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Elementary divisors of Specht modules

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Abstract

Let $\mathscr{H}_q(\mathfrak{S}_n)$ be the Iwahori-Hecke algebra of the symmetric group defined over the ring $\mathbb{Z}[q, q^{-1}]$. The q-Specht modules of $\mathscr{H}_q(\mathfrak{S}_n)$ come equipped with a natural bilinear form. In this paper we try to compute the elementary divisors of the Gram matrix of this form (which need not exist since $\mathbb{Z}[q, q^{-1}]$ is not a principal ideal domain). When they are defined, we give the relationship between the elementary divisors of the Specht modules $S_q(\lambda)$ and $S_q(\lambda')$, where λ' is the conjugate partition. We also compute the elementary divisors when λ is a hook partition and give examples to show that in general elementary divisors do not exist.

MSC: 20C08; 20G05; 33D80

Keywords: Specht modules; Elementary divisors; Hook partitions

1. Introduction and statement of main results

The irreducible representations of the symmetric groups and their Iwahori–Hecke algebras have been classified and constructed by James [6] and Dipper and James [2], yet simple properties of these modules, such as their dimensions, are still not known. Every irreducible representation of these algebras is constructed by quotienting out the radical of a bilinear form on a particular type of module, known as a Specht module. The bilinear forms on the Specht modules are the objects of our study.

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One way of determining the dimension of the simple modules would be to first find the elementary divisors of the Gram matrix over $\mathbb{Z}[q, q^{-1}]$ and then specialize. This would also give the dimensions of the subquotients of the Jantzen filtrations of the Specht modules over an arbitrary field; see [7]. In general, such an approach is not possible because, as Andersen has shown, Gram matrices need not be diagonalizable over $\mathbb{Z}[q, q^{-1}]$; see [1, Remark 5.11]. We also give some examples of non-diagonalizable Specht modules in Section 7.

Let $G(\lambda)$ be the Gram matrix of the Specht module $S(\lambda)$. Then the first result in this paper shows that $G(\lambda)$ is diagonalizable if and only if $G(\lambda')$ is diagonalizable, where λ' is the partition conjugate to λ. Moreover, if $G(\lambda)$ is divisibly diagonalizable (that is, $G(\lambda)$ is equivalent to a diagonal matrix $\mathrm{diag}(d_1, \ldots, d_m)$ such that d_i divides d_{i+1}, for $1 \leq i < m$), then so is $G(\lambda')$. In this case we can speak of elementary divisors and we show how the elementary divisors of $G(\lambda)$ and $G(\lambda')$ determine each other. This is a q-analogue of the corresponding result for the symmetric group [8].

We next consider the elementary divisors for the hook partitions. We show that when $\lambda = (n-k, 1^k)$, for $0 \leq k < n$, the Gram matrix $G(\lambda)$ is always divisibly diagonalizable over $\mathbb{Z}[q, q^{-1}]$, and we determine the elementary divisors. Again, this is a q-analogue of the corresponding result for the symmetric groups [8]; however, the proof in the Hecke algebra case is more involved and requires some interesting combinatorics.

2. The Hecke algebra and permutation modules

Fix a positive integer n and let $\mathfrak{S}_n$ be the symmetric group of degree n.

Let R be a commutative domain and let q be an invertible element in R.

The Iwahori–Hecke algebra of $\mathfrak{S}_n$ with parameter q is the unital associative algebra $\mathscr{H}$ with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leq i < n, \\ T_i T_j &= T_j T_i && \text{for } 1 \leq i < j-1 < n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leq i < n-1. \end{aligned}$$

Let $r_i = (i, i+1)$, for $i = 1, 2, \ldots, n-1$. Then $\{r_1, r_2, \ldots, r_{n-1}\}$ generate $\mathfrak{S}_n$ (as a Coxeter group). If $w \in \mathfrak{S}_n$ then $w = r_{i_1} \cdots r_{i_k}$ for some i_j with $1 \leq i_j < n$. The word $w = r_{i_1} \cdots r_{i_k}$ is reduced if k is minimal; in this case we say that w has length k and we define $\ell(w) = k$.

If $r_{i_1} \cdots r_{i_k}$ is reduced then we set $T_w = T_{i_1} \cdots T_{i_k}$. Then T_w is independent of the choice of reduced expression for w; see, for example, [10, 1.11]. Furthermore, $\mathscr{H}$ is free as an R-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$.

A composition μ of n is a sequence of non-negative integers $(\mu_1, \mu_2, \ldots)$ that sum to n. If, in addition, $\mu_1 \geq \mu_2 \geq \ldots$, then μ is a partition of n.

Let μ be a composition of n and let $\mathfrak{S}_\mu$ be the associated Young subgroup. Then $\mathscr{H}(\mathfrak{S}_\mu) = \langle T_w \mid w \in \mathfrak{S}_\mu \rangle$ is a subalgebra of $\mathscr{H}$. Given a (right) $\mathscr{H}(\mathfrak{S}_\mu)$-module V, we define the induced $\mathscr{H}$-module

$$\mathrm{Ind}_{\mathscr{H}(\mathfrak{S}_\mu)}^{\mathscr{H}}(V) = V \otimes_{\mathscr{H}(\mathfrak{S}_\mu)} \mathscr{H}.$$

Let $\mathcal{D}_\mu = \{d \in \mathfrak{S}_n \mid \ell(dr_i) > \ell(d) \text{ for all } r_i \in \mathfrak{S}_\mu\}$ be the set of distinguished right coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. Then, as an R-module,

$$\mathrm{Ind}_{\mathscr{H}(\mathfrak{S}_\mu)}^{\mathscr{H}}(V) \cong \bigoplus_{d\in\mathcal{D}_\mu} V \otimes T_d$$

by [2, Theorem 2.7].

Let $x_\mu = \sum_{w\in\mathfrak{S}_\mu} T_w$. Then $T_w x_\mu = x_\mu T_w = q^{\ell(w)} x_\mu$ for all $w \in \mathfrak{S}_\mu$. The trivial representation of $\mathscr{H}(\mathfrak{S}_\mu)$ is the free R-module $\mathbf{1}_\mu = Rx_\mu$.

Let $y_\mu = \sum_{w\in\mathfrak{S}_\mu} (-q)^{-\ell(w)} T_w$. Then $T_w y_\mu = y_\mu T_w = (-1)^{\ell(w)} y_\mu$ for all $w \in \mathfrak{S}_\mu$. The sign representation of $\mathscr{H}(\mathfrak{S}_\mu)$ is the free R-module $\mathcal{E}_\mu = Ry_\mu$.

For any composition μ we define the permutation module $M(\mu) = \mathrm{Ind}_{\mathscr{H}(\mathfrak{S}_\mu)}^{\mathscr{H}}(\mathbf{1}_\mu) \cong x_\mu\mathscr{H}$. Then $M(\mu)$ is free as an R-module of rank $[\mathfrak{S}_n : \mathfrak{S}_\mu]$ with basis $\{x_\mu T_d \mid d \in \mathcal{D}_\mu\}$. The $\mathscr{H}$-action on $M(\mu)$ is determined by

$$x_\mu T_d T_i = \begin{cases} q x_\mu T_d, & \text{if } \ell(dr_i) > \ell(d) \text{ and } dr_i \notin \mathcal{D}_\mu, \\ x_\mu T_{dr_i}, & \text{if } \ell(dr_i) > \ell(d) \text{ and } dr_i \in \mathcal{D}_\mu, \\ q x_\mu T_{dr_i} + (q-1) x_\mu T_d, & \text{otherwise.} \end{cases}$$

Note that if $\ell(dr_i) < \ell(d)$ then $dr_i \in \mathcal{D}_\mu$.

Let $* : \mathscr{H} \longrightarrow \mathscr{H}$ be the R-linear map on $\mathscr{H}$ determined by $T_w^* = T_{w^{-1}}$, for all $w \in \mathfrak{S}_n$. This defines an R-algebra anti-automorphism on $\mathscr{H}$ of order 2.

The module $M(\mu)$ carries a symmetric bilinear form $\langle , \rangle_\mu$ given by

$$\langle x_\mu T_a, x_\mu T_b\rangle_\mu = \begin{cases} q^{\ell(a)}, & \text{if } a = b, \\ 0, & \text{otherwise,} \end{cases}$$

for $a, b \in \mathcal{D}_\mu$. It follows from the formulae above that the form $\langle , \rangle_\mu$ is associative in the sense that

$$\langle xh, y\rangle_\mu = \langle x, yh^*\rangle_\mu$$

for all $x, y \in M(\mu)$ and all $h \in \mathscr{H}$.

We will need two dualities on the category of right $\mathscr{H}$-modules. Both of them come from involutions on $\mathscr{H}$. The first duality comes from the involution $*$ defined above. The second is induced from the automorphism $\# : \mathscr{H} \longrightarrow \mathscr{H}$ which is the R-linear map on $\mathscr{H}$ determined by $T_w^\# = (-q)^{l(w)} T_{w^{-1}}^{-1}$, for all $w \in \mathfrak{S}_n$. It is straightforward to check that $\#$ preserves the relations in $\mathscr{H}$ and, hence, that it is an R-algebra automorphism of order 2. Note that the involutions $\#$ and $*$ commute.

