Pseudo-consimilarity and Semi-consimilarity of Complex Matrices

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ABSTRACT

It is shown that the notion of consimilarity of n-by-n complex matrices is equivalent to each of the new concepts of pseudo-consimilarity and semi-consimilarity. This equivalence is unlike similarity in that pseudo-similarity and similarity are equivalent, while semi-similarity is a weaker relation than (pseudo-) similarity.

1. INTRODUCTION

In a series of recent papers by Y. P. Hong and R. A. Horn [8–10] and by Y. P. Hong, R. A. Horn, and C. R. Johnson [11], the notion of consimilarity of complex matrices was introduced and analyzed. Let M_n denote the set of n-by-n complex matrices. Two matrices $A, B \in M_n$ are said to be consimilar if there is a nonsingular matrix $S \in M_n$ such that $\bar{S}^{-1}BS = A$. This type of similarity appeared first in the study of semilinear transformations, as initiated by Segre [13] and Jacobson [12]. The problems of contriangularizing or condiagonalizing a given matrix $A \in M_n$, along with the simultaneous condiagonalization of more than one matrix in M_n, were studied in [8]; other simultaneous reduction problems were considered in [9] and [11]. In [10] Hong and Horn gave a canonical form for matrices under consimilarity, and proved that two matrices $A, B \in M_n$ are consimilar if and only if (a) $A\bar{A}$ is

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similar to $B\bar{B}$ and (b) $\mathrm{rank}(A) = \mathrm{rank}(B)$, $\mathrm{rank}(A\bar{A}) = \mathrm{rank}(B\bar{B})$, $\mathrm{rank}(A\bar{A}A) = \mathrm{rank}(B\bar{B}B)$, and so on, for all such alternating products with at most n terms. The notion of consimilarity is generalized by replacing the complex field with an arbitrary field F and replacing the operation of complex conjugation with an automorphism on F.

The purpose of this note is to show, through the use of the above result of Hong and Horn, that the notions of consimilarity, pseudo-consimilarity, and semi-consimilarity are equivalent. The latter two notions are slightly new, being consimilar versions of pseudo-similarity [5] and semi-similarity [6] of complex matrices. We make the definitions as follows:

Two matrices $A, B \in M_n$ are said to be *pseudo-consimilar* if there exist matrices $X, X_1, X_2 \in M_n$ such that

$$\bar{X}_1 AX = B, \qquad \bar{X}BX_2 = A, \qquad XX_1X = X, \qquad XX_2X = X. \tag{1}$$

The matrices X_1 and X_2 are called inner inverses or 1-inverses of X, and could be equal.

Two matrices $A, B \in M_n$ are said to be *semi-consimilar* if there exist matrices $X, Y \in M_n$ such that

$$\bar{Y}AX = B \quad \text{and} \quad \bar{X}BY = A. \tag{2}$$

Note that pseudo-consimilar matrices, as well as semi-consimilar matrices, have the same rank.

We develop some of our ideas (Lemma 2.1 and Theorem 2.2) along the lines of those in [6], and in particular make use of the Drazin [2] inverse of a square matrix. The Drazin inverse A^d of a matrix $A \in M_n$ is the unique solution in M_n to the following equations:

$$A^{k+1}X = A^k \qquad \text{for some} \quad k \geqslant 0, \qquad XAX = X, \quad AX = XA.$$

The smallest k for which a solution exists is called the index of A.

2. RESULTS

We start with the following preliminary results.

LEMMA 2.1. *Let $A, B, X, Y \in M_n$ with*

$$\bar{Y}AX = B \quad \textit{and} \quad \bar{X}BY = A.$$

Then

$$A = (\bar{X}\bar{Y})^k A(XY)^k, \quad B = (\bar{Y}\bar{X})^k B(YX)^k, \qquad k = 0,1,2,\ldots, \tag{3.i}$$

$$(A\bar{A})^k \bar{X} = \bar{X}(B\bar{B})^k, \quad \bar{Y}(A\bar{A})^k = (B\bar{B})^k \bar{Y}, \qquad k = 0,1,2,\ldots \tag{3.ii}$$

$$A(XY)(XY)^d = A = (\bar{X}\bar{Y})(\bar{X}\bar{Y})^d A,$$

$$B(YX)(YX)^d = B = (\bar{Y}\bar{X})(\bar{Y}\bar{X})^d B. \tag{3.iii}$$

Proof. We prove the first result in each part; the others follow by symmetry.

