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Combinatorial Properties of Ring Generated Circular Planar Nearrings

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1. INTRODUCTION

While investigating geometric and combinatorial properties of some incidence structures from planar nearrings, Clay introduced the notion of circularity in [2].

An example of a circular planar nearring is the planar nearring of the complex plane $\mathbf{C}$ with a new multiplication defined by

$$a * b = \begin{cases} 0, & \text{if } a = 0; \\ (a/|a|)\,b, & \text{if } a \neq 0, \end{cases}$$

for all $a, b \in \mathbf{C}$. The incidence structure obtained from this planar nearring is $(\mathbf{C}, \mathscr{B}^*_T, \in)$, where T is the unit circle and $\mathscr{B}^*_T$ is the set of all circles in the complex plane.

To initialize the study, Clay chooses the family of circles in $\mathscr{B}^*_T$ with a fixed radius r, $r \neq 0$, and then partitions this family into equivalence classes $E^r_c = \{Tr + b \mid b \in Tc\}$, where $c \neq 0$. Each E^r_c is the family of circles with radius r and centers on the circle Tc. Then a graph is assigned to each E^r_c in order to understand the behavior of E^r_c (cf. [4; §6]). This idea has been proven to be very useful.

In this work, we continue the study of these E^r_c's for circular planar nearrings constructed from a ring using a cyclic subgroup of order k of the

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unit group. We begin with some basic concepts concerning circular planar nearrings and graph theory in the next section. Then we describe the connection between an E_c^r and its graph. It turns out that each graph of an E_c^r can be decomposed into a union of some basic graphs. Moreover, the total number of basic graphs occurring depends on k alone. The last section is devoted to an interesting phenomenon for field generated planar nearrings. Some basic graphs always occur together as subgraphs of some E_c^r regardless of the underlying field. This behavior is not well understood yet.

An application of our results to the number of solutions of certain equations over a finite field can be found in [9, 10].

2. Preliminaries

For previous works on circular planar nearrings, the reader is referred to [2], [3], [5], [7], [11] and [12]. To be self-contained, we review a minimum of necessary concepts.

Define an equivalence relation $=_m$ on a (left) nearring $(N, +, \cdot)$ by $a =_m b$ if $ax = bx$ for all $x \in N$. Then N is planar if (1) the equation $ax = bx + c$ has a unique solution for x if $a, b \in N$ and $a \neq_m b$, and (2) $|N/=_m| \geqslant 3$. Each planar nearring $(N, +, \cdot)$ can be constructed from a Ferrero pair (N, Φ), where Φ is a regular group of automorphisms of $(N, +)$, and vice versa [1]. Thus, every mapping $\varphi \in \Phi$ is fixed point free and $-1 + \varphi$ is surjective. If $(N, +, \cdot)$ is a planar nearring and (N, Φ) is the corresponding Ferrero pair, then $(N, \mathscr{B}_\Phi^*, \in)$ is an incidence structure, where

$$\mathscr{B}_\Phi^* = \{N^*a + b \mid a, b \in N, a \neq 0\} = \{\Phi(a) + b \mid a, b \in N, a \neq 0\},$$

with $N^* = \{n \in N \mid n \neq_m 0\}$, and $\Phi(a) = \{\varphi(a) \mid \varphi \in \Phi\}$. When $|N| < \infty$, $(N, \mathscr{B}_\Phi^*, \in)$ is a BIBD [2].

A planar nearring $(N, +, \cdot)$, its corresponding Ferrero pair (N, Φ), and the incidence structure $(N, \mathscr{B}_\Phi^*, \in)$, are said to be *circular* if every three distinct points of N belong to at most one block $N^*a + b$. If furthermore, $|N| < \infty$, $(N, \mathscr{B}_\Phi^*, \in)$ is a circular BIBD. In case of circular structures we sometimes call a block $N^*a + b$ *circle* and refer to a as the radius and to b as the center of that circle.

We recall one way of obtaining Ferrero pairs from rings. Let $(R, +, \cdot)$ be a ring with the group of units $\mathscr{U}(R)$. If $\mathscr{U}(R)$ has a subgroup Φ with the property that for each $u \in \Phi \backslash \{1\}$, $-u + 1 \in \mathscr{U}(R)$, then $\bar{\Phi} = \{\bar{u} \mid u \in \Phi\}$, where $\bar{u}(x) = ux$ for all $x \in R$, is a regular group of automorphisms of $(R, +)$. Therefore, $(R, \bar{\Phi})$ is a Ferrero pair. One may identify Φ with $\bar{\Phi}$, and say that (R, Φ) is a Ferrero pair. A planar nearring constructed from (R, Φ) is referred to as a *ring generated* planar nearring.

In the special case where R is a field, we have $\mathscr{U}(R) = R^* = R\backslash\{0\}$, and the condition $-u+1 \in \mathscr{U}(R)$ is always fulfilled. So every subgroup of R^* gives a Ferrero pair. The corresponding planar nearring is then called *field generated.*

Now, we borrow some terminology from graph theory (cf. [6], [13]). For a (undirected) graph Δ, we will use $V(\Delta)$ and $E(\Delta)$ to denote the vertex set and the edge set of Δ, respectively. We say that Δ is *null* if $E(\Delta) = \emptyset$. If Δ_1 is a *subgraph* of Δ_2, i.e., $V(\Delta_1) \subseteq V(\Delta_2)$ and $E(\Delta_1) \subseteq E(\Delta_2)$, then $\Delta_1 \prec \Delta_2$ will be used to indicate the situation. If $\Delta_1 \prec \Delta_2$ and $V(\Delta_1) = V(\Delta_2)$, then Δ_1 is said to be a *spanning subgraph* of Δ_2.

The *degree* of a vertex u in a graph is the number of edges incident with u. A graph with the property that every vertex has degree n, is called a *regular graph of degree n.* A *path* in a graph is a nonempty alternating sequence of vertices edges, beginning and ending with vertices, in which each edge is preceded by one of its vertices and followed by the other. If $v_1, v_1v_2, v_2, ..., v_{s-1}, v_{s-1}v_s, v_s$ is a path, we shall denote it by $v_1v_2 ... v_{s-1}v_s$. A path $v_1v_2 ... v_{s-1}v_s$ is *closed* if $v_s = v_1$. A *cycle* is a closed path $v_1v_2 ... v_sv_1$, $s \geqslant 3$, such that the s vertices $v_1, v_2, ..., v_s$ are all distinct.