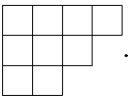
If V is an $\mathscr{H}$-module let V^* be its R-linear dual. Then V^* becomes an $\mathscr{H}$-module on letting $(\phi \cdot \xi)(v) := \phi(v\xi^*)$, where $\phi \in V^*$, $v \in V$ and $\xi \in \mathscr{H}$. With the according operation on morphisms, this defines a contravariant self-equivalence on the category of $\mathscr{H}$-modules.

If V is an $\mathscr{H}$-module let $V^\#$ be the $\mathscr{H}$-module with underlying R-module V and operation $v \cdot_\# \xi := v \cdot \xi^\#$, where $v \in V$ and $\xi \in \mathscr{H}$. With the identical operation on morphisms, this defines a covariant self-equivalence on the category of $\mathscr{H}$-modules.

3. Specht modules

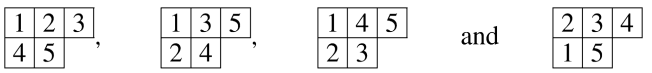
We recall some well-known facts due to Dipper and James [2].

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a composition of n. The diagram of λ is the set $[\lambda] = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq j \leq \lambda_i\}$. We identify the diagram of λ with an array of boxes in the plane. For example, if $\lambda = (4, 3, 2)$ then

$[\lambda] =$  .

The conjugate of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where $\lambda'_j = \#\{i \geq 1 \mid \lambda_i \geq j\}$ for all j; that is, λ' is the partition of n whose diagram is obtained by interchanging the rows and columns of the diagram of λ.

Formally, a λ-tableau is a bijection $\mathrm{t} : [\lambda] \longrightarrow \{1, 2, \ldots, n\}$; however, we will think of a λ-tableau as a labelling of the diagram of λ by the numbers $1, 2, \ldots, n$. Accordingly, we will speak of the rows and columns of a tableau. For example,



are all (3, 2)-tableaux.

A tableau is row standard if in each row its entries increase from left to right. A tableau is standard if it is row standard and in each column its entries increase from top to bottom. Let $\mathrm{Std}(\lambda)$ be the set of standard λ-tableaux.

All of the tableaux above are row standard; however, only the first two tableaux are standard.

The initial λ-tableau t^λ is the standard λ-tableau which has the numbers $1, 2, \ldots, n$ entered in order from left to right, and then top to bottom, along its rows. The terminal λ-tableau t_λ is the standard λ-tableau which has the numbers $1, 2, \ldots, n$ entered in order from top to bottom, and then left to right, along its columns. Of the (3, 2)-tableaux above, the first is $\mathrm{t}^{(3,2)}$ and the second is $\mathrm{t}_{(3,2)}$.

The symmetric group $\mathfrak{S}_n$ acts from the right on the set of λ-tableaux by permuting their entries. If t is a λ-tableau let $d(\mathrm{t})$ be the unique permutation such that $\mathrm{t} = \mathrm{t}^\lambda d(\mathrm{t})$. In particular, we set $w_\lambda = d(\mathrm{t}_\lambda)$.

We remark that $\mathcal{D}_\mu = \{d(\mathrm{t}) \mid \mathrm{t} \text{ is a row standard } \mu\text{-tableau}\}$.

Suppose that λ is a partition of n and let $z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'}$. The Specht module is the submodule $S(\lambda) = z_\lambda \mathscr{H}$ of $M(\lambda)$.

Let $S(\lambda)^\perp = \{x \in M(\lambda) \mid \langle x, y\rangle = 0 \text{ for all } y \in S(\lambda)\}$. As $\langle , \rangle_\lambda$ is associative, $S(\lambda)^\perp$ is an $\mathscr{H}$-submodule of $M(\lambda)$. More precisely, $S(\lambda)^\perp$ is the kernel of the $\mathscr{H}$-linear map

$$M(\lambda) \xrightarrow{\delta_\lambda} S(\lambda)^*; \qquad x_\lambda h \longmapsto \langle x_\lambda h, -\rangle_\lambda,$$

where $h \in \mathscr{H}$.

By restricting the bilinear form $\langle , \rangle_\lambda$ on $M(\lambda)$ we obtain a bilinear form on $S(\lambda)$. If R is a field then $D(\lambda) = S(\lambda)/S(\lambda) \cap S(\lambda)^\perp$ is either zero or absolutely irreducible. Moreover, all of the irreducible $\mathscr{H}$-modules arise uniquely in this way [2, Theorem 5.2].

Before we can give a basis of $S(\lambda)$ we need some more notation. If $\mathfrak{t}$ is a λ-tableau let $\mathfrak{t}'$ be the λ'-tableau obtained by interchanging the rows and columns of $\mathfrak{t}$. For example, $(\mathfrak{t}^\lambda)' = \mathfrak{t}_{\lambda'}$ and $(\mathfrak{t}_\lambda)' = \mathfrak{t}^{\lambda'}$. Finally, if $\mathfrak{t}$ is a standard λ-tableau let $v_\mathfrak{t} = z_\lambda T_{d(\mathfrak{t}')}$.

3.1 (*Dipper–James [2, Theorem 5.6]*). *The Specht module $S(\lambda)$ is free as an R-module with basis $\{v_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$.*

We call $\{v_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ the Dipper–James basis of $S(\lambda)$. Let $n_\lambda = \#\mathrm{Std}(\lambda)$ be the number of standard λ-tableaux. Then, as an R-module, $S(\lambda)$ is free of rank n_λ.

Fix an ordering of $\mathrm{Std}(\lambda)$ and let

$$G(\lambda) = (\langle v_\mathfrak{s}, v_\mathfrak{t}\rangle_\lambda)_{\mathfrak{s},\mathfrak{t}\in\mathrm{Std}(\lambda)}$$

be the Gram matrix of the bilinear form $\langle , \rangle_\lambda$, with respect to the Dipper–James basis. The matrix $G(\lambda)$ depends on the choice of ordering on $\mathrm{Std}(\lambda)$; however, all of the quantities that we are interested in will be independent of this choice. We remark that $\det G(\lambda)$ has been explicitly computed by Dipper and James [3, Theorem 4.11].

4. Diagonalizability and elementary divisors

Given an integer $m \geq 1$, an $m \times m$ matrix A with coefficients in R is diagonalizable if there exist matrices S and T in $\mathrm{GL}_m(R)$ such that SAT is a diagonal matrix. The matrix A is divisibly diagonalizable if $SAT = \mathrm{diag}(d_1, \ldots, d_m)$ is a diagonal matrix such that d_i divides d_{i+1} in R, for $1 \leq i < m$. If A is divisibly diagonalizable and $SAT = \mathrm{diag}(d_1, \ldots, d_m)$ satisfies this condition, then we call $d_1, \ldots, d_m$ the elementary divisors of A.

Given $A \in R^{m\times m}$, we let $I_k(A)$ be the ideal of the $k \times k$ minors of A, for $1 \leq k \leq m$. Note that for $B \in R^{m\times m}$, we have $I_k(AB) \subseteq I_k(A)$ and $I_k(BA) \subseteq I_k(A)$. Hence for $S, T \in \mathrm{GL}_m(R)$, we have $I_k(A) = I_k(SAT)$. Therefore, if A is divisibly diagonalizable with resulting diagonal elements $d_1, \ldots, d_m$, then $I_k(A) = I_k(\mathrm{diag}(d_1, \ldots, d_m))$ is the principal ideal generated by $d_1 d_2 \cdots d_k$. This shows that the resulting diagonal entries are independent, up to multiplication by units, of the choice of the diagonalizing matrices. In other words, the elementary divisors of a divisibly diagonalizable matrix are well-defined modulo units.

Whether or not A is diagonalizable, the ideals $I_k(A) \subseteq R$ are invariant under the equivalence relation $A \sim SAT$. It would be interesting to consider the equivalence classes within $\{A \in R^{m\times m} \mid I_k(A) = J_k \text{ for } 1 \leq k \leq m\}$ for a fixed tuple (J_i) of ideals of R.

If R is a principal ideal domain then every matrix $A \in R^{m\times m}$ is divisibly diagonalizable by the elementary divisor theorem. The resulting diagonal matrix is known as the Smith normal form.

Now the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ is not a principal ideal domain and, in fact, there are strict inclusions of the set of divisibly diagonalizable matrices in the set of diagonalizable matrices, and of the set of diagonalizable matrices in all matrices with coefficients in $\mathbb{Z}[q, q^{-1}]$. For example, the matrix $A = \left(\begin{smallmatrix} q-1 & 0 \\ 0 & q+1 \end{smallmatrix}\right)$ is diagonalizable, but not divisibly diagonalizable because $I_1(A)$, the ideal of R generated by the entries of A, is not principal.

