemma 3, we construct a non-erasing restricted context-free grammar $G'_a = (N_a, T, P_a, S', f_a)$ such that $L(G'_a) = \{a\}L_a$. By the proof of Lemma 3, we can assume that $N_a \cap N_b = \{S'\}$, for all $a, b \in T$, $a \neq b$, and $f_a(S') = (+, S')$ for all $a \in T$. It is easy to see that

$$G' = \left(\bigcup_{a \in T} N_a, T, \bigcup_{a \in T} P_a, S', f \right)$$

with $f(X) = f_a(X)$, for $X \in N_a$, is a non-erasing restricted context-free grammar with $L(G') = \bigcup_{a \in T}\{a\}L_a$. By Eq. (2), $L(G') = L$ holds. Thus, $\mathcal{L}(RC - \lambda) \subseteq \mathcal{L}(rCF - \lambda)$ is shown.

By a combination of (i), (ii), and (iii), the statement is shown. □

In the following part of this section, we prove analogous results for permitting restricted context-free grammars. However, an analogous question concerning the power of forbidding restricted context-free grammars is an open problem.

Lemma 5. *For every (non-erasing) permitting random context grammar $G = (N, T, P, S)$ and every symbol $d \in T$, there is a (non-erasing) permitting restricted context-free grammar G' such that $L(G') = \{d\}L(G)$.*

Proof. Let $G = (N, T, P, S)$ be a permitting random context grammar, and let $N = \{A_1, A_2, \ldots, A_n\}$, for some $n \geqslant 1$. For $A \in N$, let m_A be the number of rules of the form $(A \to w, Q, \emptyset)$ in P, and let $m = m_{A_1} + m_{A_2} + \cdots + m_{A_n}$. For $1 \leqslant i \leqslant m_A$, let

$$\big(A \to w_{i,A}, \{B_{i,A,1}, B_{i,A,2}, \ldots, B_{i,A,r_{i,A}}\}, \emptyset\big) \tag{3}$$

be the ith rule with a core rule with left-hand side A. We construct the permitting restricted context-free grammar $G' = (N', T, P', S', f)$ where

$$N' = N \cup \{S', Y\} \cup \{(A, i) \mid A \in N,\ 1 \leqslant i \leqslant m_A\} \cup \{(Y, A, i) \mid A \in N,\ 1 \leqslant i \leqslant m_A\}$$
$$\cup \{(A, i, j, k) \mid A \in N,\ 1 \leqslant i \leqslant m_A,\ 1 \leqslant j \leqslant r_{i,A},\ 0 \leqslant k \leqslant m_{B_{i,A,j}}\},$$

P' consists of all rules given in the following enumeration, and f is defined as in the following enumeration:

1. for S', we have the rule $S' \to YS$ and $f(S') = (+, S')$,
2. for $A \in N$, we have the rules $A \to (A, i)$ with $1 \leqslant i \leqslant m_A$ and $f(A) = (+, Y)$,
3. for Y, we have the rules $Y \to (A, i, 1, k)$ with $A \in N$, $1 \leqslant i \leqslant m_A$, $0 \leqslant k \leqslant m_{B_{i,A,1}}$, $Y \to d$ and $f(Y) = (+, Y)$,
4. for $(A, i, j, 0)$ with $1 \leqslant i \leqslant m_A$, $1 \leqslant j \leqslant r_{i,A} - 1$, we have the rules
$$(A, i, j, 0) \to (A, i, j+1, k') \quad \text{for } 0 \leqslant k' \leqslant m_{B_{i,A,j+1}} \quad \text{and} \quad f\big((A, i, j, 0)\big) = (+, B_{i,A,j}),$$
5. for $(A, i, r_{i,A}, 0)$ with $1 \leqslant i \leqslant m_A$, we have the rules
$$(A, i, r_{i,A}, 0) \to (Y, A, i) \quad \text{and} \quad f\big((A, i, r_{i,A}, 0)\big) = (+, B_{i,A,r_{i,A}}),$$
6. for (A, i, j, k) with $1 \leqslant i \leqslant m_A$, $1 \leqslant j \leqslant r_{i,A} - 1$, $1 \leqslant k \leqslant m_{B_{i,A,j}}$, we have the rules
$$(A, i, j, k) \to (A, i, j+1, k') \quad \text{for } 0 \leqslant k' \leqslant m_{B_{i,A,j+1}} \quad \text{and} \quad f\big((A, i, j, k)\big) = \big(+, (B_{i,A,j}, k)\big),$$
7. for $(A, i, r_{i,A}, k)$ with $1 \leqslant i \leqslant m_A$, $1 \leqslant k \leqslant m_{B_{i,A,r_{i,A}}}$, we have the rule
$$(A, i, r_{i,A}, k) \to (Y, A, i) \quad \text{and} \quad f\big((A, i, r_{i,A}, k)\big) = \big(+, (B_{i,A,r_{i,A}}, k)\big),$$
8. for $A \in N$ and $1 \leqslant i \leqslant m_A$, we have the rule $(A, i) \to w_{i,A}$ and $f((A, i)) = (+, (Y, A, i))$,
9. and for (Y, A, i), we have the rule $(Y, A, i) \to Y$ and $f((Y, A, i)) = (+, (Y, A, i))$.