(3.i): First

$$A = \bar{X}BY = \bar{X}(\bar{Y}AX)Y = (\bar{X}\bar{Y})A(XY),$$

which may be applied repeatedly to obtain the result.

(3.ii): From (3.i), $B = \bar{Y}\bar{X}BYX$. So $\bar{B} = YX\bar{B}\bar{Y}\bar{X}$. Hence, $A\bar{A}\bar{X} = (\bar{X}BY)(X\bar{B}\bar{Y})\bar{X} = \bar{X}B(YX\bar{B}\bar{Y}\bar{X}) = \bar{X}B\bar{B}$, that is, $(A\bar{A})\bar{X} = \bar{X}(B\bar{B})$. So

$$(A\bar{A})^2\bar{X} = (A\bar{A})\bar{X}(B\bar{B}) = \bar{X}(B\bar{B})^2,$$

and then we use induction.

(3.iii): Let the index of XY be t, so that t is also the index of $\bar{X}\bar{Y}$. Now, using (i),

$$(\bar{X}\bar{Y})^d A(XY)^d = (\bar{X}\bar{Y})^d\left[(\bar{X}\bar{Y})^{t+1}A(XY)^{t+1}\right](XY)^d$$

$$= (\bar{X}\bar{Y})^t A(XY)^t = A.$$

Then

$$(\bar{X}\bar{Y})(\bar{X}\bar{Y})^d A = (\bar{X}\bar{Y})^d(\bar{X}\bar{Y})A$$

$$= (\bar{X}\bar{Y})^d(\bar{X}\bar{Y})\left[(\bar{X}\bar{Y})^d A(XY)^d\right]$$

$$= (\bar{X}\bar{Y})^d A(XY)^d$$

$$= A.$$

Similarly,

$$\begin{aligned} A(XY)(XY)^d &= \left[(\overline{X}\overline{Y})^d A(XY)^d\right](XY)(XY)^d \\ &= (\overline{X}\overline{Y})^d A(XY)^d \\ &= A. \end{aligned}$$

■

Note that in (2) semi-consimilarity does not require $Y = X^{-1}$. However, as seen in (3.iii), the equations in (2) guarantee that

$$(\overline{X}\overline{Y})(\overline{X}\overline{Y})^d A = A,$$

so that the range of A is contained in the range of the projection $(\overline{X}\overline{Y})(\overline{X}\overline{Y})^d$. Similarly for the other equations in (3.iii). These key ideas will be used subsequently in Theorem 2.2 to show that A and B being semi-consimilar implies $A\overline{A}$ is similar to $B\overline{B}$.

Recall [5] that two matrices $A, B \in M_n$ are pseudo-similar if there exist $X, X_1, X_2 \in M_n$ such that

$$X_1 A X = B, \qquad XBX_2 = A, \qquad XX_1X = X, \qquad XX_2X = X. \tag{4}$$

For matrices over a principal-ideal domain (as well as for matrices over a more general type of ring), the complex number field in particular, it is known [4] that pseudo-similarity is equivalent to similarity. Also see [3].

THEOREM 2.2. *Let $A, B, X, Y \in M_n$ with*

$$\overline{Y}AX = B \quad \textit{and} \quad \overline{X}BY = A.$$

Then

(i) *$A\overline{A}$ is similar to $B\overline{B}$, and*
(ii) $\operatorname{rank}[(A\overline{A})^k A] = \operatorname{rank}[(B\overline{B})^k B]$, $k = 0, 1, 2, \ldots$.

Proof. (i): First note that from (3.iii) we have $\overline{B} = \overline{B}(\overline{Y}\overline{X})(\overline{Y}\overline{X})^d$. Now, from (3.ii), $\overline{Y}(A\overline{A})(\overline{X}\overline{Y})^d\overline{X} = (B\overline{B})\overline{Y}(\overline{X}\overline{Y})^d\overline{X}$. Cline and Greville [1] show $(\overline{X}\overline{Y})^d = \overline{X}[(\overline{Y}\overline{X})^d]^2\overline{Y}$, so that

$$(\overline{X}\overline{Y})^d\overline{X} = \overline{X}\left[(\overline{Y}\overline{X})^d\right]^2\overline{Y}\overline{X} = \overline{X}(\overline{Y}\overline{X})^d$$

and

$$\bar{Y}(A\bar{A})(\bar{X}\bar{Y})^d\bar{X} = (B\bar{B})\bar{Y}\bar{X}(\bar{Y}\bar{X})^d = B\left[\bar{B}(\bar{Y}\bar{X})(\bar{Y}\bar{X})^d\right] = B\bar{B}. \qquad (5)$$