Proving that a matrix is not diagonalizable is slightly harder. For example, we claim that the matrix $B = \begin{pmatrix} q+1 & 2 \\ 0 & q+1 \end{pmatrix}$ is not diagonalizable over $\mathbb{Z}[q, q^{-1}]$. To see this, notice that over $\mathbb{Q}[q, q^{-1}]$ the matrix B has elementary divisors 1 and $(q+1)^2$. Therefore, if B is diagonalizable over $\mathbb{Z}[q, q^{-1}]$ then one of these diagonal entries must be a unit in $\mathbb{Q}[q, q^{-1}]$; that is, of the form aq^b with $a, b \in \mathbb{Z}$. Reducing modulo 2 this shows that one of the elementary divisors of B over $\mathbb{F}_2[q, q^{-1}]$ is zero or a unit. However, this is a contradiction because the elementary divisors of B over $\mathbb{F}_2[q, q^{-1}]$ are $q+1$ and $q+1$.

In proving that certain Gram matrices $G(\lambda)$ are divisibly diagonalizable over $\mathbb{Z}[q, q^{-1}]$, we shall make use of the following simple lemma.

Lemma 4.1. *Let A be an $m \times m$ matrix with coefficients in R, and suppose that there exist invertible matrices $S, T \in \mathrm{GL}_m(R)$ such that*

$$SAT = \begin{pmatrix} d_1 & b_{12} & \cdots & b_{1m} \\ 0 & d_2 & \cdots & b_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_m \end{pmatrix},$$

where $d_1 \mid d_2 \mid \ldots \mid d_m$ and d_i divides b_{ij} for all j. Then A is divisibly diagonalizable and $d_1, d_2, \ldots, d_m$ are the elementary divisors of A.

Proof. The matrix SAT can be written as the product of $\mathrm{diag}(d_1, \ldots, d_m)$ with a matrix in $\mathrm{GL}_m(R)$. □

As we saw with the non-diagonalizable matrix $\begin{pmatrix} q+1 & 2 \\ 0 & q+1 \end{pmatrix}$ above, the requirement that d_i divides b_{ij} for all j is not superfluous.

5. Elementary divisors for conjugate partitions

Let $R = \mathbb{Z}[q, q^{-1}]$. Let λ be a partition of n. In this section we relate the Gram matrices $G(\lambda)$ and $G(\lambda')$. We start with some mild generalizations of some results about Specht modules which were proved by Dipper and James [2] over a field.

Recall that if Y is a submodule of an R-free module X then Y is pure if the quotient module X/Y is R-free.

Lemma 5.1. *Suppose that λ is a partition. Then the Specht module $S(\lambda)$ is a pure submodule of $M(\lambda)$.*

Proof. Using the Dipper–James basis of $S(\lambda)$, and the basis $\{x_\lambda T_d \mid d \in \mathcal{D}_\lambda\}$ of $M(\lambda)$, suitably ordered, the matrix representing the embedding $S(\lambda) \longrightarrow M(\lambda)$ $\mathbb{Z}[q, q^{-1}]$-linearly becomes triangular with 1's on the diagonal [2, Theorem 5.8]. □

Corollary 5.2. *The map $M(\lambda) \xrightarrow{\delta_\lambda} S(\lambda)^*$ is surjective.*

Proof. The map δ_λ is the composition of the map

$$M(\lambda) \xrightarrow{\sim} M(\lambda)^*; \qquad \xi \longmapsto \langle \xi, - \rangle_\lambda$$

with the dual of the inclusion map $S(\lambda) \longrightarrow M(\lambda)$. This is surjective by Lemma 5.1. □

Let $\alpha(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}$. Note that $\alpha(\lambda) = \ell(w_{0,\lambda'})$, where $w_{0,\lambda'}$ is the unique element of longest element in $\mathfrak{S}_{\lambda'}$. The next lemma is well known; see, for example, [13, Prop. 2.2]. We include a proof for completeness.

Recall that automorphism #, and the corresponding operation on the module category of $\mathscr{H}$, were defined at the end of Section 2.

Lemma 5.3. *We have $x_\lambda^\# = q^{\alpha(\lambda')}y_\lambda$ and $y_\lambda^\# = q^{-\alpha(\lambda')}x_\lambda$.*

Proof. As # is an involution the two equalities are equivalent, so we prove only the first. For any integer i, with $1 \leq i < n$, we have $x_\lambda^\# T_i = (x_\lambda T_i^\#)^\# = -x_\lambda^\#$. Write $x_\lambda^\# = \sum_{w\in\mathfrak{S}_\lambda} a_w T_w$, for some $a_w \in \mathbb{Z}[q,q^{-1}]$. Comparing coefficients on both sides of the equation $x_\lambda^\# T_i = -x_\lambda^\#$ shows that $a_{wr_i} = (-q)a_w$ for each w that has a reduced expression ending in r_i; compare [10, Cor. 1.7]. Hence, $x_\lambda^\#$ is a scalar multiple of y_λ. Then $T_{w_{0,\lambda}}^\# = (-1)^{\ell(w_{0,\lambda})}T_{w_{0,\lambda}}$ plus a linear combination of T_v where $v \in \mathfrak{S}_\lambda$ and $\ell(v) < \ell(w_{0,\lambda})$. Therefore, comparing the coefficient of $T_{w_{0,\lambda}}$ in $x_\lambda^\#$ and y_λ gives the result. □

Recall that $S(\lambda) = z_\lambda\mathscr{H}$, where $z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'}$. The importance of z_λ, and the irreducibility of $S(\lambda)$ in the semisimple case, follow from the following simple fact.

5.4 (*Dipper–James [2, Lemma 4.1]*). *Suppose that $w \in \mathfrak{S}_n$. Then*

$$x_\lambda T_w y_{\lambda'} = \begin{cases} \pm q^a z_\lambda, & \text{if } w \in \mathfrak{S}_\lambda w_\lambda \mathfrak{S}_{\lambda'}, \\ 0, & \text{otherwise}, \end{cases}$$

for some integer a.

The proof of this result amounts to the observation that $\mathfrak{S}_\lambda \cap w\mathfrak{S}_{\lambda'}w^{-1} = \{1\}$ if and only if $w \in \mathfrak{S}_\lambda w_\lambda \mathfrak{S}_{\lambda'}$.

Lemma 5.5 (*The Submodule Theorem*). *If U is a pure submodule of $M(\lambda)$, then $S(\lambda) \subseteq U$ or $U \subseteq S(\lambda)^\perp$.*

Proof. For all $u \in U$, we have $uy_{\lambda'} = \alpha_u z_\lambda$ for some $\alpha_u \in \mathbb{Z}[q,q^{-1}]$ by 5.4.

Case 1. $\alpha_u = 0$ for all $u \in U$. Therefore, if $u \in U$ and $h \in \mathscr{H}$ then we have $\langle u, z_\lambda h\rangle_\lambda = \langle uh^* y_{\lambda'}, x_\lambda T_{w_\lambda}\rangle_\lambda$, since $y_{\lambda'}^* = y_{\lambda'}$. But $uh^* \in U$, so $uh^* y_{\lambda'} = 0$ and $u \in S(\lambda)^\perp$. Hence, $U \subseteq S(\lambda)^\perp$.

Case 2. $\alpha_u \neq 0$ for some $u \in U$. Now $U \ni uy_{\lambda'} = \alpha_u z_\lambda$ implies $z_\lambda \in U$ since $U \subseteq M^\lambda$ is a pure submodule. Therefore, $S(\lambda) \subseteq U$. □

Note that the right ideal $y_{\lambda'}T_{w_\lambda}^{-1}x_\lambda\mathscr{H}$ is isomorphic to $S(\lambda)^\#$ via $\xi \longmapsto \xi^\#$. Composing left multiplication by $y_{\lambda'}T_{w_\lambda}^{-1}$ with this isomorphism, and using Lemma 5.3, we obtain a surjective $\mathscr{H}$-linear map

$$M(\lambda) = x_\lambda\mathscr{H} \xrightarrow{\theta_\lambda} S(\lambda')^\#; \qquad x_\lambda h \longmapsto z_{\lambda'} \cdot_\# h,$$

where $h \in \mathscr{H}$.

Lemma 5.6. *We have* $\operatorname{Kern}\theta_\lambda = S(\lambda)^\perp$. *This induces an isomorphism*

$$S(\lambda')^{\#} \xrightarrow{\psi_\lambda}_{\sim} S(\lambda)^*; \qquad z_{\lambda'} \cdot_{\#} h \longmapsto \langle x_\lambda h, - \rangle_\lambda,$$

where $h \in \mathscr{H}$.