Let $U = T \cup N \cup \{S'\} \cup \{(A, i) \mid A \in N,\ 1 \leqslant i \leqslant m_A\}$. The rule of group 1 derives YS from S', S derives words of $L(G)$ and Y derives d. Consider a sentential form Yx with $x \in U^*$ such that $g(x)$ is a sentential form of G for the homomorphism $g : U^* \to (N^* \cup \{S'\})$ defined as $g(A) = A$, for $A \in T \cup N \cup \{S'\}$, and $g((A, i)) = A$, for $A \in N$ and $1 \leqslant i \leqslant m_A$. As long as Y is present we can apply the rules of group 2, i.e., we replace some nonterminals A by (A, i) with $1 \leqslant i \leqslant m_A$. Now assume that we change Y according to the rules of group 3. Then, we obtain a sentential form which starts with $(A, i, 1, k)$, for some $A \in N$, $1 \leqslant i \leqslant m_A$ and $0 \leqslant k \leqslant m_{B_{i,A,1}}$, or with d. Assume that we apply $Y \to (A, i, 1, k)$, which means that we choose a nonterminal A and the ith rule with a core rule with left-hand side A. This ith rule has the form given in (3). Now we essentially have to apply rules of groups 4–9. Rules of groups 4–5 check the presence of nonterminals

$$B_{i,A,1}, B_{i,A,2}, \ldots, B_{i,A,r_{i,A}}$$

and rules of groups 6–7 check the presence of nonterminals

$$(B_{i,A,1}, k_1), (B_{i,A,2}, k_2), \ldots, (B_{i,A,r_{i,A}}, k_{r_{i,A}})$$

for certain numbers k_u, $1 \leqslant u \leqslant r_{i,A}$. Essentially, we check that all the symbols $B_{i,A,1}, B_{i,A,2}, \ldots, B_{i,A,r_{i,A}}$ occur in $g(x)$. If all these checks are positive, then we can replace (A, i) by $w_{i,A}$ by the rule of group 8 simulating the rule $A \to w_{i,A}$. Now the only applicable rule is a rule of group 9 replacing (Y, A, i) by Y. Thus, we have simulated one derivation step of G in G'.

Note that we can replace some nonterminals (A, i) by rules of group 8 which means that we simulate some applications of $A \to w_{i,A}$. If no rule with left-hand side (A, i) is applied, then, by the application of a rule of group 9, we return to the sentential form we started from.

If we use the rule $Y \to d$ of group 3, we get the sentential form dw. In that case, however, if w contains a nonterminal, the derivation cannot terminate (see the rules of group 8). Therefore, every terminating derivation in G' has the form $S' \Longrightarrow YS \Longrightarrow^* Yy_1 \Longrightarrow^* Yy_2 \Longrightarrow^* \cdots \Longrightarrow^* Yy_z \Longrightarrow dy_z$ where $S \Longrightarrow^* g(y_1) \Longrightarrow^* g(y_2) \Longrightarrow^* \cdots \Longrightarrow^* g(y_z) = y_z$ is a terminating derivation in G. Hence, $L(G') = \{d\}L(G)$ is shown. □

Theorem 6. *$\mathcal{L}(P - \lambda) = \mathcal{L}(rCF_+ - \lambda)$ and $\mathcal{L}(P) = \mathcal{L}(rCF_+)$.*

Proof. (i) Let $G = (N, T, P, S, f)$ be a (non-erasing) permitting restricted context-free grammar. We construct the (non-erasing) permitting random context grammar $G' = (N, T, P', S)$ with

$$P' = \{(A \to w, \{B\}, \emptyset) \mid A \to w \in P,\ f(A) = (+, B),\ A \neq B\} \cup \{(A \to w, \emptyset, \emptyset) \mid A \to w \in P,\ f(A) = (+, A)\}.$$

It is easy to see that $x_1Ax_2 \Longrightarrow x_1wx_2$ holds in G' if and only if it holds in G. Hence $L(G') = L(G)$, which proves the inclusions $\mathcal{L}(rCF_+) \subseteq \mathcal{L}(P)$ and $\mathcal{L}(rCF_+ - \lambda) \subseteq \mathcal{L}(P - \lambda)$.