For two graphs Δ_1 and Δ_2 with disjoint vertex sets and disjoint edge sets, $\Delta_1 \cup \Delta_2$ is the graph with vertex set $V(\Delta_1) \cup V(\Delta_2)$ and edge set $E(\Delta_1) \cup E(\Delta_2)$, and is called the *disjoint union* of Δ_1 and Δ_2. On the other hand, if Δ_1 and Δ_2 are two graphs with the *same* vertex set, then $\Delta_1 \vee \Delta_2$, called the *union* of Δ_1 and Δ_2, is the graph with vertex set $V(\Delta_1) = V(\Delta_2)$ and edge set $E(\Delta_1) \cup E(\Delta_2)$.

The *cyclic graph on n vertices*, denoted by C_n, is the graph having a cycle containing each vertex and each edge. We call a graph *complete* if every two vertices in the graph are connected by an edge. The complete graph with n vertices is denoted by K_n.

3. Basic Graphs

Although some of the following discussion can be made on any kind of planar nearrings, *we consider, throughout the rest of this paper, a ring* $(N, +, \cdot)$, *not necessarily commutative, and a cyclic subgroup* $\Phi = \langle \varphi \rangle$ *of* $\mathscr{U}(N)$ *of finite order k, such that* (N, Φ) *is a circular Ferrero pair.*

For $r, c \in N\backslash\{0\}$, define

$$E_c^r = \{\Phi r + b \mid b \in \Phi c\}.$$

One may visualize E_c^r as the set of circles with radius r and centers on the circle Φc. To describe E_c^r, we assign to it a graph $\Gamma(E_c^r)$ whose vertex set is Φc, and whose edge set is

$$\{c_1c_2 \mid c_1, c_2 \in \Phi c, c_1 \neq c_2, \text{and } (\Phi r + c_1) \cap (\Phi r + c_2) \neq \emptyset\}.$$

Since N is circular, we have $|(\Phi r+b_1)\cap(\Phi r+b_2)|\leqslant 2$ for any $b_1, b_2\in\Phi c$ with $b_1\neq b_2$. To impose this fact on the graph $\Gamma(E_c^r)$, we decompose $E(\Gamma(E_c^r))$ into a disjoint union of two subsets $E_1(\Gamma(E_c^r))$ and $E_2(\Gamma(E_c^r))$:

$$E_1(\Gamma(E_c^r))=\{c_1c_2\in E(\Gamma(E_c^r)) \mid |(\Phi r+c_1)\cap(\Phi r+c_2)|=1\},$$

$$E_2(\Gamma(E_c^r))=\{c_1c_2\in E(\Gamma(E_c^r)) \mid |(\Phi r+c_1)\cap(\Phi r+c_2)|=2\}.$$

We say that an edge of $\Gamma(E_c^r)$ is *odd* if it is in $E_1(\Gamma(E_c^r))$, *even* if it is in $E_2(\Gamma(E_c^r))$.

Next, we define for each E_c^r, $r, c\in N\backslash\{0\}$, a sequence of $k-1$ entries of 0, 1 and 2. Let $\varphi\in\Phi$ be a generator of Φ. Let $r, c\in N\backslash\{0\}$. For $l\in\{1, 2, ..., k\}$, the sequence $s_l(r, c)=(i_1i_2\ldots i_{k-1})$ is defined by

$$i_j=|(\Phi r+\varphi^l c)\cap(\Phi r+\varphi^{l+j}c)|, \tag{3.1}$$

where $j=1, 2, ..., k-1$. Therefore, each i_j of $s_l(r, c)$, $1\leqslant j\leqslant k-1$, is either 0, 1, or 2.

(3.1) LEMMA. *If r, c and $s_l(r, c)$ are given as above, then*

(1) *$i_j=i_{k-j}$ for $j\in\{1, 2, ..., k-1\}$;*

(2) *if $l'\in\{1, 2, ..., k-1\}$, then $s_{l'}(r, c)=s_l(r, c)$.*

Proof. From the definition of $s_l(r, c)$, we have

$$\begin{aligned}
i_j&=|(\Phi r+\varphi^l c)\cap(\Phi r+\varphi^{l+j}c)|\\
&=|(\Phi r+\varphi^j\varphi^{-j}\varphi^l c)\cap(\Phi r+\varphi^j\varphi^{-j}\varphi^{l+j}c)|\\
&=|\varphi^j(\Phi r+\varphi^{l-j}c)\cap\varphi^j(\Phi r+\varphi^l c)|\\
&=|\varphi^j((\Phi r+\varphi^{l+k-j}c)\cap(\Phi r+\varphi^l c))|\\
&=|(\Phi r+\varphi^{l+k-j}c)\cap(\Phi r+\varphi^l c)|\\
&=i_{k-j},
\end{aligned}$$

where the last equality is from Eq. (3.1). This proves (1).

Let $s_{l'}(r, c)=(i'_1i'_2\ldots i'_{k-1})$. For any j, $1\leqslant j\leqslant k-1$, we have

$$\begin{aligned}
i_j&=|(\Phi r+\varphi^l c)\cap(\Phi r+\varphi^{l+j}c)|\\
&=|(\Phi r+\varphi^{l-l'}\varphi^{l'-l}\varphi^l c)\cap(\Phi r+\varphi^{l-l'}\varphi^{l'-l}\varphi^{l+j}c)|\\
&=|\varphi^{l-l'}(\Phi r+\varphi^{l'}c)\cap\varphi^{l-l'}(\Phi r+\varphi^{l'+j}c)|
\end{aligned}$$

$$= |\varphi^{l-l'}((\Phi r + \varphi^{l'} c) \cap (\Phi r + \varphi^{l'+j} c))|$$

$$= |(\Phi r + \varphi^{l'} c) \cap (\Phi r + \varphi^{l'+j} c)|$$

$$= i'_j.$$

Therefore, $s_l(r, c) = s_{l'}(r, c)$. This proves (2). ∎

The lemma justifies that we define $s(r, c) = s_1(r, c)$ to be the sequence corresponding to E^r_c. It also shows that $s(r, c)$ does not depend on the choice of the generator φ.

The following lemma gives a connection between a sequence $s(r, c)$ and the edge set of the graph of $\Gamma(E^r_c)$.