Proof. Both $\operatorname{Kern}\theta_\lambda$ and $S(\lambda)^\perp$ are pure submodules of $M(\lambda)$. Over $\mathbb{Z}[q, q^{-1}]$, both $S(\lambda')^{\#}$ and $S(\lambda)^*$ are free of rank n_λ, so it suffices to prove that $\operatorname{Kern}\theta_\lambda \subseteq S(\lambda)^\perp$. By Lemma 5.5 this is equivalent to showing that $S(\lambda) \not\subseteq \operatorname{Kern}\theta_\lambda$. So it is enough to show that $z_\lambda \theta_\lambda \neq 0$. The bilinear form $\langle , \rangle_{\lambda'}$ is associative, so

$$\begin{aligned}\langle z_\lambda \theta_\lambda, x_{\lambda'} \rangle_{\lambda'} &= q^{-\alpha(\lambda)} \langle z_{\lambda'} T_{w_{\lambda'}}^{-1} x_{\lambda'}, x_{\lambda'} \rangle_{\lambda'} \\ &= q^{-\alpha(\lambda')} \left(\sum_{w \in \mathfrak{S}_{\lambda'}} q^{\ell(w)} \right) \langle z_{\lambda'}, x_{\lambda'} T_{w_\lambda}^{-1} \rangle_{\lambda'}.\end{aligned}$$

Now, $z_{\lambda'} = \sum_{v \in \mathfrak{S}_\lambda} (-q)^{-\ell(v)} x_{\lambda'} T_{w_{\lambda'} v}$, where each $w_{\lambda'} v$ is a distinguished coset representative for $\mathfrak{S}_{\lambda'}$. In contrast, $T_{w_\lambda}^{-1}$ is equal to $T_{w_{\lambda'}}$ plus a linear combination of terms T_u, where $u \in \mathfrak{S}_n$ with $\ell(u) < \ell(w_{\lambda'})$. Thus $\langle z_{\lambda'}, x_{\lambda'} T_{w_\lambda}^{-1} \rangle_{\lambda'} = \langle x_{\lambda'} T_{w_{\lambda'}}, x_{\lambda'} T_{w_{\lambda'}} \rangle_\lambda = q^{\ell(w_{\lambda'})}$. Hence, $\langle z_\lambda \theta_\lambda, x_{\lambda'} \rangle_{\lambda'} \neq 0$.

A comparison of the short exact sequences

$$0 \longrightarrow \operatorname{Kern}\theta_\lambda \longrightarrow M(\lambda) \xrightarrow{\theta_\lambda} S(\lambda')^{\#} \longrightarrow 0$$

and

$$0 \longrightarrow S(\lambda)^\perp \longrightarrow M(\lambda) \xrightarrow{\delta_\lambda} S(\lambda)^* \longrightarrow 0$$

yields the isomorphism ψ_λ. □

For each node $(i, j) \in [\lambda]$, we let $h_{i,j} = (\lambda_i - j) + (\lambda'_j - i) + 1$ be the corresponding hook length and set $h_\lambda(q) = \prod_{(i,j)\in[\lambda]} [h_{i,j}]_q$. The next lemma follows from results of Murphy [11].

Lemma 5.7. *We have* $z_\lambda T_{w_\lambda}^{-1} z_\lambda = q^{n-\alpha(\lambda)} h_\lambda(q) z_\lambda$.

Proof. For the purpose of this proof, we may assume $R = \mathbb{Q}(q)$. By [11, p. 510–511], there exists an element $\Psi_{\mathfrak{t}_\lambda}^* = T_{w_{\lambda'}} + \sum_{\ell(v)<\ell(w_{\lambda'})} r_v T_v \in \mathscr{H}$, for some $r_v \in R$, such that

$$z_\lambda \Psi_{\mathfrak{t}_\lambda}^* = q^{n-\alpha(\lambda)+\ell(w_{\lambda'})} h_\lambda(q) E_\lambda,$$

where E_λ is a primitive idempotent such that $E_\lambda \mathscr{H} = z_\lambda \mathscr{H} = S(\lambda)$. In particular $E_\lambda z_\lambda = z_\lambda$. (Note that $z_\lambda = z_{\lambda t}$ in Murphy's notation; see [11, p. 496, p. 498].)

Note that $T_{w_\lambda}^{-1} = q^{-\ell(w_\lambda)} T_{w_{\lambda'}} + \sum_{\ell(v)<\ell(w_{\lambda'})} r_v T_v$, for some $r_v \in R$. Now, if $\ell(v) < \ell(w_{\lambda'})$ then $v \notin \mathfrak{S}_{\lambda'} w_{\lambda'} \mathfrak{S}_\lambda$, so $y_{\lambda'} T_v x_\lambda = (x_\lambda T_v^* y_{\lambda'})^* = 0$ by 5.4. Consequently, $z_\lambda T_v z_\lambda = 0$. Therefore,

$$\begin{aligned} z_\lambda T_{w_\lambda}^{-1} z_\lambda &= q^{-\ell(w_\lambda)} z_\lambda T_{w_{\lambda'}} z_\lambda \\ &= q^{-\ell(w_\lambda)} z_\lambda \Psi_{\mathfrak{t}_\lambda}^* z_\lambda \end{aligned}$$

$$= q^{n-\alpha(\lambda)} h_\lambda(q) E_\lambda z_\lambda$$
$$= q^{n-\alpha(\lambda)} h_\lambda(q) z_\lambda. \quad \square$$

Consider the $\mathscr{H}$-linear map

$$S(\lambda) \xrightarrow{\gamma_\lambda} S(\lambda)^*; \qquad \xi \longmapsto \langle \xi, - \rangle_\lambda.$$

Lemma 5.8. *The composition*

$$S(\lambda) \xrightarrow{\gamma_\lambda} S(\lambda)^* \xrightarrow{\psi_\lambda^{-1}}_{\sim} S(\lambda')^{\#} \xrightarrow{\gamma_{\lambda'}^{\#}} S(\lambda')^{*,\#} \xrightarrow{(\psi_{\lambda'}^{\#})^{-1}}_{\sim} S(\lambda)$$

is equal to scalar multiplication by $(-q)^{\ell(w_{\lambda'})} q^{n-\alpha(\lambda)-\alpha(\lambda')} h_\lambda(q)$.

Proof. The element z_λ is mapped via γ_λ to $\langle z_\lambda, - \rangle_\lambda$, which is mapped via ψ_λ^{-1} to $z_{\lambda'} \cdot_{\#} T_{w_\lambda} y_{\lambda'} = z_{\lambda'} T_{w_\lambda}^{\#} y_{\lambda'}^{\#}$, which in turn goes to $\langle z_{\lambda'} T_{w_\lambda}^{\#} y_{\lambda'}^{\#}, - \rangle_\lambda$ via $\gamma_{\lambda'}^{\#}$, and finally to

$$z_\lambda \cdot_{\#} T_{w_{\lambda'}} y_\lambda T_{w_\lambda}^{\#} y_{\lambda'}^{\#} = (-q)^{\ell(w_{\lambda'})} q^{-\alpha(\lambda')} z_\lambda T_{w_\lambda}^{-1} z_\lambda$$
$$= (-q)^{\ell(w_{\lambda'})} q^{n-\alpha(\lambda)-\alpha(\lambda')} h_\lambda(q) z_\lambda$$

via $(\psi_{\lambda'}^{\#})^{-1}$, by Lemma 5.7. $\square$

Let I_m be the $m \times m$ identity matrix. Recall that $n_\lambda = \#\mathrm{Std}(\lambda)$ is the dimension of the Specht module $S(\lambda)$.

Proposition 5.9. *Suppose that* λ *is a partition of* n.

(1) *There exist invertible matrices* $A, B \in \mathrm{GL}_{n_\lambda}(\mathbb{Z}[q, q^{-1}])$ *such that*

$$G(\lambda) \cdot A \cdot G(\lambda') \cdot B = h_\lambda(q) \cdot I_{n_\lambda}.$$

(2) $G(\lambda)$ *is diagonalizable to the diagonal matrix* D *if and only if* $G(\lambda')$ *is diagonalizable to the diagonal matrix* $h_\lambda(q) D^{-1}$.

(3) $G(\lambda)$ *is divisibly diagonalizable if and only if* $G(\lambda')$ *is divisibly diagonalizable. In this case, the product of the* i^{th} *elementary divisor of* $G(\lambda)$ *and the* $(n_\lambda + 1 - i)^{th}$ *elementary divisor of* $G(\lambda')$ *is equal to* $h_\lambda(q)$.

Recall that elementary divisors are only well defined up to a unit in $\mathbb{Z}[q, q^{-1}]$; the same is true of their product in (3).

Proof. (1) The R-linear map γ_λ is represented by the matrix $G(\lambda)$ with respect to the Dipper–James basis and its dual basis. Thus the assertion follows by Lemma 5.8.

(2) If $G(\lambda) = SDT$ with $S, T \in \mathrm{GL}_{n_\lambda}(\mathbb{Z}[q, q^{-1}])$ and $D \in \mathbb{Z}[q, q^{-1}]^{n_\lambda \times n_\lambda}$ is a diagonal matrix, then $G(\lambda') = A^{-1} T^{-1} (h_\lambda(q) D^{-1}) S^{-1} B^{-1}$. Since $G(\lambda')$ has coefficients in $\mathbb{Z}[q, q^{-1}]$, so does $h_\lambda(q) D^{-1}$.

(3) Repeat the argument of (2). $\square$

We remark that all of the results in this section hold more generally when the Hecke algebra $\mathscr{H}$ is defined over an integral domain R such that $\mathscr{H} \otimes_R Q$ is semisimple, where Q is the field of fractions of R. (We need semisimplicity over Q only when we apply Murphy's results in the proof of Lemma 5.7.) In particular, Proposition 5.9 holds when $R = F[q, q^{-1}]$ and F is any field. Notice that $G(\lambda)$ is always diagonalizable in this case because $F[q, q^{-1}]$ is a principal ideal domain.