(ii) Let G be a permitting random context grammar. According to Lemma 5, we construct the restricted context-free grammar G' where we use the rule $Y \to \lambda$ instead of $Y \to d$ in group 3. Then, $L(G') = L(G)$. Thus, $\mathcal{L}(P) \subseteq \mathcal{L}(rCF_+)$ is shown.

(iii) Let $L \subseteq T^*$ be a language generated by a non-erasing permitting random context grammar. For $a \in T$, let $L_a = \{w \mid aw \in L\}$. Obviously, $L = \bigcup_{a \in T}\{a\}L_a$. By [2, Corollary of Theorem 1.3.2], L_a is generated by a non-erasing permitting random context grammar G_a. By Lemma 5, we construct a non-erasing permitting restricted context-free grammar $G'_a = (N_a, T, P_a, S', f_a)$ such that $L(G'_a) = \{a\}L_a$. We can assume that $N_a \cap N_b = \{S'\}$, for all $a, b \in T$, $a \neq b$, and $f_a(S') = (+, S')$, for all $a \in T$. Then, $G' = (\bigcup_{a \in T} N_a,\ T, \bigcup_{a \in T} P_a, S', f)$ with $f(X) = f_a(X)$, for $X \in N_a$, is a non-erasing permitting restricted context-free grammar satisfying $L(G') = \bigcup_{a \in T}\{a\}L_a = L$. This proves $\mathcal{L}(P - \lambda) \subseteq \mathcal{L}(rCF_+ - \lambda)$. □

As an immediate consequence of Theorems 4 and 6, we have the following corollary.

Corollary 7. *Let $\mathcal{L}(CF)$ and $\mathcal{L}(REC)$ denote the families of context-free and recursive languages, respectively. Then, the following holds*:

1. $\mathcal{L}(CF) \subset \mathcal{L}(rCF_+) = \mathcal{L}(P) = \mathcal{L}(P - \lambda) = \mathcal{L}(rCF_+ - \lambda) \subset \mathcal{L}(RC - \lambda)$,
2. $\mathcal{L}(CF) \subset \mathcal{L}(rCF_-) \subseteq \mathcal{L}(F) \subset \mathcal{L}(REC)$, *and*
3. $\mathcal{L}(CF) \subset \mathcal{L}(rCF_- - \lambda) \subseteq \mathcal{L}(F - \lambda) \subset \mathcal{L}(RC - \lambda)$.

Proof. First, note that the equation $\mathcal{L}(P) = \mathcal{L}(P - \lambda)$ has recently been shown in [16]. In addition, it is not hard to construct a permitting random context grammar G' generating a non-context-free language $L(G') = \{a^nb^nc^n \mid n \geqslant 1\}$ (see [2, Example 1.1.7]). By Lemma 5, there is a permitting restricted context-free grammar G such that $L(G) = \{da^nb^nc^n \mid n \geqslant 1\}$, for $d \in \{a, b, c\}$. Thus, we have $\mathcal{L}(CF) \subset \mathcal{L}(rCF_+ - \lambda)$. Proofs of the remaining proper inclusions can be found in [1,3,4,15]. □

3. Consequences concerning normal forms

In this section, we present some consequences for normal forms of random context and matrix grammars. We first start with the definition of the latter type of grammars.

A *matrix grammar* is a construct $G = (N, T, M, S, F)$ where N, T, and S are as in a restricted context-free grammar, M is a finite set of sequences (*matrices*) of the form $[r_1, r_2, \ldots, r_n]$, where $n \geqslant 1$ and r_i is a context-free rule, for $1 \leqslant i \leqslant n$, and F is a finite set of context-free rules. For $u \in (N \cup T)^+$, $v \in (N \cup T)^*$, and $[r_1, r_2, \ldots, r_n] \in M$, $u \Longrightarrow v$ holds if and only if there are words $x_0, x_1, \ldots, x_{n-1} \in (N \cup T)^+$ and $x_n \in (N \cup T)^*$ such that

1. $x_0 = u$ and $x_n = v$,
2. for $1 \leqslant i \leqslant n$, $x_{i-1} \Longrightarrow x_i$ by an application of r_i, or r_i is not applicable to x_{i-1}, $r_i \in F$ and $x_i = x_{i-1}$.

The language of G is defined as $L(G) = \{z \in T^* \mid S \Longrightarrow^* z\}$, where $\Longrightarrow^*$ is the reflexive and transitive closure of the relation $\Longrightarrow$.

Intuitively, in a matrix grammar, the rules have to be applied in the order given by the matrices (and rules of F can be ignored if they are not applicable).

