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Numerical ranges as circular discs[☆]

Pei Yuan Wu

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan

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ABSTRACT

We prove that if a finite matrix A of the form $\begin{bmatrix} aI & B \\ 0 & C \end{bmatrix}$ is such that its numerical range $W(A)$ is a circular disc centered at a, then a must be an eigenvalue of C. As consequences, we obtain, for any finite matrix A, that (a) if $\partial W(A)$ contains a circular arc, then the center of this circle is an eigenvalue of A with its geometric multiplicity strictly less than its algebraic multiplicity, and (b) if A is similar to a normal matrix, then $\partial W(A)$ contains no circular arc.

For an n-by-n complex matrix A, its *numerical range* $W(A)$ is, by definition, the subset $\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the plane, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$, respectively. It is known that $W(A)$ is a nonempty compact convex subset of $\mathbb{C}$. For its other properties, the reader may consult [1, Chapter 1].

A prominent result as regards the numerical range is the one obtained by Anderson in the early 1970s.

Anderson's Theorem. *If A is an n-by-n matrix with $W(A)$ contained in a closed circular disc D such that $\partial W(A) \cap \partial D$ has more than n points, then $W(A) = D$ and the center of D is an eigenvalue of A with algebraic multiplicity at least 2.*

His proof, never published, is based on Kippenhahn's result ($W(A)$ is the convex hull of the real points (x, y) satisfying $q(x, y, 1) = 0$, where $q(x, y, z) = 0$ is the dual, in the projective plane, of the curve $\det(x \operatorname{Re} A + y \operatorname{Im} A + zI_n) = 0$, $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/(2i)$ being the real and imaginary parts of A) and Bézout's theorem (if two projective curves $p(x, y, z) = 0$ and $q(x, y, z) = 0$ of degrees m and n, respectively, intersect at more than mn points, then p and q have a common factor). For other related results, see [2, Theorem], [3, Theorem 1], [4] and [5, Theorem 4.12 and Corollary 4.4]. For the first assertion of Anderson's theorem, the author discovered (in [6, Lemma 6]) another proof by using the Riesz–Fejér theorem on nonnegative trigonometric polynomials and the fundamental theorem of algebra. The purpose of this note is to utilize this approach to prove a generalization of Anderson's theorem. The following is our main result.

Theorem 1. *If A is an n-by-n matrix of the form*

$$\begin{bmatrix} aI_m & B \\ 0 & C \end{bmatrix}$$

($0 \le m < n$) such that $W(A)$ is contained in the closed circular disc D centered at a and $\partial W(A) \cap \partial D$ has more than $n - m$ points, then $W(A) = D$ and a is an eigenvalue of C.

To prove this theorem, we need the following lemma. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc.

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E-mail address: pywu@math.nctu.edu.tw.

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Lemma 2. *Let A be an n-by-n matrix. Then the following hold:*

(a) $W(A) \subseteq \overline{\mathbb{D}}$ *if and only if* $\mathrm{Re}\,(e^{i\theta}A) \le I_n$ *for all real* θ.
(b) *Assume that* $W(A) \subseteq \overline{\mathbb{D}}$ *and* θ_0 *is some real number. Then* $e^{i\theta_0}$ *is in* $W(A)$ *if and only if* 1 *is an eigenvalue of* $\mathrm{Re}\,(e^{i\theta_0}A)$.

These assertions are easy consequences of the fact that the numerical range $W(B)$ of a Hermitian matrix B is the closed interval $[a, b]$ of the real line, where a (resp., b) is the smallest (resp., largest) eigenvalue of B.

Proof of Theorem 1. Assume that D has radius r. Replacing A by $(A - aI_n)/r$ and D by $(D - a)/r$, we may assume, without loss of generality, that $a = 0$ and $D = \overline{\mathbb{D}}$. Since $W(A) \subseteq \overline{\mathbb{D}}$, we have $\mathrm{Re}\,(e^{i\theta}A) \le I_n$ for all real θ by Lemma 2(a) and thus $p(e^{i\theta}) \equiv \det(I_n - \mathrm{Re}\,(e^{i\theta}A)) \ge 0$ for all θ. On the other hand, that $\partial W(A) \cap \partial\mathbb{D}$ contains more than $n - m$ points implies, by Lemma 2(b), the existence of distinct $\theta_1, \ldots, \theta_{n-m+1}$ in $[0, 2\pi)$ such that $p(e^{i\theta_j}) = 0$ for all j, $1 \le j \le n - m + 1$. Note that