Also, from (3.ii) and (3.iii) respectively,

$$(\bar{X}\bar{Y})^d\bar{X}(B\bar{B})\bar{Y} = (\bar{X}\bar{Y})^d\bar{X}\bar{Y}(A\bar{A}) = A\bar{A}. \qquad (6)$$

Next observe that $[(\bar{X}\bar{Y})^d\bar{X}]\bar{Y}[(\bar{X}\bar{Y})^d\bar{X}] = (\bar{X}\bar{Y})^d\bar{X}$, so that $\bar{Y}$ is a 1-inverse of $(\bar{X}\bar{Y})^d\bar{X}$. Then, with (5)–(6), we have that $A\bar{A}$ and $B\bar{B}$ are pseudo-similar [here we are taking $X_1 = X_2$ in (4)]. Thus, $A\bar{A}$ and $B\bar{B}$ are similar.

(ii): Using (3.ii), we have

$$\bar{Y}(A\bar{A})^k AX = (B\bar{B})^k\bar{Y}AX = (B\bar{B})^k B$$

and

$$\bar{X}(B\bar{B})^k BY = (A\bar{A})^k\bar{X}BY = (A\bar{A})^k A.$$

Hence we see that $(A\bar{A})^kA$ and $(B\bar{B})^kB$ are semi-consimilar and the rank result follows. ■

We next use some techniques in [4] to prove part of our equivalences.

THEOREM 2.3. *Let $A, B \in M_n$. Then A is consimilar to B if and only if A is pseudo-consimilar to B.*

Proof. Assume A is consimilar to B, so that $\bar{S}^{-1}BS = A$ for some nonsingular matrix $S \in M_n$. Then $\bar{S}AS^{-1} = B$, $\overline{S^{-1}}BS = A$, $S^{-1}SS^{-1} = S^{-1}$, so that A is pseudo-consimilar to B.

Conversely, assume that A is pseudo-consimilar to B, so that the equations in (1) hold for some $X, X_1, X_2 \in M_n$. We proceed similarly to the proof, given in Section III of [4], that pseudo-similarity implies similarity.

Let $\operatorname{rank}(X) = r$. Then there exist nonsingular matrices $P, Q \in M_n$ and conformable complex matrices L, Z_2, Z_3, Z_4 such that

$$PXQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_1 = Q\begin{pmatrix} I_r & 0 \\ 0 & L \end{pmatrix}P, \qquad X_2 = Q\begin{pmatrix} I_r & Z_2 \\ Z_3 & Z_4 \end{pmatrix}P.$$

Also, write

$$A = \bar{P}^{-1}\begin{pmatrix} W & W_2 \\ W_3 & W_4 \end{pmatrix} P$$

for some matrices W, W_2, W_3, W_4, where W is r-by-r. Now $\bar{X}_1 AX = B$ gives

$$\begin{aligned} B &= \bar{Q}\begin{pmatrix} I_r & 0 \\ 0 & \bar{L} \end{pmatrix}\begin{pmatrix} W & W_2 \\ W_3 & W_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} \\ &= \bar{Q}\begin{pmatrix} W & 0 \\ \bar{L}W_3 & 0 \end{pmatrix} Q^{-1}, \end{aligned}$$

and then $\bar{X}BX_2 = A$ yields

$$A = \bar{P}^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W & 0 \\ \bar{L}W_3 & 0 \end{pmatrix}\begin{pmatrix} I & Z_2 \\ Z_3 & Z_4 \end{pmatrix} P = \bar{P}^{-1}\begin{pmatrix} W & WZ_2 \\ 0 & 0 \end{pmatrix} P.$$

So $W_3 = 0$, and hence

$$B = \bar{Q}\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}.$$

Finally, letting

$$T = P^{-1}\begin{pmatrix} I_r & -Z_2 \\ 0 & I_{n-r} \end{pmatrix} Q^{-1},$$

we have

$$\bar{T}^{-1} = \bar{Q}\begin{pmatrix} I_r & \bar{Z}_2 \\ 0 & I_{n-r} \end{pmatrix} \bar{P},$$

and it is easy to check by direct multiplication that $\bar{T}^{-1}AT = B$. Thus, A is consimilar to B. ∎

By using a theorem of Hong and Horn [10] we can now go full circle with our equivalences, and give our major result.