(3.2) LEMMA. *Let $E^r_c \subseteq \mathscr{B}^*_\Phi$ in $(N, \mathscr{B}^*_\Phi, \in)$ and let $v_j = \varphi^j c$ for each $j \in \{0, 1, ..., k-1\}$, where φ is a generator of Φ. If $s(r, c) = (i_1 i_2 \ldots i_{k-1})$, then*:

(1) *$\Gamma(E^r_c)$ is null if and only if $i_j = 0$ for all j;*

(2) *if $\Gamma(E^r_c)$ is not null, then*

$$E(\Gamma(E^r_c)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\}, \tag{3.2}$$

where $j + t$ is carried out modulo k.

Proof. (1) is obvious.

From the definition of $s(r, c)$, we have

$$E(\Gamma(E^r_c)) = \bigcup_{j=0}^{k-1} \{v_j v_{j+t} \mid i_t \neq 0, 1 \leqslant t \leqslant k-1\}$$

$$= \bigcup_{\substack{1 \leqslant t \leqslant k-1 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\} = A \cup B, \tag{3.3}$$

where

$$A = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\}$$

and

$$B = \bigcup_{\substack{k/2 < t \leqslant k-1 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\}.$$

If $k/2 < t' \leqslant k-1$ and $i_{t'} \neq 0$, then $k-t' \leqslant k/2$ and $i_{k-t'} = i_{t'} \neq 0$ by (3.1(1)). Since $\varphi^j c = \varphi^{j+k} c = \varphi^{(j+t')+(k-t')} c$, we have $v_j = v_{(j+t')+(k-t')}$; hence $v_j v_{j+t'} = v_{j+t'} v_j = v_{j+t'} v_{(j+t')+(k-t')}$. If we let $j' \equiv j+t' \pmod k$, then $v_j v_{j+t'} = v_{j'} v_{j'+(k-t')} \in A$. Therefore, $B \subseteq A$, and Eq. (3.3) becomes

$$E(\Gamma(E_c^r)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\}.$$

This proves (2). ∎

The relationship between a $\Gamma(E_c^r)$ and the sequence $s(r, c)$ can be generalized. Let $s = (i_1 i_2 \ldots i_{k-1})$ be a sequence of 0, 1 and 2 such that $i_j = i_{k-j}$ for $j \in \{1, 2, ..., k-1\}$. Let $\Gamma(s)$ be a graph with vertex set $V(\Gamma(s)) = \{v_0, v_1, ..., v_{k-1}\}$ of k arbitrary elements, and define the edge set of $\Gamma(s)$ by

$$E(\Gamma(s)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, ..., k-1\}.$$

Thus, we see that $\Gamma(E_c^r) = \Gamma(s(r, c))$.

From (3.2), we derive the concept of a basic graph. Notice that if we set $E_t = \{v_j v_{j+t} \mid j = 1, 2, ..., k-1\}$ for $t \in \{1, 2, ..., k-1\}$, then (3.2) can be rewritten as

$$E(\Gamma(E_c^r)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} E_t.$$

Moreover, if $i_t \neq 0$, then $(\Phi c, E_t)$ is a spanning subgraph of $\Gamma(E_c^r)$. Each $(\Phi c, E_t)$ with $i_t \neq 0$ can be viewed as a "basic" component of $\Gamma(E_c^r)$. We shall formalize this concept in the following.

Let o be the sequence $(o_1 o_2 \ldots o_{k-1})$ with $o_i = 0$ if $i \notin \{j, k-j\}$, and $o_j = o_{k-j} = 1$. We denote the graph $\Gamma(o)$ by Γ_j^k, and call it the jth odd basic k-graph. On the other hand, let $e = (e_1 e_2 \ldots e_{k-1})$ be a sequence satisfying $e_i = 0$ if $i \notin \{j, k-j\}$, and $e_j = e_{k-j} = 2$. Denote the graph $\Gamma(e)$ by Π_j^k, and call it the jth even basic k-graph.

(3.3) Lemma. *Let $\Delta \in \{\Gamma_j^k, \Pi_j^k\}$, $j \in \{1, 2, ..., k-1\}$, and $V(\Delta) = \{v_0, v_1, ..., v_{k-1}\}$. Then $E(\Delta) = \{v_i v_{i+j} \mid i = 0, 1, ..., k-1\}$, where $i+j$ is carried out modulo k.* ∎

From (3.3), we obtain the following description of basic graphs.

(3.4) Theorem. *Let $\Delta \in \{\Gamma_j^k, \Pi_j^k\}$, where $j \in \{1, 2, ..., k-1\}$. Let $l = (k, j)$, the greatest common divisor of k and j. Then:*

(1) *If* $l \neq k/2$, *then* Δ *is isomorphic to a disjoint union of* l *copies of* $C_{k/l}$, *the cyclic graph on* k/l *vertices.*

(2) *If* $l = k/2$, *then* Δ *is isomorphic to a disjoint union of* l *copies of* K_2, *the complete graph on 2 vertices.*

In particular, if $(k, j) = 1$, *then* Δ *is isomorphic to* C_k.

Proof. Let $V(\Delta) = \{v_0, v_1, ..., v_{k-1}\}$. By the above lemma $E(\Delta) = \{v_i v_{i+j} \mid i = 0, 1, ..., k-1\}$. Let $i \in \{0, 1, ..., l-1\}$, $r > 0$, and consider the path

$$p_i = v_i v_{i+j} v_{i+2j} \cdots v_{i+rj}.$$

Since $V(\Delta)$ is finite, there is a minimal r such that $v_i = v_{i+rj}$. It follows that $k \mid rj$. Since $(k/l, j/l) = 1$, we have $(k/l) \mid r$. By the minimality of r, we get $r = k/l$. Therefore, $v_i v_{i+j} v_{i+2j} \cdots v_{i+(k/l)j}$ is a closed path of Δ with distince vertices, for every $i \in \{0, 1, ..., l-1\}$.