$$\begin{aligned} p(e^{i\theta}) &= \det\begin{bmatrix} I_m & -e^{i\theta}B/2 \\ -e^{-i\theta}B^*/2 & I_{n-m} - \mathrm{Re}(e^{i\theta}C) \end{bmatrix} \\ &= \det\left(I_{n-m} - \mathrm{Re}\,(e^{i\theta}C) - \frac{1}{4}(-e^{-i\theta}B^*)(-e^{i\theta}B)\right) \\ &= \det\left(I_{n-m} - \mathrm{Re}\,(e^{i\theta}C) - \frac{1}{4}B^*B\right). \end{aligned}$$

We may assume that $C = [c_{jk}]_{j,k=1}^{n-m}$ is upper triangular ($c_{jk} = 0$ for $j > k$) and $B^*B/4 = [b_{jk}]_{j,k=1}^{n-m}$. Then $I_{n-m} - \mathrm{Re}\,(e^{i\theta}C) - (B^*B/4) = [d_{jk}]_{j,k=1}^{n-m}$, where

$$d_{jk} = \begin{cases} 1 - \mathrm{Re}(e^{i\theta}c_{jj}) - b_{jj} & \text{if } j = k, \\ -(e^{i\theta}c_{jk}/2) - b_{jk} & \text{if } j < k, \\ -(e^{-i\theta}\overline{c_{kj}}/2) - b_{jk} & \text{if } j > k. \end{cases}$$

Hence $p(e^{i\theta})$ is a trigonometric polynomial of the form $\sum_{l=-(n-m)}^{n-m} u_l e^{il\theta}$ with $u_{n-m} = (-1)^{n-m}c_{11}\cdots c_{n-m,n-m}/2^{n-m}$ and $\overline{u_{-l}} = u_l$ for all l. Since $p(e^{i\theta}) \ge 0$ for all θ, the classical Riesz–Fejér theorem implies that $p(e^{i\theta}) = |q(e^{i\theta})|^2$ for some polynomial q of degree at most $n-m$ (cf. [7, p. 77, Problem 40]). On the other hand, $p(e^{i\theta}) = 0$ for $\theta = \theta_j$, $1 \le j \le n-m+1$, yields the same for $q(e^{i\theta})$. Applying the fundamental theorem of algebra to q, we obtain that $p(e^{i\theta}) = |q(e^{i\theta})|^2 = 0$ for all θ. Thus 1 is an eigenvalue of $\mathrm{Re}\,(e^{i\theta}A)$ for all θ and therefore $W(A) = \overline{\mathbb{D}}$ by Lemma 2(b). From $p \equiv 0$, we have $u_{n-m} = 0$ and hence $c_{jj} = 0$ for some j, $1 \le j \le n - m$. This shows that 0 is an eigenvalue of C as asserted. □

Note that in the preceding theorem, the case $m = 0$ corresponds to Anderson's theorem. Also note that if we assume that $D \subseteq W(A)$ instead of $W(A) \subseteq D$, then $W(A)$ may not equal D even in the context of Anderson's theorem. This is seen from the n-by-n ($n \ge 3$) matrix $A = [1] \oplus J_{n-1}$, where J_{n-1} is the $(n-1)$-by-$(n-1)$ Jordan block

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

since in this case $W(A)$ equals the convex hull of the point 1 and the circular disc $\{z \in \mathbb{C} : |z| \le \cos(\pi/n)\}$ (cf. [8, Proposition 1]). Using arguments analogous to those in the proof of the preceding theorem but without invoking the Riesz–Fejér theorem, we can easily show the following corollary, whose proof we omit.

Corollary 3. *Let A be an n-by-n matrix of the form*

$$\begin{bmatrix} aI_m & B \\ 0 & C \end{bmatrix}$$

($0 \le m < n$*) and D be a closed circular disc centered at a. If* $\partial W(A) \cap \partial D$ *contains more than* $2(n - m)$ *points, then a is an eigenvalue of* C.

The next corollary gives spectral information on the center of a circular numerical range. It generalizes [5, Corollary 4.4].