6. The elementary divisors for hook partitions

Throughout this section we fix an integer k, with $0 \leq k < n$, and consider the Specht module $S(\lambda)$, where $\lambda = (n-k, 1^k)$. We will show that $G(\lambda)$ is divisibly diagonalizable by explicitly constructing two bases of $S(\lambda)$ which transform $G(\lambda)$ an upper triangular matrix satisfying the requirements of Lemma 4.1. In particular, this will allow us to determine the elementary divisors of $S(\lambda)$.

The Specht module $S(\lambda)$ is defined as a submodule of the permutation module $M(\lambda)$; however, to compute the elementary divisors we will work inside a different permutation module.

By definition, $S(\lambda) = x_\lambda T_{w_\lambda} y_{\lambda'} \mathscr{H} = x_{(n-k,1^k)} T_{w_{(n-k,1^k)}} y_{(k+1,1^{n-k-1})}$. We first need to understand the permutation $w_\lambda = w_{(n-k,1^k)}$ a little better. This requires some new notation. For integers non-negative i and j define

$$r_{i,j} = \begin{cases} 1, & \text{if } i = 0 \text{ or } j = 0, \\ r_i r_{i+1} \cdots r_j, & \text{if } 0 < i \leq j, \\ r_i r_{i-1} \cdots r_j, & \text{if } i > j > 0, \end{cases}$$

and set $T_{i,j} = T_{r_{i,j}}$. Next, let a and b be non-negative integers such $a + b \leq n$. If either $a = 0$ or $b = 0$ then set $w_{a,b} = 1$. If both a and b are non-zero then define $w_{a,b} = (r_{a+b-1,1})^b$; then one can check that, in two-line notation,

$$w_{a,b} = \begin{pmatrix} 1 & 2 & \cdots & a \\ b+1 & b+2 & \cdots & a+b \end{pmatrix} \begin{pmatrix} a+1 & a+2 & \cdots & a+b \\ 1 & 2 & \cdots & b \end{pmatrix}.$$

It is not hard to see that $w_{b,a} = w_{a,b}^{-1}$ and that $w_{a,b} = r_{a,a+b-1} w_{a-1,b}$ and $\ell(w_{a,b}) = \ell(r_{a,a+b-1}) + \ell(w_{a-1,b})$; see [4]. Consequently,

$$w_{a,b} = r_{a,a+b-1} r_{a-1,a+b-2} \cdots r_{1,b} = r_{1,a} r_{2,a+2} \cdots r_{b,a+b-1}$$

with the lengths adding in both cases. Hence, $\ell(w_{a,b}) = ab$.

The permutation $w_{(n-k,1^k)}$ is essentially one of these permutations because

$$w_{(n-k,1^k)} = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ 1 & k+2 & \cdots & n & 2 & \cdots & k+1 \end{pmatrix}.$$

Hence, $w_{(n-k,1^k)} = r_{n-k,n-1} r_{n-k-1,n-2} \cdots r_{2,k+1}$, with the lengths adding. So

$$T_{w_{(n-k,1^k)}} = T_{n-k,n-1} \cdots T_{2,k+1},$$

Notice also that $T_{w_{n-k.k}} = T_{w_{(n-k,1^k)}} T_{1,k}$.

If $w \in \mathfrak{S}_{(k,n-k)} \cong \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ then we write $w = (u, v)$, where $u \in \mathfrak{S}_k$ and $v \in \mathfrak{S}_{(1^k,n-k)}$ are the unique permutations such that $w = uv = vu$. Set

$$x_{(k|n-k)} = y_{(k,1^{n-k})} x_{(1^k,n-k)} = \sum_{(u,v)\in\mathfrak{S}_{(k,n-k)}} (-q)^{-\ell(u)} T_{uv}.$$

Then it is easy to see that $Rx_{(k|n-k)}$ is an $\mathscr{H}(\mathfrak{S}_\mu)$-module on which the subalgebras $\mathscr{H}(\mathfrak{S}_k)$ and $\mathscr{H}(\mathfrak{S}_{(1^k,n-k)})$ act via their sign and trivial representations, respectively. Let

$$M(k|n-k) = \operatorname{Ind}_{\mathscr{H}(\mathfrak{S}_{(k,n-k)})}^{\mathscr{H}(\mathfrak{S}_n)} (Rx_{(k|n-k)}) \cong x_{(k|n-k)} \mathscr{H}.$$

As in Section 2, the induced module $M(k|n-k)$ is free as an R-module with basis $\{x_{(k|n-k)}T_d \mid d \in \mathcal{D}_{(k,n-k)}\}$. Furthermore, $M(k|n-k)$ possesses a natural non-degenerate associative bilinear form $\langle , \rangle_{(k|n-k)}$ which is determined by

$$\langle x_{(k|n-k)}T_u, x_{(k|n-k)}T_v\rangle_{(k|n-k)} = \begin{cases} q^{\ell(u)}, & \text{if } u = v, \\ 0, & \text{otherwise,} \end{cases}$$

for $u, v \in \mathcal{D}_{(k,n-k)}$. Donkin [5] calls $M(k|n-k)$ a trivial source module.

Let $y'_{k+1} = 1 + \sum_{j=1}^{k}(-q)^{j-k-1}T_{k,j} = 1 - q^{-1}T_k + q^{-2}T_{k,k-1} + \cdots + (-q)^{-k}T_{k,1}$. This is a sum over the right coset representatives of $\mathfrak{S}_k$ in $\mathfrak{S}_{k+1}$. Consequently, it follows that $y_{(k+1,1^{n-k-1})} = y_{(k,1^{n-k})}y'_{k+1}$. The reason for introducing the module $M(k|n-k)$ is the following result.

Given a non-negative integer $k > 1$ let $[k]_q = 1 + q + \cdots + q^{k-1}$ and $[k]_q^! = [1]_q[2]_q \cdots [k]_q$. Notice that if $q = 1$ then $[k]_1 = k$ and $[k]_1^! = k!$.

Proposition 6.1. *Let $\lambda = (n-k, 1^k)$. The map*

$$\pi_k : S(\lambda) \longrightarrow M(k|n-k); z_\lambda h \longmapsto x_{(k|n-k)}y'^{}_{k+1}h$$

is an injective $\mathscr{H}$-module homomorphism. Moreover,

$$\langle x, y\rangle_\lambda = q^{\frac{k}{2}(2n-3k-1)}[k]_q^!\langle \pi(x), \pi(y)\rangle_{(k|n-k)},$$

for all $x, y \in S(\lambda)$.

Proof. By definition, $S(\lambda) = x_{(n-k,1^k)}T_{w_\lambda}y_{(k+1,1^{n-k-1})}\mathscr{H}$. As remarked above, $w_{n-k,k} = w_{(n-k,1^k)}r_{1,k}$ with the lengths adding. Therefore, since $r_{1,k} \in \mathfrak{S}_{\lambda'}$,

$$\begin{aligned} x_{(n-k,1^k)}T_{w_{(n-k,1^k)}}y_{(k+1,1^{n-k-1})} &= (-1)^k x_{(n-k,1^k)}T_{w_{(n-k,1^k)}}T_{1,k}y_{(k+1,1^{n-k-1})} \\ &= (-1)^k x_{(n-k,1^k)}T_{w_{n-k,k}}y_{(k+1,1^{n-k-1})} \\ &= (-1)^k T_{w_{n-k,k}}x_{(1^k,n-k)}y_{(k+1,1^{n-k-1})} \\ &= (-1)^k T_{w_{n-k,k}}x_{(1^k,n-k)}y_{(k,1^{n-k})}y'_{k+1} \\ &= (-1)^k T_{w_{n-k,k}}x_{(k|n-k)}y'_{k+1}. \end{aligned}$$

Therefore, $\pi(x) = (-1)^k T^{-1}_{w_{n-k,k}}x$, for all $x \in S(\lambda)$. As $T_{w_{n-k,k}}$ is invertible, the first claim now follows.