We claim that the sets of the vertices of p_i, $0 \leqslant i \leqslant l-1$, are all disjoint. Let $s, t \in \{0, 1, ..., l-1\}$ with $s \leqslant t$, and $m, n \in \{0, 1, ..., k/l-1\}$. If $v_{s+mj} = v_{t+nj}$, then $s + mj \equiv t + nj \pmod k$. Therefore, $t - s \equiv j(m-n) \pmod k$, and so $l \mid (t-s)$. Since $0 \leqslant t - s < l$, this can be true only if $t - s = 0$. Hence $s = t$, and so the paths p_s and p_t are the same. This proves the claim. Therefore, $V(\Delta)$ is a disjoint union of the l vertex sets of the paths $p_0, p_1, ..., p_{l-1}$.

If $l \neq k/2$, each path p_i, $i \in \{0, 1, ..., l-1\}$, is a k/l-cycle $C_{k/l}$. This is (1). If $l = k/2$, each path p_i, $i \in \{0, 1, ..., l-1\}$, is isomorphic to K_2. This proves (2). ∎

(3.5) Corollary.

(1) *Let* $\Delta \in \{\Gamma_j^k, \Pi_j^k\}$. *Then* Δ *is regular of degree* 2 *if* $j \neq k/2$, *and* Δ *is regular of degree* 1 *if* $j = k/2$.

(2) $(k, j) = (k, j')$ *if and only if* Γ_j^k *is isomorphic to* $\Gamma_{j'}^k$, *and* Π_j^k *is isomorphic to* $\Pi_{j'}^k$.

Proof. If $j \neq k/2$, then each vertex $v \in \Delta$ belongs exactly to a cycle $v_t v_{t+j} \cdots v_{t+(k/l-1)j} v_t$ for some $t \in \{0, 1, ..., l-1\}$. Therefore, $\deg(v) = 2$. On the other hand, if $j = k/2$, then each vertex $v \in V(\Delta)$ is the vertex of a complete graph on 2 vertices; therefore $\deg(v) = 1$. This is (1).

Now, (2) follows immediately from the above theorem. ∎

4. Decomposition of $\Gamma(E_c^r)$

We remind again that N is a (not necessarily commutative) ring with $\Phi \leqslant \mathcal{U}(N)$ such that Φ is cyclic with a generator φ of order k, (N, Φ) is a Ferrero pair, and that $(N, \mathscr{B}^*_\Phi, \in)$ is a circular incidence structure. Since Φ is a regular group of automorphisms of N, we have for any $\psi, \lambda \in \Phi$ and $r \in N\backslash\{0\}$, if $\psi r = \lambda r$, then $\psi = \lambda$.

(4.1) THEOREM. *If $\Gamma(E^r_c)$ is not null, then it is a union of spanning subgraphs, and each of them is isomorphic to an even basic k-graph, or an odd basic k-graph.*

Proof. Let $s = (i_1 i_2 \ldots i_{k-1})$ be the sequence for E^r_c. By (3.2), the edge set of $\Gamma(E^r_c)$ is

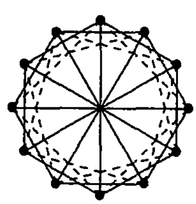
$$E(\Gamma(E^r_c)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} \{v_j v_{j+t} \mid j = 0, 1, \ldots, k-1\}.$$

For each t, $1 \leqslant t \leqslant k/2$, such that $i_t \neq 0$, define $E_t = \{v_j v_{j+t} \mid j = 0, 1, \ldots, k-1\}$. Then $E_t \cap E_{t'} = \varnothing$ if $i_t \neq 0$, $i_{t'} \neq 0$, and $t \neq t'$. Moreover,

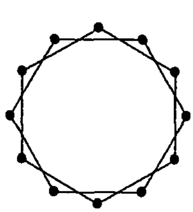
$$E(\Gamma(E^r_c)) = \bigcup_{\substack{1 \leqslant t \leqslant k/2 \\ i_t \neq 0}} E_t.$$

By (3.4), if $i_t \neq 0$, then $(\Phi c, E_t)$ is a spanning subgraph of $\Gamma(E^r_c)$ isomorphic to either Γ^k_t or Π^k_t. Hence the result. ▮

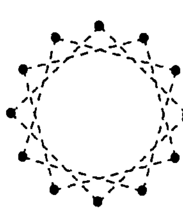
In the following example we put $R = \mathbf{Z}_{229}$ and $k = 12$, and the figure illustrates the decomposition of $\Gamma(E^1_6)$ into basic graphs. A broken line indicates an odd edge, while a solid line denotes an even edge.



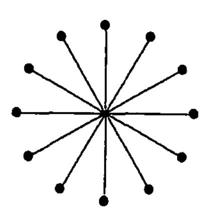
$\Gamma(E^1_6) = \Pi^{12}_2 \vee \Gamma^{12}_3 \vee \Pi^{12}_6$



Π^{12}_2



Γ^{12}_3



Π^{12}_6

Next, we ask the question, "How often does a basic graph occur as a subgraph of the graphs $\Gamma(E_c^r)$?" To answer this question is the same as to find out the following two numbers:

$$\gamma_j(r) = |\{E_c^r \mid \Gamma_j^k \prec \Gamma(E_c^r)\}|$$

and

$$\pi_j(r) = |\{E_c^r \mid \Pi_j^k \prec \Gamma(E_c^r)\}|.$$

Surprisingly enough, when k is even, these two numbers turn out to be constants, and depend on $k = |\Phi|$ only. We also suspect that this is true for odd k, but we can only give a proof for a restricted situation.

The following lemma will be used from time to time without reference.

(4.2) LEMMA. (1) *For any* $\psi, \lambda \in \Phi$, *we have* $\psi(\lambda-1)^{-1} = (\lambda-1)^{-1}\psi$.

(2) *Let* $r, c, r', c' \in N\backslash\{0\}$. *Then* $E_c^r = E_{c'}^{r'}$ *if and only if* $\Phi r = \Phi r'$ *and* $\Phi c = \Phi c'$.

(3) *If* $2 \mid k$, *then* $-1 = \varphi^{k/2} \in \Phi$.

Proof. Using the fact that Φ is an abelian subgroup of $\mathcal{U}(N)$, we derive the equality directly:

$$\begin{aligned}\psi(\lambda-1)^{-1} &= ((\lambda-1)\psi^{-1})^{-1} = (\lambda\psi^{-1} - \psi^{-1})^{-1} \\ &= (\psi^{-1}\lambda - \psi^{-1})^{-1} = (\psi^{-1}(\lambda-1))^{-1} \\ &= (\lambda-1)^{-1}\psi.\end{aligned}$$

This is (1).