We denote the families of languages generated by matrix grammars with and without erasing rules by $\mathcal{L}(M)$ and $\mathcal{L}(M-\lambda)$, respectively. It is well known (see [2]) that

$$\mathcal{L}(M) = \mathcal{L}(RC) = \mathcal{L}(RE) \quad \text{and} \quad \mathcal{L}(M-\lambda) = \mathcal{L}(RC-\lambda) \subset \mathcal{L}(CS). \tag{4}$$

We recall here the following known normal form for matrix grammars (see [2, Lemmas 1.2.3 and 1.3.1]). For every $L \in \mathcal{L}(RE)$ ($L \in \mathcal{L}(M-\lambda)$), there is a (non-erasing) matrix grammar $G = (N \cup \{Z\}, T, M, S, F)$, for some $Z \notin N \cup T$, such that $L(G) = L$, all matrices have the form

- $[A \to w]$ or $[X \to Y, A \to w]$ with $A, X, Y \in N$, $w \in (N \cup T)^*$, and $|w| \leqslant 2$, and
- F consists only of rules of the form $A \to Z$, where $A \in N$.

Using the simulation of matrix grammars by random context grammars (see [2, Proof of Theorem 1.2.3]), we obtain that, for any language $L \in \mathcal{L}(RE)$ (resp. $L \in \mathcal{L}(RC-\lambda)$), there is a (non-erasing) random context grammar $G = (N, T, P, S)$ with $L(G) = L$ such that all rules are of the form $(A \to w, Q, R)$ with $A \in N$, $w \in (N \cup T)^*$, $|w| \leqslant 2$, and R, Q are two disjoint subsets of N.

In [8], another normal form is given where the length of the right-hand sides is restricted by 3, but the forms of the chain rules in matrix grammars and the permitting and forbidding contexts in random context grammars are more restricted than in the above cases. We now prove that both these features (the length at most two and a further restriction to the rules/contexts) can be combined.

Corollary 8. *For every language $L \in \mathcal{L}(RE)$ (resp. $L \in \mathcal{L}(RC-\lambda)$), there is a random context grammar $G = (N, T, P, S)$ such that $L(G) = L$ with the following properties*:

- *if $(A \to w, Q, R) \in P$, then $|w| \leqslant 2$, $Q \cap R = \emptyset$, and $\#(Q \cup R) = 1$, where $\#(Q \cup R)$ denotes the cardinality of $Q \cup R$, and*
- *if $(A \to w_1, Q_1, R_1) \in P$ and $(A \to w_2, Q_2, R_2) \in P$, then $Q_1 = Q_2$ and $R_1 = R_2$.*

Proof. Let $L \in \mathcal{L}(RE)$ (resp. $L \in \mathcal{L}(RC-\lambda)$). Then, there is a (non-erasing) random context grammar $G' = (N', T, P', S')$ such that all rules $(A \to w, Q, R) \in P'$ satisfy $|w| \leqslant 2$. We now construct the restricted context-free grammar $G'' = (N'', T, P'', S'')$ from G' as done in proofs of Lemma 3 and Theorem 4. For all rules $A \to w \in P''$, $|w| \leqslant 2$ holds. From G'' we construct a random context grammar G according to the proof of Theorem 4, part (i). It is easy to see that G satisfies all the requirements of the statement. □

Note that there is no simpler normal form for random context grammars with respect to the following parameters: the number of contexts (Q, R) associated with a nonterminal or a rule, the size of the context, and the length of the right-hand side of the core rules. If we restrict the length of the right-hand sides by one, then we can generate only sentential forms of length at most one, i.e., not all random context languages. If we add no contexts to the nonterminals or rules or the size of the context, i.e., $\#(Q \cup R)$ is bounded by 0, we get only context-free grammars and languages. Moreover, if we omit forbidding or permitting contexts, then we only get permitting or forbidding random context grammars which are weaker than random context grammars.

Corollary 9. *For every language $L \in \mathcal{L}(RE)$ (resp. $L \in \mathcal{L}(M-\lambda)$), there is a (non-erasing) matrix grammar $G = (N \cup \{Z\}, T, M, S, F)$, for some $Z \notin N \cup T$, such that $L(G) = L$ with the following conditions*:

- *every matrix has the form $[A \to A, B \to w]$ with $A, B \in N$, $|w| \leqslant 2$ or $[A \to Z, B \to w]$ with $A, B \in N$, $|w| \leqslant 2$,*
- *$[r_1, A \to w_1]$ and $[r_2, A \to w_2]$ imply that $r_1 = r_2$, and*
- *F consists of all rules of the form $A \to Z$ occurring in matrices of M.*

Proof. By (4), there is a random context grammar $G' = (N, T, P, S)$ which satisfies the requirement of Corollary 8. Now we replace

- any rule of the form $(B \to w, \{A\}, \emptyset)$ by the matrix $[A \to A, B \to w]$, and
- any rule of the form $(B \to w, \emptyset, \{A\})$ by the matrix $[A \to Z, B \to w]$.