To prove the second claim we first suppose that $R = \mathbb{Z}[q, q^{-1}]$. If $x, y \in S(\lambda)$ then, by extending scalars, we may assume that x and y are elements of $S(\lambda)_{\mathbb{Q}(q)} = S(\lambda)_{\mathbb{Z}[q,q^{-1}]} \otimes \mathbb{Q}(q)$. Now $S(\lambda)_{\mathbb{Q}(q)} \cong \pi(S(\lambda)_{\mathbb{Q}(q)})$ is irreducible so, up to a scalar, there is a unique associative bilinear form on $S(\lambda)_{\mathbb{Q}(q)}$. To determine this scalar it is enough to compare the two inner products on z_λ and $\pi(z_\lambda)$. Using associativity,

$$\begin{aligned} \langle z_\lambda, z_\lambda\rangle_\lambda &= \langle x_\lambda T_{w_\lambda}y_{\lambda'}, x_\lambda T_{w_\lambda}y_{\lambda'}\rangle_\lambda = \langle x_\lambda T_{w_\lambda}y^2_{\lambda'}, x_\lambda T_{w_\lambda}\rangle_\lambda \\ &= q^{-\binom{k+1}{2}}[k+1]_q^!\langle x_\lambda T_{w_\lambda}y_{\lambda'}, x_\lambda T_{w_\lambda}\rangle_\lambda \\ &= q^{-\binom{k+1}{2}}[k+1]_q^!\sum_{v\in\mathfrak{S}_{\lambda'}}(-q)^{-\ell(v)}\langle x_\lambda T_{w_\lambda v}, x_\lambda T_{w_\lambda}\rangle_\lambda \end{aligned}$$

$$= q^{\ell(w_\lambda)-\binom{k+1}{2}}[k+1]_q^!$$
$$= q^{\frac{k}{2}(2n-3k-3)}[k+1]_q^!.$$

Similarly,

$$\langle \pi(z_\lambda), \pi(z_\lambda)\rangle_{(k|n-k)} = \langle x_{(k|n-k)}y'_{k+1}, x_{(k|n-k)}y'_{k+1}\rangle_{(k|n-k)} = q^{-k}[k+1]_q.$$

This proves that $\langle x, y\rangle_\lambda = q^{\frac{k}{2}(2n-3k-1)}[k]_q^!\langle \pi(x), \pi(y)\rangle_{(k|n-k)}$, for all $x, y \in S(\lambda)$ when $R = \mathbb{Z}[q, q^{-1}]$. The general case now follows by specialization. □

Corollary 6.2. *Suppose that* $\lambda = (n-k, 1^k)$. *Then* $[k]_q^!$ *divides* $\langle x, y\rangle_\lambda$, *for all* $x, y \in S(\lambda)$.

Let $S'(\lambda) = \pi(S(\lambda)) = x_{(k|n-k)}y'_{k+1}\mathscr{H}$. Then $S'(\lambda) \cong S(\lambda)$ by the Proposition. We will work with $S'(\lambda)$ in what follows rather than working with $S(\lambda)$ directly.

As a first step we need a basis of $S'(\lambda)$. For *any* λ-tableau $\mathfrak{t}$ define

$$v'_\mathfrak{t} = \pi(v_\mathfrak{t}) = x_{(k|n-k)}y'_{k+1}T_{d(\mathfrak{t})}.$$

The Dipper–James basis of $S(\lambda)$, 3.1, combined with Proposition 6.1, give us the following.

Corollary 6.3. *The module* $S'(\lambda)$ *is R-free with basis* $\{v'_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$.

In order to exploit this basis we introduce another type of tableaux. For our purposes we could get by using $(k, n-k)$-tableaux; however, we use the notation from the theory of trivial source modules.

The diagram of $(k|n-k)$ is the ordered pair of diagrams $[k|n-k] = ([k], [n-k])$. A $(k|n-k)$-tableau is a bijection from $[k|n-k]$ to $\{1, 2, \ldots, n\}$. Once again, we will think of a $(k|n-k)$-tableau as being a labelling of $[k|n-k]$. Accordingly, we will write a $(k|n-k)$-tableau as an ordered pair $(\mathfrak{a}|\mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are suitable labellings of the diagrams of the partitions (k) and $(n-k)$ respectively. We refer to $\mathfrak{a}$ and $\mathfrak{b}$ as the first and second components of $(\mathfrak{a}|\mathfrak{b})$.

A $(k|n-k)$-tableau $(\mathfrak{a}|\mathfrak{b})$ is (row) standard if the entries in $\mathfrak{a}$ increase from left to right and the entries in $\mathfrak{b}$ increase from left to right. Let $\mathrm{Std}(k|n-k)$ be the set of standard $(k|n-k)$-tableaux. For example, the standard $(1|3)$-tableaux are

$$\left(\begin{array}{|c|}\hline 1\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}\right),\quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array}\right),\quad \left(\begin{array}{|c|}\hline 3\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\right) \text{ and } \left(\begin{array}{|c|}\hline 4\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right).$$

Let $\mathfrak{t}^{(k|n-k)}$ be the standard $(k|n-k)$-tableau with $1, \ldots, k$ entered in order, from left to right, in the first component and the numbers $k+1, \ldots, n$ in the second. The first of the tableaux above is $\mathfrak{t}^{(1|3)}$.

Two $(k|n-k)$-tableaux $(\mathfrak{a}|\mathfrak{b})$ and $(\mathfrak{s}|\mathfrak{t})$ are row equivalent if $\mathfrak{a}$ and $\mathfrak{s}$ contain the same entries up to reordering (in which case, $\mathfrak{b}$ and $\mathfrak{t}$ also contain the same set of entries). As with ordinary tableaux, the symmetric group acts from the right on the set of $(k|n-k)$-tableaux. If $(\mathfrak{a}|\mathfrak{b})$ is a $(k|n-k)$-tableau we define $d(\mathfrak{a}|\mathfrak{b})$ to be the unique permutation such that $(\mathfrak{a}|\mathfrak{b}) = \mathfrak{t}^{(k|n-k)}d(\mathfrak{a}|\mathfrak{b})$. Then $(\mathfrak{a}|\mathfrak{b})$ and $(\mathfrak{s}|\mathfrak{t})$ are row equivalent if and only if $d(\mathfrak{a}|\mathfrak{b}) = wd(\mathfrak{s}|\mathfrak{t})$ for some $w \in \mathfrak{S}_{(k,n-k)}$. Consequently,

$$\mathcal{D}_{(k,n-k)} = \{d(\mathfrak{a}|\mathfrak{b}) \mid (\mathfrak{a}|\mathfrak{b}) \in \mathrm{Std}(k|n-k)\}.$$

So the standard $(k|n-k)$-tableaux index a basis of $M(k|n-k)$.

For future reference, notice that if $(\mathfrak{a}|\mathfrak{b})$ is a standard $(k|n-k)$-tableau then $\ell(d(\mathfrak{a}|\mathfrak{b}))$ is equal to the number of pairs of integers (i, j) where i appears in $\mathfrak{a}$, j appears in $\mathfrak{b}$ and $i > j$. This follows because if $w \in \mathfrak{S}_n$ then $\ell(w)$ is equal to the number of pairs $a < b$ with $i = a^w > j = b^w$, and the entries in $\mathfrak{a}$ are the images of $1, \dots, k$ under $d(\mathfrak{a}|\mathfrak{b})$, whereas the entries in $\mathfrak{b}$ are the images of $k+1, \dots, n$.

For any $(k|n-k)$-tableau $(\mathfrak{a}|\mathfrak{b})$ define $x_{(\mathfrak{a}|\mathfrak{b})} = x_{(k|n-k)}T_{d(\mathfrak{a}|\mathfrak{b})}$. Here we do not assume that $(\mathfrak{a}|\mathfrak{b})$ is standard. The following lemma is easily verified.

Lemma 6.4. *Suppose that* $0 \le k < n$.

(i) *$M(k|n-k)$ is free as an R-module with basis* $\{x_{(\mathfrak{a}|\mathfrak{b})} | (\mathfrak{a}|\mathfrak{b}) \in \mathrm{Std}(k|n-k)\}$.
(ii) *Suppose that* $(\mathfrak{a}|\mathfrak{b}) \in \mathrm{Std}(k|n-k)$ *and* $1 \le i < n$. *Then*

$$x_{(\mathfrak{a}|\mathfrak{b})}T_i = \begin{cases} -x_{(\mathfrak{a}|\mathfrak{b})}, & \text{if } i \text{ and } i+1 \text{ are both contained in } \mathfrak{a}, \\ qx_{(\mathfrak{a}|\mathfrak{b})}, & \text{if } i \text{ and } i+1 \text{ are both contained in } \mathfrak{b}, \\ x_{(\mathfrak{a}_i|\mathfrak{b}_i)}, & \text{if } i \text{ is in } \mathfrak{a} \text{ and } i+1 \text{ is in } \mathfrak{b}, \\ qx_{(\mathfrak{a}_i|\mathfrak{b}_i)} + (q-1)x_{(\mathfrak{a}|\mathfrak{b})}, & \text{otherwise,} \end{cases}$$

where $(\mathfrak{a}_i|\mathfrak{b}_i) = (\mathfrak{a}|\mathfrak{b})r_i$.

The action of $\mathscr{H}$ on $M(k|n-k)$ is completely determined by (ii).

We now show how to write the basis $\{v'_{\mathfrak{t}}\}$ of $S'(\lambda)$ in terms of this basis of $M(k|n-k)$. To do this, if $\mathfrak{t}$ is a λ-tableau and $(\mathfrak{a}|\mathfrak{b})$ is a $(k|n-k)$-tableau write $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}$ if $(\mathfrak{a}|\mathfrak{b})$ is standard and all of the entries in $\mathfrak{a}$ are contained in the first column of $\mathfrak{t}$. Finally, if $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}$ we set $I_{\mathfrak{t}}(\mathfrak{a}|\mathfrak{b}) = i$, the index of $(\mathfrak{a}|\mathfrak{b})$ in $\mathfrak{t}$, if the number in row i of $\mathfrak{t}$ does not appear in $\mathfrak{a}$.