Suppose $E_c^r = E_{c'}^{r'}$. Then $\Phi r + c \in E_{c'}^{r'}$, and so there is a $\varphi \in \Phi$ such that $\Phi r + c = \Phi r' + \varphi c'$. By [8; (1.4)], we have $\Phi r = \Phi r'$ and $c = \varphi c'$. Thus $\Phi c = \Phi \varphi c' = \Phi c'$.

Conversely, suppose $\Phi r = \Phi r'$ and $\Phi c = \Phi c'$. Then $c' = \psi c$ for some $\psi \in \Phi$. Let $\Phi r + \varphi c \in E_c^r$. Then

$$\Phi r + \varphi c = \Phi r' + (\varphi\psi^{-1})\,\psi c = \Phi r' + \varphi\psi^{-1} c' \in E_{c'}^{r'}.$$

Therefore, $E_c^r \subseteq E_{c'}^{r'}$. Changing the roles of E_c^r and $E_{c'}^{r'}$, we also have $E_{c'}^{r'} \subseteq E_c^r$, hence $E_c^r = E_{c'}^{r'}$. This completes the proof of (2).

As for (3), we note that $-1 + \varphi^{k/2} \in \mathcal{U}(N)$ since Φ is regular. Therefore, from the identity

$$0 = 1 - \varphi^k = (-1 + \varphi^{k/2})(-1 - \varphi^{k/2}),$$

we get $-1 - \varphi^{k/2} = 0$, and so $-1 = \varphi^{k/2} \in \Phi$ as stated. ∎

(4.3) LEMMA. *Let* $\Delta \in \{\Gamma_j^k, \Pi_j^k\}$. *Then* $\Delta \prec \Gamma(E_c^r)$ *if and only if* $c \in \Phi(\varphi^j - 1)^{-1}(\psi - 1)r$ *for some* $\psi \in \Phi \backslash \{1\}$.

Proof. Suppose $\Delta \prec \Gamma(E_c^r)$. From the definition of an edge, this occurs exactly when

$$(\Phi r + c) \cap (\Phi r + \varphi^j c) \neq \emptyset.$$

In this case, there exist λ, $\delta \in \Phi$ such that $\lambda r + c = \delta r + \varphi^j c$. Therefore,

$$\begin{aligned} c &= (\varphi^j - 1)^{-1}(\lambda - \delta) r = (\varphi^j - 1)^{-1} \delta(\delta^{-1}\lambda - 1) r \\ &= \delta(\varphi^j - 1)^{-1}(\delta^{-1}\lambda - 1) r = \delta(\varphi^j - 1)^{-1}(\psi - 1) r, \end{aligned}$$

where $\psi = \delta^{-1}\lambda$. Thus

$$c \in \Phi(\varphi^j - 1)^{-1}(\psi - 1) r.$$

Conversely, if $c = \delta'(\varphi^j - 1)^{-1}(\psi' - 1) r$, where $\delta', \psi' \in \Phi$ and $\psi' \neq 1$, then $(\varphi^j - 1) c = \delta'(\psi' - 1) r$. Therefore, $\varphi^j c - c = \delta'\psi' r - \delta' r$, and so

$$\delta' r + \varphi^j c = \delta'\psi' r + c \in (\Phi r + \varphi^j c) \cap (\Phi r + c);$$

hence a jth basic k-graph is a subgraph of the graph $\Gamma(E_c^r)$. ▮

For $i, j \in \{1, 2, ..., k-1\}$, define $c_{j,i} = (\varphi^j - 1)^{-1}(\varphi^i - 1) r$. Since $E_c^r = E_{\delta c}^r$ for all $\delta \in \Phi$, we may assume that $c = c_{j,i}$ for some $i \in \{1, 2, ..., k-1\}$ if a jth basic k-graph is a subgraph of $\Gamma(E_c^r)$.

(4.4) THEOREM. *Suppose* $2 | k$. *If* $\Delta \in \{\Gamma_j^k, \Pi_j^k\}$ *and* $\Delta \prec \Gamma(E_c^r)$, *where* $c = (\varphi^j - 1)^{-1}(\psi - 1) r$, *and* $\psi \neq 1$, *then* $\Delta = \Gamma_j^k$ *if and only if* $\psi = -1$. *Moreover, if* $\Delta = \Pi_j^k$, *then* $(\Phi r + \varphi^j c) \cap (\Phi r + c) = \{a, b\}$, *where* $a = r + \varphi^j c = \psi r + c$, *and* $b = -\psi r + \varphi^j c = -r + c$.

Proof. From $c = (\varphi^j - 1)^{-1}(\psi - 1) r$, we get $(\varphi^j - 1) c = (\psi - 1) r$, and so $\psi r + c = r + \varphi^j c$. Let $a = \psi r + c = r + \varphi^j c$. Then $a \in (\Phi r + c) \cap (\Phi r + \varphi^j c)$. Let $b = -r + c = -\psi r + \varphi^j c$. Since $2 | k$, we have $-1 \in \Phi$, and so $b \in (\Phi r + c) \cap (\Phi r + \varphi^j c)$, also.

Suppose $\psi \neq -1$. Then $-r + c \neq \psi r + c$; hence $a \neq b$, and so we have $(\Phi r + c) \cap (\Phi r + \varphi^j c) = \{a, b\}$ by the circularity of $(N, \mathscr{B}_\Phi^*, \in)$. This proves the "if" part and the last statement.

Conversely, suppose $\psi = -1$. We want to show $(\Phi r + c) \cap (\Phi r + \varphi^j c) = \{b\}$. Take $d = \lambda_1 r + c = \lambda_2 r + \varphi^j c \in (\Phi r + c) \cap (\Phi r + \varphi^j c)$, where $\lambda_1, \lambda_2 \in \Phi$, and assume that $d \neq b$, hence $\lambda_2 \neq 1$. Let $e = -\lambda_2 r + c = -\lambda_1 r + \varphi^j c$. Then $e \in (\Phi r + c) \cap (\Phi r + \varphi^j c)$. By the circularity of $(N, \mathscr{B}_\Phi^*, \in)$, we have $e \in \{b, d\}$.

Case 1. If $e=b$, then $-\lambda_2 r+c=-r+c$, and so $\lambda_2=1$, a contradiction.