Let M be the set of all matrices and F the set of all rules of the form $A \to Z$ obtained in this way. It follows (see the construction in the beginning of the proof of [2, Theorem 1.2.3]) that the matrix grammar $G = (N \cup \{Z\}, T, M, S, F)$ generates $L(G')$ and hence L. Obviously, G satisfies the required conditions. □

4. String restricted context-free grammars

The case of regulated context-free grammars where a presence and absence of a string instead of a symbol is required in the sentential form has also been widely discussed in the literature. As mentioned above, it is sufficient to consider strings of length no more than two. This motivates the following discussion.

Definition 10. A *string restricted context-free grammar* is a quintuple $G = (N, T, P, S, f)$ where N is an alphabet of nonterminals, T is an alphabet of terminals such that $N \cap T = \emptyset$, $S \in N$ is the start symbol, P is a finite set of context-free rules, and $f : N \to \{+, -\} \times (N \cup NN)$ is a function which maps every nonterminal to a signed string of length one or two. We say that x directly derives y in G, written as $x \Longrightarrow y$, if the following two conditions are satisfied:

1. $x = x_1 A x_2$, $y = x_1 w x_2$, $A \to w \in P$,
2. $f(A) = (+, b)$ implies that x contains b as a substring, and $f(A) = (-, b)$ implies that x does not contain b as a substring.

The language generated by G is defined as $L(G) = \{z \in T^* \mid S \Longrightarrow^* z\}$, where $\Longrightarrow^*$ is the reflexive and transitive closure of the relation $\Longrightarrow$.

We denote the families of languages generated by string restricted context-free grammars and non-erasing string restricted context-free grammars by $\mathcal{L}(srCF)$ and $\mathcal{L}(srCF - \lambda)$, respectively. Note that from Theorem 4 it immediately follows that $\mathcal{L}(srCF) = \mathcal{L}(RE)$. Thus, only the family $\mathcal{L}(srCF - \lambda)$ is of interest. In this section, we show that this family coincides with the family of context-sensitive languages.

Recall that a *monotone grammar* is a quadruple $G = (N, T, P, S)$ where N is an alphabet of nonterminals, T is an alphabet of terminals such that $N \cap T = \emptyset$, $S \in N$ is the start symbol, and P is a finite set of rules of the form $u \to v$, where $u \in (N \cup T)^* N (N \cup T)^*$, $v \in (N \cup T)^+$ and $|u| \leqslant |v|$. A sentential form $x = x_1 u x_2$ directly derives a word $y = x_1 v x_2$ in G, written as $x \Longrightarrow y$, if there is a rule $u \to v \in P$. The language generated by G is $L(G) = \{z \in T^* \mid S \Longrightarrow^* z\}$.

A monotone grammar $G = (N, T, P, S)$ is in *Penttonen normal form* if all its rules are of the following forms: (1) $AB \to AC$, (2) $A \to BC$, (3) $A \to a$, where $A, B, C \in N$ and $a \in T$. It is well known that monotone grammars characterize the family of context-sensitive languages, and that any monotone grammar can be transformed to an equivalent monotone grammar in Penttonen normal form, see [12].

Lemma 11. *For every monotone grammar $G = (N, T, P, S)$ in Penttonen normal form and every symbol $d \in T$, there is a non-erasing string restricted context-free grammar G' such that $L(G') = \{d\}L(G)$.*

Proof. Let $G = (N, T, P, S)$ be a monotone grammar, and assume that $N = \{A_1, A_2, \ldots, A_n\}$, for some $n \geqslant 1$. For $AB \in NN$, let m_{AB} be the number of rules of the form $AB \to AX$ in P, for $X \in N$. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m_{A_i B}$, let

$$A_i B \to A_i X_{(i,j)} \tag{5}$$

be the jth rule with left-hand side $A_i B$, and let $m_B = m_{A_1 B} + m_{A_2 B} + \cdots + m_{A_n B}$. Let π_B be a bijection defined so that

$$\pi_B(i, j) = k = \sum_{r=1}^{i-1} m_{A_r B} + j,$$

then the rule $A_i B \to A_i X_{(i,j)}$ is the kth rule with left-hand side in $N\{B\}$. Let $m = \sum_{i=1}^{n} m_{A_i}$. We construct the string restricted context-free grammar $G' = (N', T, P', S', f)$ where

$$N' = N \cup \{S', Y\} \cup \{(B, i) \mid B \in N,\ 1 \leqslant i \leqslant m_B\} \cup \{(B, i)' \mid B \in N, 1 \leqslant i \leqslant m_B\}$$
$$\cup \{(B, i, k) \mid B \in N,\ 1 \leqslant i \leqslant m_B,\ 1 \leqslant k \leqslant 2m + 2\},$$

P' consists of all rules given below, and f is defined as below:

1. for S', we have the rule $S' \to YS$ and $f(S') = (+, S')$,
2. for $A \to w \in P$, we have the rule $A \to w$ and $f(A) = (+, Y)$,
3. for $B \in N$, we have the rules $B \to (B, i)$ with $1 \leqslant i \leqslant m_B$, and $f(B) = (+, Y)$,
4. for (B, k) with $B \in N$, $1 \leqslant k \leqslant m_B$, we have the rule $(B, k) \to (B, k)'$ and $f((B, k)) = (-, (B, k)')$,
5. for $(B, k)'$ with $B \in N$, $1 \leqslant k \leqslant m_B$, we have the rule
$$(B, k)' \to X_{\pi_B^{-1}(k)} \quad \text{and} \quad f((B, k)') = (+, (B, k, 2m + 2)),$$
6. for Y, we have the rules $Y \to (A, i, 1)$ with $A \in N$, $1 \leqslant i \leqslant m_A$, $Y \to d$ and $f(Y) = (+, Y)$,

7. for (A, i, j) with $A \in N$, $1 \leqslant i \leqslant m_A$, $\sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant \sum_{t=1}^{k} m_{A_t}$, $1 \leqslant k \leqslant n$, we have the rule

$$(A, i, j) \to (A, i, j+1) \quad \text{and} \quad f\big((A, i, j)\big) = \left(-, \left(A_k, j - \sum_{t=1}^{k-1} m_{A_t}\right)\right),$$

8. for (A_h, i, j) with $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant m + \sum_{t=1}^{k} m_{A_t}$, $1 \leqslant k \leqslant h-1$, we have the rule

$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\big((A_h, i, j)\big) = \left(-, \left(A_k, j - m - \sum_{t=1}^{k-1} m_{A_t}\right)'\right),$$

9. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{h-1} m_{A_t} + 1 \leqslant j \leqslant m + i - 1 + \sum_{t=1}^{h-1} m_{A_t}$, we have the rule

$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\big((A_h, i, j)\big) = \left(-, \left(A_h, j - m - \sum_{t=1}^{h-1} m_{A_t}\right)'\right),$$

10. for $(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t})$ with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, we have the rule

$$\left(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t}\right) \to \left(A_h, i, m + i + 1 + \sum_{t=1}^{h-1} m_{A_t}\right) \quad \text{and}$$

$$f\left(\left(A_h, i, m + i + \sum_{t=1}^{h-1} m_{A_t}\right)\right) = \big(+, (A_h, i)'\big),$$

11. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{h-1} m_{A_t} + 1 + i \leqslant j \leqslant m + \sum_{t=1}^{h} m_{A_t}$, we have the rule

$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\big((A_h, i, j)\big) = \left(-, \left(A_h, j - m - \sum_{t=1}^{h-1} m_{A_t}\right)'\right),$$

12. for (A_h, i, j) with $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant m_{A_h}$, $m + \sum_{t=1}^{k-1} m_{A_t} + 1 \leqslant j \leqslant m + \sum_{t=1}^{k} m_{A_t}$, $h+1 \leqslant k \leqslant n$, we have the rule

$$(A_h, i, j) \to (A_h, i, j+1) \quad \text{and} \quad f\big((A_h, i, j)\big) = \left(-, \left(A_k, j - m - \sum_{t=1}^{k-1} m_{A_t}\right)'\right),$$

13. for $(B, k, 2m+1)$ with $1 \leqslant k \leqslant m_B$ and $\pi_B^{-1}(k) = (i, j)$, we have the rule

$$(B, k, 2m+1) \to (B, k, 2m+2) \quad \text{and} \quad f\big((B, k, 2m+1)\big) = \big(+, A_i(B, k)'\big),$$

14. and for $(B, k, 2m+2)$ with $1 \leqslant k \leqslant m_A$, we have the rule

$$(B, k, 2m+2) \to Y \quad \text{and} \quad f\big((B, k, 2m+2)\big) = \big(-, (B, k)'\big).$$