Lemma 6.5. *Suppose that* $\mathfrak{t}$ *is a standard* λ*-tableau. Then*

$$v'_{\mathfrak{t}} = \sum_{(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}} (-1)^{k+1-I_{\mathfrak{t}}(\mathfrak{a}|\mathfrak{b})} q^{\ell(d(\mathfrak{t}'))-\ell(d(\mathfrak{a}|\mathfrak{b}))} x_{(\mathfrak{a}|\mathfrak{b})}.$$

Proof. First consider $v_{\mathfrak{t}_\lambda}$. Looking at the definitions we see that

$$\begin{aligned} v'_{\mathfrak{t}_\lambda} &= x_{(k|n-k)}y'_{k+1} = x_{\mathfrak{t}^{(k|n-k)}}(1 - q^{-1}T_k + q^{-2}T_{k,k-1} - \cdots + (-q)^{-k}T_{k,1}) \\ &= \sum_{(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}_\lambda} (-q)^{-\ell(d(\mathfrak{a}|\mathfrak{b}))} x_{(\mathfrak{a}|\mathfrak{b})}. \end{aligned}$$

As $\ell(d(\mathfrak{t}_\lambda)') = \ell(d(\mathfrak{t}^{\lambda'})) = 0$ and $\ell(d(\mathfrak{a}|\mathfrak{b})) = k + 1 - I_{\mathfrak{t}_\lambda}(\mathfrak{a}|\mathfrak{b})$, when $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}_\lambda$, the Lemma follows in this case.

Now suppose that $\mathfrak{t}$ is an arbitrary standard λ-tableaux. If $\mathfrak{t} \ne \mathfrak{t}_\lambda$ then we can find another standard λ-tableau $\mathfrak{s}$ and an integer i in the first column of $\mathfrak{s}$ such that $\mathfrak{t} = \mathfrak{s}r_i$ and $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{s})) - 1$. (That is, $\mathfrak{t} \trianglerighteq \mathfrak{s}$ where $\trianglerighteq$ is the dominance order on tableaux; see, for example, [10].) Therefore, by induction,

$$v'_{\mathfrak{t}} = v'_{\mathfrak{s}}T_i = \sum_{(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{s}} (-1)^{k+1-I_{\mathfrak{s}}(\mathfrak{a}|\mathfrak{b})} q^{\ell(d(\mathfrak{s}'))-\ell(d(\mathfrak{a}|\mathfrak{b}))} x_{(\mathfrak{a}|\mathfrak{b})}T_i.$$

Since $\mathfrak{s}$ and $\mathfrak{t}$ are standard, i is in the first column of $\mathfrak{s}$ and the first row of $\mathfrak{t}$ and $i+1$ is in the first row of $\mathfrak{s}$ and the first column of $\mathfrak{t}$. Therefore, if $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{s}$ then the entries in the first component of $(\mathfrak{a}|\mathfrak{b})r_i$ are still in increasing order and the entries in the second component

are in increasing order unless i and $i+1$ both appear in $\mathfrak{b}$. So, $\ell(d(\mathfrak{a}|\mathfrak{b})r_i) = \ell(d(\mathfrak{a}|\mathfrak{b}))+1$ and by Lemma 6.4(ii) we have

$$x_{(\mathfrak{a}|\mathfrak{b})}T_i = \begin{cases} qx_{(\mathfrak{a}|\mathfrak{b})}, & \text{if } i \text{ and } i+1 \text{ both appear in } \mathfrak{b}, \\ x_{(\mathfrak{a}|\mathfrak{b})r_i}, & \text{otherwise.} \end{cases}$$

In the first case, when i and $i+1$ both appear in $\mathfrak{b}$, we have that $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}$. Also, $\ell(d(\mathfrak{t}')) - \ell(d(\mathfrak{a}|\mathfrak{b})) = \ell(d(\mathfrak{s}')) - \ell(d(\mathfrak{a}|\mathfrak{b})) + 1$ and $I_{\mathfrak{t}}(\mathfrak{a}|\mathfrak{b}) = I_{\mathfrak{s}}(\mathfrak{a}|\mathfrak{b})$ so $x_{(\mathfrak{a}|\mathfrak{b})}$ has the required coefficient in $v'_{\mathfrak{t}}$.

In the second case, i appears in $\mathfrak{a}$ and $i+1$ appears in $\mathfrak{b}$, so $(\mathfrak{a}r_i|\mathfrak{b}r_i) = (\mathfrak{a}|\mathfrak{b})r_i \prec \mathfrak{t}$, $I_{\mathfrak{t}}(\mathfrak{a}r_i|\mathfrak{b}r_i) = I_{\mathfrak{s}}(\mathfrak{a}|\mathfrak{b})$ and $\ell(d(\mathfrak{t}')) - \ell(d(\mathfrak{a}|\mathfrak{b})r_i) = \ell(d(\mathfrak{s}')) - \ell(d(\mathfrak{a}|\mathfrak{b}))$. Hence, once again, $x_{(\mathfrak{a}|\mathfrak{b})r_i}$ has the predicted coefficient in $v'_{\mathfrak{t}}$.

As there are exactly k standard $(k|n-k)$-tableaux $(\mathfrak{a}|\mathfrak{b})$ satisfying $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}$, this completes the proof. □

In order to compute the elementary divisors of $S(\lambda)$ we need a second basis of $S'(\lambda)$. Let

$$x_{n-k} = 1 + T_1 + \cdots + T_{1,n-k-1} = \sum_{j=0}^{n-k-1} T_{1,j}.$$

(Note that $r_{1,0} = 1$.) As with y'_{k+1}, we have $x_{(n-k,1^k)} = x_{(1,n-k-1,1^{k-1})}x_{n-k}$. Now, for any standard $(n-k, 1^k)$-tableau $\mathfrak{t}$ we define

$$w'_{\mathfrak{t}} = \begin{cases} v'_{\mathfrak{t}(1,n)}, & \text{if } n \text{ appears in row 1 of } \mathfrak{t}, \\ v'_{\mathfrak{t}^\lambda}x_{n-k}T_{d(\mathfrak{t})}, & \text{otherwise.} \end{cases}$$

We remark that it is not obvious that the set of elements $\{w'_{\mathfrak{t}} | \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a basis of $S'(\lambda)$. We will prove this below.

Lemma 6.5 gives an explicit description of the basis $\{v'_{\mathfrak{t}}\}$. We need to do the same for the basis $\{w'_{\mathfrak{t}}\}$, and for this we need some more notation. If $\mathfrak{t}$ is a standard λ-tableau let $\mathfrak{t}^* = \mathfrak{t}(1,n)$. If $(\mathfrak{a}|\mathfrak{b})$ is a $(k|n-k)$-tableau write $(\mathfrak{a}|\mathfrak{b}) \prec_n \mathfrak{t}$ if $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}$ and n is contained in $\mathfrak{a}$. Finally, if n appears in the first row of $\mathfrak{t}$ then we define $(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})$ to be the unique standard $(k|n-k)$-tableau such that $(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}}) \prec \mathfrak{t}^*$ and 1 appears in $\mathfrak{b}^*_{\mathfrak{t}}$. So $(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}}) \prec \mathfrak{t}^*$ and $(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}}) \prec \mathfrak{t}$.

Lemma 6.6. *Suppose that $\mathfrak{t}$ is a standard λ-tableau and that n appears in the first row of $\mathfrak{t}$. Then*

$$w'_{\mathfrak{t}} = (-1)^k q^{2n-2k-3} x_{(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})} + \sum_{(\mathfrak{a}|\mathfrak{b}) \prec_n \mathfrak{t}^*} r_{\mathfrak{a}\mathfrak{b}} x_{(\mathfrak{a}|\mathfrak{b})},$$

for some scalars $r_{\mathfrak{a}\mathfrak{b}} \in \mathbb{Z}[q, q^{-1}]$.

Proof. We now argue by downwards induction on $\mathfrak{t}$ beginning with $\mathfrak{t} = \mathfrak{t}_\lambda$, this is an unpleasant calculation. Now,

$$w'_{\mathfrak{t}_\lambda} = x_{(k|n-k)}y'_{k+1}T_{1,n-1}T_{n-2,1} = (-1)^k x_{(k|n-k)}y'_{k+1}T_{k+1,n-1}T_{n-2,1}$$

since $x_{(k|n-k)}y'_{k+1} = x_{(1^k,n-k)}y_{(k+1,1^{n-k-1})}$. Therefore, using the definitions together with the braid relations,

$$\begin{aligned}
w'_{\mathfrak{t}_\lambda} &= (-1)^k x_{(k|n-k)} y'_{k+1} T_{n-1}\cdots T_{k+2}T_{k+1,n-1}T_{k,1}\\
&= (-1)^k x_{(k|n-k)} T_{n-1}\cdots T_{k+2} y'_{k+1} T_{k+1,n-1}T_{k,1}\\
&= (-1)^k q^{n-k-2} x_{(k|n-k)} y'_{k+1} T_{k+1,n-1}T_{k,1}\\
&= (-1)^k q^{n-k-2} x_{(k|n-k)} \left\{1 + \sum_{j=1}^{k} (-q)^{j-k-1} T_{k,j}\right\} T_{k+1,n-1}T_{k,1}\\
&= (-1)^k q^{n-k-2} x_{(k|n-k)} \left\{q^{n-k-1} + \sum_{j=1}^{k} (-q)^{j-k-1} T_{k,j}T_{k+1,n-1}\right\} T_{k,1}\\
&= (-1)^k q^{n-2k-3} x_{(k|n-k)} \left\{q^{n} T_{k,1} + \sum_{j=1}^{k} (-q)^{j} T_{k,j}T_{k+1,1}T_{k+2,n-1}\right\}\\
&= (-1)^k q^{n-2k-3} x_{(k|n-k)} \left\{q^{n} T_{k,1} + \sum_{j=1}^{k} (-q)^{j} T_{k+1,1}T_{k+1,j+1}T_{k+2,n-1}\right\}\\
&= (-1)^k q^{n-2k-3} x_{(k|n-k)} \left\{q^{n} T_{k,1} - \sum_{j=1}^{k} (-q)^{j+1} T_{k,1}T_{k+1,j+1}T_{k+2,n-1}\right\}\\
&= (-1)^k q^{n-2k-3} x_{(k|n-k)} T_{k,1} \left\{q^{n} - \sum_{j=2}^{k+1} (-q)^{j} T_{k+1,j}T_{k+2,n-1}\right\}.
\end{aligned}$$