Case 2. If $e=d$, then $-\lambda_2 r+c=\lambda_1 r+c$, making $-\lambda_2=\lambda_1$. From $e=-\lambda_2 r+c=-\lambda_1 r+\varphi^j c$,. we get $\lambda_1 r+c=-\lambda_1 r+\varphi^j c$, and so $2\lambda_1 r=(\varphi^j-1)c$. Therefore, $(\varphi^j-1)^{-1}(-2)r=c=(\varphi^j-1)^{-1}2\lambda_1 r$, and so $-1\cdot 2r=\lambda_1\cdot 2r$, which puts $\lambda_1=-1$. (Note that $2\neq 0$ since $-1+(-1)$ is a unit in N.) But then $\lambda_2=-\lambda_1=1$, a contradiction again.

Therefore, $(\Phi r+c)\cap(\Phi r+\varphi^j c)=\{b\}$, and the proof is complete. ∎

(4.5) Corollary. *Let* $2|k$ *and let* $j\in\{1,2,...,k-1\}$. *Then*

(1) $\Pi_j^k\prec(E_c^r)$ *if and only if* $E_c^r=E_{c_{j,i}}^r$ *for some* $i\in\{1,2,...,k-1\}\setminus\{k/2\}$;

(2) $\Gamma_j^k\prec\Gamma(E_c^r)$ *if and only if* $E_c^r=E_{c_{j,k/2}}^r$.

(4.6) Lemma. *If* $2|k$, *then* $E_{c_{j,i_1}}^r=E_{c_{j,i_2}}^r$ *if and only if* $i_2=i_1$ *or* $i_2=k-i_1$.

Proof. Let $c_1=c_{j,i_1}$ and $c_2=c_{j,i_2}$. First, suppose $i_2=k-i_1$. Then

$$\begin{aligned}c_2=c_{j,i_2}=c_{j,k-i_1}&=(\varphi^j-1)^{-1}(\varphi^{k-i_1}-1)r\\&=-(\varphi^j-1)^{-1}(1-\varphi^{k-i_1})r=-(\varphi^j-1)^{-1}\,\varphi^{k-i_1}(\varphi^{i_1-k}-1)r\\&=-\varphi^{k-i_1}(\varphi^j-1)^{-1}(\varphi^{i_1}-1)r=-\varphi^{-i_1}c_{j,i_1}=-\varphi^{-i_1}c_1\in\Phi c_1.\end{aligned}$$

Therefore, $E_{c_1}^r=E_{c_2}^r$.

Conversely, suppose $E_{c_1}^r=E_{c_2}^r$ with $c_1\neq c_2$, hence $i_1\neq k/2$. Therefore, it follows from (4.4) that $\Pi_j^k\prec\Gamma(E_{c_1}^r)=\Gamma(E_{c_2}^r)$. By (4.3), we have

$$(\Phi r+\varphi^j c_1)\cap(\Phi r+c_1)=\{a,b\},$$

$$(\Phi r+\varphi^j c_2)\cap(\Phi r+c_2)=\{d,e\},$$

where $a=r+\varphi^j c_1=\varphi^{i_1}r+c_1$, $b=-\varphi^{i_1}r+c_1=-r+c_1$, $d=r+\varphi^j c_2=\varphi^{i_2}r+c_2$, and $e=-\varphi^{i_2}r+c_2=-r+c_2$. Since $E_{c_1}^r=E_{c_2}^r$, there is a $\lambda\in\Phi$ such that $\lambda c_1=c_2$. Thus,

$$d=r+\varphi^j\lambda c_1=\varphi^{i_2}r+\lambda c_1,$$

and so

$$\lambda^{-1}d=\lambda^{-1}r+\varphi^j c_1=\lambda^{-1}\varphi^{i_2}r+c_1.$$

But then

$$\lambda^{-1}d\in(\Phi r+\varphi^j c_1)\cap(\Phi r+c_1)=\{a,b\}.$$

Suppose $\lambda^{-1}d=a$. Then $\lambda^{-1}r+\varphi^j c_1=r+\varphi^j c_1$, and so $\lambda^{-1}r=r$. Hence $\lambda=1$, and so $c_1=c_2$, which has been excluded. On the other hand, if $\lambda^{-1}d=b$, then

$$\lambda^{-1}\varphi^{i_2}r+c_1=\lambda^{-1}r+\varphi^j c_1=-\varphi^{i_1}r+\varphi^j c_1=-r+c_1,$$

and so

$$\lambda^{-1}\varphi^{i_2}r+c_1=-r+c_1$$

$$\lambda^{-1}r+\varphi^j c_1=-\varphi^{i_1}r+\varphi^j c_1,$$

hence

$$\varphi^{i_2}=-\lambda$$

$$\varphi^{i_1}=-\lambda^{-1}.$$

Therefore, $\varphi^{i_1}\varphi^{i_2}=1$, and so $i_2=k-i_1$. ∎

Immediately from (4.4), (4.5) and (4.6), we obtain the desired result.

(4.7) THEOREM. *If* $2|k$, *then* $\gamma_j(r)=1$ *and* $\pi_j(r)=k/2-1$ *for any* $j\in\{1,2,...,k-1\}$ *and* $r\in N\backslash\{0\}$.

Now let's turn to the case when k is odd.

(4.8) LEMMA. *Let k be odd. If* $E^r_{c_{j,i_1}}=E^r_{c_{j,i_2}}$ *and* $\Gamma^k_j\prec\Gamma(E^r_{c_{j,i_1}})=\Gamma(E^r_{c_{j,i_2}})$, *then* $c_{j,i_1}=c_{j,i_2}$.