By the rule of group 1 we get YS from S', where S derives words of $L(G)$ and Y derives d. Consider the sentential form Yx with a sentential form x of G. As long as Y is present we can apply the rules of group 2 and 3, i.e., to simulate context-free rules or replace nonterminals A by (A, k) with $1 \leqslant k \leqslant m_A$. Moreover, we can replace exactly one occurrence of (A, k) by $(A, k)'$, see group 4. Thus, the sentential form can contain at most one occurrence of $(A, k)'$ for each $A \in N$ and $1 \leqslant k \leqslant m_A$. Then, we rewrite Y according to the rules of group 6. We obtain a sentential form which starts with $(A, k, 1)$, for $A \in N$ and $1 \leqslant k \leqslant m_A$, or with d. Assume that we applied $Y \to (A_h, k, 1)$, i.e., we choose a nonterminal A_h and the kth rule with left-hand side in $N\{A_h\}$. For $\pi_{A_h}^{-1}(k) = (i, j)$, this kth rule has the form given in (5). Now we essentially have to apply the rules given in groups 7–14. Rules of group 7 check the absence of $(A_1, 1)$, $(A_1, 2)$, $\ldots$, (A_1, m_{A_1}), $(A_2, 1)$, $(A_2, 2)$, $\ldots$, (A_2, m_{A_2}), $\ldots$, $(A_n, 1)$, $(A_n, 2)$, $\ldots$, (A_n, m_{A_n}). These checks have not been done in succession because we can replace (A, k) by $(A, k)'$ as long as the absence of (A, k) was not checked. Rules of group 8 check the absence of $(A_1, 1)'$, $(A_1, 2)'$, $\ldots$, $(A_1, m_{A_1})'$, $(A_2, 1)'$, $\ldots$, $(A_2, m_{A_2})'$, $\ldots$, $(A_{h-1}, 1)'$, $\ldots$, $(A_{h-1}, m_{A_{h-1}})'$. Rules of group 9 check the absence of $(A_h, 1)'$, $(A_h, 2)'$, $\ldots$, $(A_h, k-1)'$. The rule in group 10 checks the presence of $(A_h, k)'$. Rules of group 11 check the absence of $(A_h, k+1)'$, $(A_h, k+2)'$, $\ldots$, $(A_h, m_{A_h})'$. Rules of group 12 check the absence of $(A_{h+1}, 1)'$, $\ldots$, $(A_{h+1}, m_{A_{h+1}})'$, $(A_{h+2}, 1)'$, $\ldots$, $(A_{h+2}, m_{A_{h+2}})'$, $\ldots$, $(A_n, 1)'$, $\ldots$, $(A_n, m_{A_n})'$. If all these checks are positive, then the derivation before the checks was of the form $Yw = Yx_1 A_h x_2 \Longrightarrow Yx_1 (A_h, k) x_2 \Longrightarrow Yx_1 (A_h, k)' x_2$, where the last step can be performed until the absence of (A_h, k) is checked by a rule of group 7. Now rules of group 13 check that the left context of A_h is the correct context for the rule $A_i A_h \to A_i X_{\pi_{A_h}^{-1}(k)}$. Then, the only applicable rule is that of group 5 which gives the following sentential form

$(A_h, k, 2m+2)x_1 A_i X_{\pi_{A_h}^{-1}(k)} x_2$. Then, the rule of group 14 has to be applied, which gives the sentential form $Yx_1 A_i X_{\pi_{A_h}^{-1}(k)} x_2$. Thus, we have simulated one derivation step of a rule of G in G'.

Now, assume that we applied the rule $Y \to d$ of group 6, which gives the sentential form dw. If w contains a nonterminal, we can perform replacements according to the rules of group 4, but the derivation cannot terminate (see group 5). Therefore, every terminating derivation in G' has the form $S' \Longrightarrow YS \Longrightarrow^* Yy_1 \Longrightarrow^* Yy_2 \Longrightarrow^* \cdots \Longrightarrow^* Yy_z \Longrightarrow dy_z$ where $S \Longrightarrow^* y_1 \Longrightarrow^* y_2 \Longrightarrow^* \cdots \Longrightarrow^* y_z$ is a terminating derivation in G. Hence, $L(G') = \{d\}L(G)$. □

The following theorem characterizes the language family $\mathcal{L}(srCF - \lambda)$.

Theorem 12. $\mathcal{L}(CS) = \mathcal{L}(srCF - \lambda)$.

Proof. Let $G = (N, T, P, S, f)$ be a non-erasing string restricted context-free grammar. Using a standard technique, we construct a linear bounded automaton [5] accepting $L(G)$. Thus, $\mathcal{L}(srCF - \lambda) \subseteq \mathcal{L}(CS)$.

On the other hand, let $L \subseteq T^*$ be a language generated by a monotone grammar. For $a \in T$, let $L_a = \{w \mid aw \in L\}$. Then, $L = \bigcup_{a \in T} \{a\}L_a$. It is known that L_a is generated by a monotone grammar G_a. By Lemma 11, we construct a non-erasing string restricted context-free grammar $G'_a = (N_a, T, P_a, S', f_a)$ such that $L(G'_a) = \{a\}L_a$, and we can assume that $N_a \cap N_b = \{S'\}$, for all $a, b \in T$, $a \neq b$, and $f_a(S') = (+, S')$, for all $a \in T$. Then, $G' = (\bigcup_{a \in T} N_a,\ T, \bigcup_{a \in T} P_a, S', f)$ with $f(X) = f_a(X)$, for $X \in N_a$, is a non-erasing string restricted context-free grammar with $L(G') = \bigcup_{a \in T} \{a\}L_a = L$. Thus, it is shown that $\mathcal{L}(CS) \subseteq \mathcal{L}(srCF - \lambda)$. □

Finally, we mention the following immediate corollary of the previous construction.