Corollary 4. *If A is an n-by-n matrix such that* $\partial W(A)$ *contains more than* $2n$ *points of a circle centered at a, then a is an eigenvalue of A with its geometric multiplicity strictly less than its algebraic multiplicity. In this case, the number* $2n$ *is sharp.*

Proof. That a is an eigenvalue of A follows from case $m = 0$ of Corollary 3. Let n_1 (resp., n_2) be the geometric (resp., algebraic) multiplicity of a. We obviously have $n_1 \leq n_2$. If $n_1 = n_2 \equiv m$, then A is unitarily equivalent to a matrix of the form $\begin{bmatrix} aI_m & B \\ 0 & C \end{bmatrix}$ on $\mathbb{C}^n = \ker(A - aI_n) \oplus (\ker(A - aI_n))^{\perp}$ with $C - aI_{n-m}$ one-to-one. However, Corollary 3 implies that a is an eigenvalue of C. This is a contradiction. Thus $n_1 < n_2$ as asserted.

The sharpness of $2n$ is seen from the n-by-n diagonal matrix $A = \mathrm{diag}(1+\epsilon, \omega_n(1+\epsilon), \omega_n^2(1+\epsilon), \ldots, \omega_n^{n-1}(1+\epsilon))$, where $\omega_n = e^{2\pi i/n}$ and $\epsilon > 0$ is sufficiently small. In this case, $W(A)$ is the regular n-gonal region with vertices $\omega_n^j(1+\epsilon)$, $0 \leq j \leq n-1$, and $\partial W(A) \cap \partial \mathbb{D}$ containing exactly $2n$ points. □

For a matrix similar to a normal one, we can say slightly more of its numerical range.

Corollary 5. *Let A be an n-by-n matrix which is similar to a normal one. Then the following hold:*

(a) *$\partial W(A)$ contains no circular arc.*
(b) *If $W(A) \subseteq D$ or $D \subseteq W(A)$, where D is a closed circular disc, then $\partial W(A) \cap \partial D$ contains at most n points. In this case, the number n is sharp.*

Proof. (a) This follows from Corollary 4 since under our assumption the geometric and algebraic multiplicities of every eigenvalue of A are equal to each other.

(b) If $W(A) \subseteq D$, then the assertion regarding $\partial W(A) \cap \partial D$ is a consequence of Theorem 1 (or Anderson's theorem) and (a). Next assume that $D \subseteq W(A)$ and $\partial W(A) \cap \partial D$ has more than n points. If $n = 2$, then $W(A)$ is an elliptic disc. Our assumptions imply, via Anderson's theorem, by interchanging the roles of $W(A)$ and D, that $W(A)$ and D coincide. Hence both eigenvalues of A are a, the center of D. It follows that A is similar to the scalar matrix aI_2 and, therefore, is equal to aI_2 itself. Thus $W(A) = \{a\}$, which contradicts $W(A) = D$. On the other hand, if $n \geq 3$, then [4, Theorem 2.5 (b)] implies that $\partial W(A)$ contains an arc of ∂D. This contradicts (a). This shows that in any case $\partial W(A) \cap \partial D$ contains at most n points.

The sharpness of n is seen from the n-by-n diagonal matrix $A = \mathrm{diag}(1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1})$, where $\omega_n = e^{2\pi i/n}$, and $D = \overline{\mathbb{D}}$ (for $W(A) \subseteq D$) or $D = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$ (for $D \subseteq W(A)$). □

Corollary 6. *If A is the product of two positive semidefinite matrices, then $\partial W(A)$ contains no circular arc.*

Proof. Our assumption on A implies that it is similar to a positive semidefinite matrix (cf. [9, Theorem 2.2]). The assertion then follows from Corollary 5(a). □

We conclude this note by remarking that most of the results here are no longer valid for operators on infinite-dimensional spaces. As an example, if A is the bilateral weighted shift with weights $\ldots, 1, 1, \underline{w}, 1, 1, \ldots$, where $|w| > 1$:

$$A(\ldots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \ldots) = (\ldots, x_{-2}, \underline{x_{-1}}, wx_0, x_1, \ldots)$$

on $l^2(\mathbb{Z})$ (both the 0th weight and the 0th component of a vector are underlined), then it can be shown that $W(A) = \{z \in \mathbb{C} : |z| \leq (|w|^2 + 1)/(2|w|)\}$ by using the method developed in [10, Section III, p. 500] (cf. also [11, Theorem 4.9 (b)] for an alternative proof). In this case, A is similar to the (simple) bilateral shift (the bilateral weighted shift with weights all equal to 1) by [12, Theorem 2 (a)], and hence has its spectrum $\sigma(A)$ equal to $\partial \mathbb{D}$. In particular, 0, the center of $W(A)$, is not in $\sigma(A)$.

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