Proof. Let $c_1=c_{j,i_1}$ and $c_2=c_{j,i_2}$. Since $\Gamma^k_j\prec\Gamma(E^r_{c_1})=\Gamma(E^r_{c_2})$, we have

$$(\Phi r+\varphi^j c_1)\cap(\Phi r+c_1)=\{a\}$$

and

$$(\Phi r+\varphi^j c_2)\cap(\Phi r+c_2)=\{b\}.$$

From the definition of c_1 and c_2, we find $r+\varphi^j c_1=\varphi^{i_1}r+c_1$ and $r+\varphi^j c_2=\varphi^{i_2}r+c_2$. Therefore $a=r+\varphi^j c_1=\varphi^{i_1}r+c_1$ and $b=r+\varphi^j c_2=\varphi^{i_2}r+c_2$. Since $E^r_{c_{j,i_1}}=E^r_{c_{j,i_2}}$, there is a $\lambda\in\Phi$ such that $\lambda c_1=c_2$. Thus $b=r+\varphi^j\lambda c_1=\varphi^{i_2}+\lambda c_1$, and so

$$\lambda^{-1}b=\lambda^{-1}r+\varphi^j c_1=\lambda^{-1}\varphi^{i_2}r+c_1\in(\Phi r+\varphi^j c_1)\cap(\Phi r+c_1).$$

Therefore, $\lambda^{-1}b=a$, or equivalently, $b=\lambda a$. But then $r+\varphi^j c_2=b=\lambda a=\lambda(r+\varphi^j c_1)=\lambda r+\varphi^j\lambda c_1=\lambda r+\varphi^j c_2$; hence $r=\lambda r$, and so $\lambda=1$. Therefore, $c_1=c_2$. ∎

Now, we need to put a restriction on the planar nearrings we are dealing with in order to get a satisfactory result. By [12] there are lots of examples. So, for the next two results, we assume that $k=|\Phi|$ is odd, and that there is a Ferrero pair (N, Ψ) with cyclic $\Psi \leqslant \mathcal{U}(N)$ such that Φ is a subgroup of Ψ of index 2. Let $\Psi=\langle \psi \rangle$ such that $\varphi=\psi^2$. We will show that if $(N, \mathscr{B}^*_{\Psi}, \in)$ is also circular, then

(1) $\pi_j(r)=|\{E^r_c \mid \Pi^k_j \prec \Gamma(E^r_c)\}|=0$;

(2) $\gamma_j(r)=|\{E^r_c \mid \Gamma^k_j \prec \Gamma(E^r_c)\}|=k-1$.

(4.9) Theorem. *If* $\Delta \in \{\Gamma^k_j, \Pi^k_j\}$ *and* $\Delta \prec \Gamma(E^r_c)$ *in* $(N, \mathscr{B}^*_{\Phi}, \in)$, *then* $\Delta \in \{\Gamma^k_j\}$. *Therefore,* $\pi_j(r)=0$.

Proof. Assume that Π^k_j is a subgraph of $\Gamma(E^r_c)$ in $(N, \mathscr{B}^*_{\Phi}, \in)$, and let $(\Phi r+\varphi^j c) \cap (\Phi r+c)=\{a, b\}$, where $a=\lambda_1 r+\varphi^j c=\lambda_2 r+c$ and $b=\lambda_3 r+\varphi^j c=\lambda_4 r+c$ such that $a \neq b$. Also, let $d=-\lambda_1 r+c=-\lambda_2 r+\varphi^j c$ and $e=-\lambda_3 r+c=-\lambda_4 r+\varphi^j c$.

We claim that $|\{a, b, d, e\}|=4$. First, we cannot have $d=e$, otherwise, $\lambda_1=\lambda_3$ and $\lambda_2=\lambda_4$, and so $a=b$, contradicting $a \neq b$. If $a=d$, then $\lambda_2 r+c=a=d=-\lambda_1 r+c$; hence $\lambda_2=-\lambda_1$, and so $-1=\lambda_2\lambda_1^{-1} \in \Phi$, a contradiction. If $b=d$, then $\lambda_4 r+c=-\lambda_1 r+c$; hence $\lambda_4=-\lambda_1$, and so $-1=\lambda_4\lambda_1^{-1} \in \Phi$, a contradiction again. If $a=e$, then $\lambda_2 r+c=-\lambda_3 r+c$; hence $\lambda_2=-\lambda_3$, and so $-1=\lambda_2\lambda_3^{-1} \in \Phi$, which cannot be. Finally, if $b=e$, then $\lambda_4 r+c=-\lambda_3 r+c$; hence $\lambda_4=-\lambda_3$, and so $-1=\lambda_4\lambda_3^{-1} \in \Phi$, a contradiction. Therefore, $|\{a, b, d, e\}|=4$ as claimed.

Since $|\Psi|=2k$, we have $-1 \in \Psi$. But then

$$\{a, b, d, e\} \subseteq (\Psi r+c) \cap (\Psi r+\psi^{2j} c),$$

contradicting the circularity of $(N, \mathscr{B}^*_{\Psi}, \in)$. This shows that an even basic graph cannot be a subgraph of the graph $\Gamma(E^r_c)$ in $(N, \mathscr{B}^*_{\Phi}, \in)$. Hence the result follows. ∎

(4.10) Theorem. $\gamma_j(r)=k-1$.

Proof. From (4.3), each $\Gamma(E^r_{c_{j,i}})$ contains a jth basic graph, Δ, say. By (4.9), $\Delta \neq \Pi^k_j$. Therefore, $\Delta=\Gamma^k_j$. Together with (4.8), we see that the $k-1$ many $E^r_{c_{j,i}}$, $1 \leqslant i \leqslant k-1$, are all distinct. Therefore, we have $\gamma_j(r)=k-1$. ∎

(4.11) *Remark.* (1) The evidence from the data we gathered on field generated circular planar nearrings shows that the results (4.9) and (4.10) may be true even without the requirement for a circular $(N, \mathscr{B}^*_{\Psi}, \in)$. We still cannot prove it, though.

(2) In case when N is a finite field of characteristic p, Theorem 8 of [12] guarantees that the restriction we put on the results (4.9) and (4.10) excludes only finitely many values of p for each k.

5. Overlaps

One phenomenon among the graphs $\Gamma(E_c^r)$ of the field generated circular planar nearrings is still a mystery to us now. That is, some basic graphs always occur together as subgraphs of some $\Gamma(E_c^r)$. In this case, the basic graphs are said to *overlap*.

In this section, we consider a field $(F, +, \cdot)$, and a subgroup $\Phi = \langle \varphi \rangle$ of F^* of *even* order k. We also assume that $(F, \mathscr{B}_\Phi^*, \in)$ is circular. Fix an $r \in F^*$.

Directly from (4.3) and (4.4), we have the following result.