Corollary 13. *Every context-sensitive language is generated by a string restricted context-free grammar $G = (N, T, P, S, f)$ where $A \to w \in P$ implies that $|w| \leqslant 2$, and for all $A \in N$,*

- *if $f(A) = (-, x)$, then $|x| = 1$, and*
- *if $f(A) = (+, x)$, then $1 \leqslant |x| \leqslant 2$.*

5. Conclusion

In this paper, we have discussed the simplest restriction placed on context-free grammars so that the derivation is controlled by the structure of the sentential form. We have shown the following characterization of these systems based on context-free rules that check for a presence and absence of symbols or strings in the sentential form. If such a restricted context-free grammar can check only for a presence and absence of nonterminal symbols, then the generative power of context-free grammars is increased to the power of van der Walt's random context grammars or, equivalently, to the power of matrix or programmed grammars. However, if such a restricted context-free grammar can check for a presence of nonterminals or strings of nonterminals of length two, and for an absence of only nonterminals, then the generative power of non-erasing context-free grammars is increased to the generative power of monotone grammars.

As a consequence of the definition of restricted context-free grammars and the main results of this paper, two new normal forms for random context grammars and matrix grammars have been discussed. Recall also that the presented normal form for random context grammars is the simplest possible.

Finally, the discussion concerning the generative power of special variants of these grammars, namely permitting and forbidding restricted context-free grammars, is (except for the case of permitting restricted context-free grammars) left as an open problem for the future investigation. Note, however, that from the theoretical point of view, the generative power of these special cases is of particular interest especially in the case of the forbidding variants of (string) restricted context-free grammars.

Acknowledgments

The authors gratefully acknowledge very useful suggestions and comments of the anonymous referees.

References

[1] H. Bordihn, H. Fernau, Accepting grammars and systems: An overview, in: Developments in Language Theory – DLT 1995, pp. 199–208, Technical Report 9/94, Universität Karlsruhe, Fakultät für Informatik, 1994.
[2] J. Dassow, G. Păun, Regulated Rewriting in Formal Language Theory, Springer-Verlag, Berlin, 1989.
[3] S. Ewert, A.P.J. van der Walt, A pumping lemma for random permitting context languages, Theoret. Comput. Sci. 270 (2002), 959–967.
[4] Z. Gazdag, A note on restricted context-free rewriting systems, in: Proceedings of the 13th International Multiconference Information Society – IS, vol. A, Ljubljana, Slovenia, 2010, pp. 405–408.
[5] J.E. Hopcroft, R. Motwani, J.D. Ullman, Introduction to Automata Theory, Languages, and Computation, third ed., Addison–Wesley, Boston, MA, 2006.

[6] T. Masopust, A note on the generative power of some simple variants of context-free grammars regulated by context conditions, in: A.H. Dediu, A.M. Ionescu, C. Martín-Vide (Eds.), LATA 2009 Proceedings, in: Lecture Notes in Comput. Sci., vol. 5457, Springer-Verlag, Berlin, 2009, pp. 554–565.
[7] T. Masopust, Comparison of two context-free rewriting systems with simple context-checking mechanisms, CoRR abs/1004.3635, 2010.
[8] T. Masopust, Simple restriction in context-free rewriting, J. Comput. System Sci. 76 (2010) 837–846.
[9] T. Masopust, A. Meduna, On context-free rewriting with a simple restriction and its computational completeness, Theor. Inform. Appl. 43 (2009) 365–378.
[10] O. Mayer, Some restrictive devices for context-free grammars, Inform. and Control 20 (1972) 69–92.
[11] A. Meduna, M. Švec, Grammars with Context Conditions and Their Applications, John Wiley & Sons, New York, 2005.
[12] M. Penttonen, One-sided and two-sided context in formal grammars, Inform. and Control 25 (1974) 371–392.
[13] G. Păun, A variant of random context grammars: Semi-conditional grammars, Theoret. Comput. Sci. 41 (1985) 1–17.
[14] A.P.J. van der Walt, Random context grammars, in: Proc. IFIP Congress, North-Holland Publ. Co., 1971, pp. 66–68.
[15] A.P.J. van der Walt, S. Ewert, A shrinking lemma for random forbidding context languages, Theoret. Comput. Sci. 237 (2000) 149–158.
[16] G. Zetzsche, On erasing productions in random context grammars, in: S. Abramsky, C. Gavoille, C. Kirchner, F.M. auf der Heide, P.G. Spirakis (Eds.), ICALP (2), in: Lecture Notes in Comput. Sci., vol. 6199, Springer, 2010, pp. 175–186.