THEOREM 2.4. *The following statements are equivalent:*

(i) A *is consimilar to* B.
(ii) A *is pseudo-consimilar to* B.

(iii) *A is semi-consimilar to B.*
(iv) *$A\bar{A}$ is similar to $B\bar{B}$ and*

$$\operatorname{rank}\left[(A\bar{A})^k A\right] = \operatorname{rank}\left[(B\bar{B})^k B\right], \qquad k = 0,1,2,\dots.$$

Proof. (i) if and only if (ii): Theorem 2.3.
(i) implies (iii): Clear.
(iii) implies (iv): Theorem 2.2.
(iv) implies (i): This result is contained in [10]; also see the doctoral dissertation of Hong [7]. ∎

3. REMARKS AND CONCLUSIONS

Similarity and consimilarity are two special cases of the transformation $f(A) = (S^{-1})\hat{}\ AS$, where $(\cdot)\hat{}$ is *any* involutory automorphism on M_n, i.e. any map $(\cdot)\hat{}$ on M_n that satisfies $(A+B)\hat{} = A\hat{} + B\hat{}$, $(AB)\hat{} = A\hat{}\ B\hat{}$, and $A\hat{}\hat{} = A$. It is convenient to call such a map a "conjugation." The particular choices of $A\hat{} = A$ (the identity) and $A\hat{} = \bar{A}$ (complex conjugation) lead to similarity and consimilarity, as studied in this paper. Likewise for pseudo- (semi-) similarity and pseudo- (semi-) consimilarity. These are not the only conjugations that can be considered on M_n. A more general class of conjugations is given by $A\hat{} = L^{-1}\bar{A}L$, where $\bar{L} = L^{-1}$ and $\overline{(\cdot)}$ is either the identity map or complex conjugation on M_n.

For the identity conjugation it is known (for matrices over any principal-ideal domain) that similarity and pseudo-similarity are equivalent, but that semi-similarity [defined by removing the conjugation symbol in (2)] is a weaker relation. For example, $-A$ is semi-similar to A via $-I$ and I, while $-A$ and A need not be similar in general. On the other hand, Theorem 2.4 shows that for the complex conjugation, consimilarity and semi-consimilarity are equivalent. For example, $e^{i\alpha}A$, α real, is not only semi-consimilar to A via $e^{i\alpha}I$ and I, but also consimilar to A via $e^{-i\alpha/2}I$.

Theorem 2.4 may at first seem somewhat surprising. However, it merely underscores the fact that the identity conjugation and the complex conjugation differ considerably. For the "stronger" identity conjugation, part (iv) of Theorem 2.4 does not imply part (i), yet for the "weaker" complex conjugation there are enough distinct elements of the form $\bar{c}c^{-1}$, ensuring (iv) implies (i). It will be shown elsewhere that this increase in "entropy" allows one to construct more invertible matrix pencils of the form $cA + \bar{c}B$, which in turn can be used to give a shorter, more general proof of Theorem 2.4.

For complex conjugation, consimilarity is not only weaker than similarity, but also less discriminating. It was shown in [8] and [9] that if λ is a coneigenvalue of A, then so is $\lambda e^{i\alpha}$ for all real α. One interpretation of Theorem 2.4 is that semi-consimilarity does not involve a further weakening of consimilarity.

More generally, let $\overline{(\cdot)}$ be a conjugation on a field F and let $(\cdot)^{\wedge}$ be a conjugation on F_n which collapses to $\overline{(\cdot)}$ on F. If there are $n+1$ distinct elements $\bar{c}_k c_k^{-1}$ in F, then it can be shown that $AA^{\wedge}$ similar to $BB^{\wedge}$ and $\operatorname{rank}[(AA^{\wedge})^k A] = \operatorname{rank}[(BB^{\wedge})^k B]$ for all k imply that A is $(\cdot)^{\wedge}$-similar to B, that is, there is a nonsingular X in F_n such that $X^{-1}AX^{\wedge} = B$.

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