(5.1) Theorem. *Let $1 \leqslant i < j \leqslant k/2$. Then the following statements are equivalent.*

(1) *There is a $c \in F^*$ such that*

$$\Pi_i^k \vee \Pi_j^k \prec \Gamma(E_c^r), \quad \textit{resp.} \quad \Pi_i^k \vee \Gamma_j^k \prec \Gamma(E_c^r)$$

(2) *There exist $u, v, s \in \mathbf{N}$, $u \neq k/2$ such that*

$$(\varphi^i - 1)^{-1}(\varphi^u - 1) = \varphi^s(\varphi^j - 1)^{-1}(\varphi^v - 1) \quad \textit{and} \quad v \neq k/2 \textit{ resp. } v = k/2.$$

As an easy consequence by taking $u = i$, $v = j$ and $s = 1$ in the above theorem, we derive the following

(5.2) Corollary. *If k is even, then $\Gamma(E_r^r)$ is complete.*

The same result for finite prime fields can be found in [7; IV.4].

(5.3) Corollary. *Let K be an extension field of F. Then $(K, \mathscr{B}_\Phi^*, \in)$ is also circular, and an overlap occurs in $\Gamma(E_c^r)$ over F if and only if it occurs over K.*

Proof. From [4; (5.21)] or [12], we know that $(K, \mathscr{B}_\Phi^*, \in)$ is circular. The second statement is obvious. ∎

(5.4) Theorem. *Let $6 \mid k$. Put $c_i = (\varphi^{2i} - 1)^{-1}(\varphi^{k/2 - i} - 1)r$ and $d_i = (\varphi^i - 1)^{-1}(\varphi^{k/6} - 1)r$, for every i, $1 \leqslant i < k/2$. Then $c_i \in \Phi d_i$, and so $E_{c_i}^r = E_{d_i}^r$. Consequently, we have*

$$\Pi_i^k \vee \Pi_{2i}^k \prec \Gamma(E_{c_i}^r) = \Gamma(E_{d_i}^r),$$

and

$$\begin{cases} \Pi^k_{k/2-i} \vee \Pi^k_{k/6} \prec \Gamma(E^r_{c_i^{-1}}) = \Gamma(E^r_{d_i^{-1}}) & \text{if } 2i \neq k/2; \\ \Gamma^k_{k/4} \vee \Pi^k_{k/6} \prec \Gamma(E^r_{c_i^{-1}}) = \Gamma(E^r_{d_i^{-1}}) & \text{if } 2i = k/2. \end{cases}$$

Proof. Let $f(x) = x^{k/3} - x^{k/6} + 1$. Then $(x^{k/2} - 1)(x^{k/6} + 1) f(x) = x^k - 1$. Since $\varphi^k = 1$, $\varphi^{k/2} = -1$ and $\varphi^{k/6} \neq -1$, we conclude that $f(\varphi) = 0$, i.e.,

$$\begin{aligned} 0 &= \varphi^{k/3} - \varphi^{k/6} + 1 \\ &= \varphi^{k/6}(\varphi^{k/6} - 1) + 1 \\ &= \varphi^{k/6-i}\varphi^i(\varphi^{2i} - 1)(\varphi^{2i} - 1)^{-1}(\varphi^{k/6} - 1) + 1 \\ &= \varphi^{k/6-i}\varphi^i(\varphi^{2i} - 1)(\varphi^i - 1)^{-1}(\varphi^i + 1)^{-1}(\varphi^{k/6} - 1) + 1 \\ &= \varphi^{k/6-i}(\varphi^i - 1)^{-1}(\varphi^{k/6} - 1) \cdot (\varphi^{2i} - 1)\, \varphi^i(\varphi^i + 1)^{-1} + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi^{k/6-i} d_i &= \varphi^{k/6-i}(\varphi^i - 1)^{-1}(\varphi^{k/6} - 1) \\ &= (\varphi^{2i} - 1)^{-1}(-\varphi^{-i})(\varphi^i + 1) \\ &= (\varphi^{2i} - 1)^{-1}(-1 - \varphi^{-i}) \\ &= (\varphi^{2i} - 1)^{-1}(\varphi^{k/2-i} - 1) = c_i. \end{aligned}$$

Therefore, $c_i \in \Phi d_i$, and the result follows from (5.1). ∎

(5.5) *Remark.* If $i = k/3$ in the above theorem, then $\Pi^k_i = \Pi^k_{2i}$ and $\Pi^k_{k/2-i} = \Pi^k_{k/6}$. Therefore, the overlap situation does not occur in this case. But this is the only exception in the theorem.

The data we have obtained by running Maple on a Sun Workstation for the field of complex numbers with $k \leqslant 300$ showed overlaps only when $6 \mid k$, and overlaps other than the ones in (5.4) have only been found if 5 or 7 is also a divisor of k.

Conjecture 1. Let $F = \mathbf{C}$, the field of complex numbers, and consider a finite subgroup Φ of the unit circle with $|\Phi| = k$. Let $\mathscr{B}^*_k = \mathscr{B}^*_\Phi$. Certainly, $(\mathbf{C}, \mathscr{B}^*_k, \in)$ is circular. Let Δ_1 and Δ_2 be two distinct basic k-graphs. If $c \notin \Phi r$ and $\Delta_1 \vee \Delta_2 \prec \Gamma(E^r_c)$ for some $E^r_c \subseteq \mathscr{B}^*_k$, then $6 \mid k$.

We point out that Conjecture 1 does not hold for every field generated circular planar nearring as one can easily find counterexamples in a finite prime field $\mathbf{Z}_p$ with a "small" p. In fact, there are quite a few overlaps in circular planar nearrings generated from the finite prime fields. (See Appendix C of [7].) The only "explanation" we have is that when p is "small," there is "no room" for these circles to separate from each other. However, our computer generated data suggest that the following conjecture is also true.

Conjecture 2. Let $F = \mathbf{Z}_p$, where p is a prime. For each k, $k \geqslant 4$, there is an $n_k \in \mathbf{N}$ such that if $p > n_k$ and if $(F, \mathscr{B}^*_\Phi, \in)$ is circular, then the following is true:

> Let Δ_1 and Δ_2 be two distinct basic k-graphs. If $\Delta_1 \vee \Delta_2 \prec \Gamma(E^1_c)$ for some $E^1_c \subseteq \mathscr{B}^*_\Phi$ over F, then $\Delta_1 \vee \Delta_2 \prec \Gamma(E^1_{c'})$ for some $E^1_{c'} \subseteq \mathscr{B}^*_k$ over $\mathbf{C}$.

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