Now, $\mathfrak{t}^{(k|n-k)}r_{k,1} = \left(\begin{array}{|c|c|c|}\hline 2&\cdots&k+1\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|}\hline 1&k+2&\cdots&n\\\hline\end{array}\right) = (\mathfrak{a}^*_{\mathfrak{t}_\lambda}|\mathfrak{b}^*_{\mathfrak{t}_\lambda})$ and, consequently, $\mathfrak{t}^{(k|n-k)}r_{k,1}$ $r_{k+1,j}r_{k+2,n-1} = \left(\begin{array}{|c|c|c|c|c|}\hline 2&\cdots&j-1&j+1&\cdots&k+1&n\\\hline\end{array}\,\middle|\,\begin{array}{|c|c|c|c|c|}\hline 1&j&k+2&\cdots&n-1\\\hline\end{array}\right)$, for $j = 2,\dots,k+1$. This completes the proof for $w'_{\mathfrak{t}_\lambda}$.

Now suppose that $\mathfrak{t}$ is an arbitrary standard λ-tableau which has n in its first row. Then $d(\mathfrak{t}') \in \mathfrak{S}_{(1,n-2,1)}$ so $d(\mathfrak{t}')$ and $(1,n)$ commute and $\ell(d(\mathfrak{t}')(1,n)) = \ell(d(\mathfrak{t}')) + \ell(1,n)$. Therefore, $w'_{\mathfrak{t}} = w'_{\mathfrak{t}_\lambda}T_{d(\mathfrak{t})}$. To complete the proof now argue by induction, as in the proof of Lemma 6.5; we leave the details to the reader. (Indeed, this shows that $r_{\mathfrak{ab}} = \pm q^a$ for some integer a.) □

For convenience we now write $\langle,\rangle = \langle,\rangle_{(k|n-k)}$. In terms of the standard basis of $M(k|n-k)$, the bilinear form $\langle,\rangle$ on $M(k|n-k)$ is determined by

$$\langle x_{(\mathfrak{a}|\mathfrak{b})}, x_{(\mathfrak{s}|\mathfrak{t})}\rangle = \begin{cases} q^{\ell(d(\mathfrak{a}|\mathfrak{b}))}, & \text{if } (\mathfrak{a}|\mathfrak{b}) = (\mathfrak{s}|\mathfrak{t}),\\ 0, & \text{otherwise,}\end{cases}$$

for standard $(k|n-k)$-tableaux $(\mathfrak{a}|\mathfrak{b})$ and $(\mathfrak{s}|\mathfrak{t})$.

Corollary 6.7. *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ-tableaux which have n in their first row. Then*

$$\langle w'_{\mathfrak{t}}, v'_{\mathfrak{s}}\rangle = \begin{cases} q^{2n-2k-3+\ell(d(\mathfrak{t}'))}, & \text{if } \mathfrak{s}=\mathfrak{t},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By Lemmas 6.5 and 6.6 we have

$$v'_{\mathfrak{s}} = \sum_{(\mathfrak{a}_{\mathfrak{s}}|\mathfrak{b}_{\mathfrak{s}})\prec\mathfrak{s}} (-1)^{k+1-I_{\mathfrak{s}}(\mathfrak{a}_{\mathfrak{s}}|\mathfrak{b}_{\mathfrak{s}})} q^{\ell(d(\mathfrak{s}'))-\ell(d(\mathfrak{a}_{\mathfrak{s}}|\mathfrak{b}_{\mathfrak{s}}))} x_{(\mathfrak{a}_{\mathfrak{s}}|\mathfrak{b}_{\mathfrak{s}})}$$

and, by Lemma 6.6,

$$w'_{\mathfrak{t}} = (-1)^k q^{2n-2k-3} x_{(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})} + \sum_{(\mathfrak{a}_{\mathfrak{t}}|\mathfrak{b}_{\mathfrak{t}})\prec_n \mathfrak{t}^*} r_{\mathfrak{a}_{\mathfrak{t}}\mathfrak{b}_{\mathfrak{t}}} x_{(\mathfrak{a}_{\mathfrak{t}}|\mathfrak{b}_{\mathfrak{t}})}.$$

Now, all of the tableaux appearing in $w'_{\mathfrak{t}}$ have 1 appearing in their second component. In contrast, the only tableau in $v'_{\mathfrak{s}}$ which has 1 in its second component is the tableau $(\mathfrak{a}^*_{\mathfrak{s}}|\mathfrak{b}^*_{\mathfrak{s}})$. Therefore,

$$\begin{aligned}\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle &= (-1)^{2k+1-I_{\mathfrak{s}}(\mathfrak{a}^*_{\mathfrak{s}}|\mathfrak{b}^*_{\mathfrak{s}})} q^{2n-2k-3+\ell(d(\mathfrak{s}'))-\ell(d(\mathfrak{a}^*_{\mathfrak{s}}|\mathfrak{b}^*_{\mathfrak{s}}))} \langle x_{(\mathfrak{a}^*_{\mathfrak{s}}|\mathfrak{b}^*_{\mathfrak{s}})}, x_{(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})}\rangle\\ &= \begin{cases}(-1)^{1+I_{\mathfrak{t}}(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})} q^{2n-2k-3+\ell(d(\mathfrak{t}'))}, & \text{if } \mathfrak{s}=\mathfrak{t},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Finally, the sign vanishes when $\mathfrak{s}=\mathfrak{t}$ because $I_{\mathfrak{t}}(\mathfrak{a}^*_{\mathfrak{t}}|\mathfrak{b}^*_{\mathfrak{t}})=1$. □

We need one more result before we can produce the elementary divisors of $S(\lambda)$.

Lemma 6.8. *Let $(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)$ be the unique standard $(k|n-k)$-tableau which has the numbers $n-k+1,\dots,n$ in $\mathfrak{a}^+_\lambda$. Then*

$$w'_{\mathfrak{t}^\lambda} = (-1)^k q^{\ell(w_{\lambda'})} \Bigg\{ q^{-\ell(d(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda))}[n-k]_q x_{(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)} + \sum_{\substack{(\mathfrak{a}|\mathfrak{b})\prec\mathfrak{t}^\lambda\\(\mathfrak{a}|\mathfrak{b})\neq(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)}} (-1)^{1-I_{\mathfrak{t}^\lambda}(\mathfrak{a}|\mathfrak{b})} q^{-\ell(d(\mathfrak{a}|\mathfrak{b}))} \sum_{j=0}^{n-k-1} x_{(\mathfrak{a}|\mathfrak{b})r_{1,j}} \Bigg\}.$$

Proof. By definition $w'_{\mathfrak{t}^\lambda} = v'_{\mathfrak{t}^\lambda} x_{n-k}$. Also, $d((\mathfrak{t}^\lambda)') = d(\mathfrak{t}_{\lambda'}) = w_{\lambda'}$ so, by Lemma 6.5,

$$\begin{aligned} w'_{\mathfrak{t}^\lambda} = v'_{\mathfrak{t}^\lambda} x_{n-k} &= \sum_{(\mathfrak{a}|\mathfrak{b})\prec\mathfrak{t}^\lambda} (-1)^{k+1-I_{\mathfrak{t}^\lambda}(\mathfrak{a}|\mathfrak{b})} q^{\ell(w_{\lambda'})-\ell(d(\mathfrak{a}|\mathfrak{b}))} x_{(\mathfrak{a}|\mathfrak{b})} x_{n-k}\\ &= \sum_{(\mathfrak{a}|\mathfrak{b})\prec\mathfrak{t}^\lambda} (-1)^{k+1-I_{\mathfrak{t}^\lambda}(\mathfrak{a}|\mathfrak{b})} q^{\ell(w_{\lambda'})-\ell(d(\mathfrak{a}|\mathfrak{b}))} x_{(\mathfrak{a}|\mathfrak{b})}(1+T_1+\cdots+T_{1,n-k-1}).\end{aligned}$$

Let $(\mathfrak{a}|\mathfrak{b})$ be one of the tableaux appearing in this sum. If $(\mathfrak{a}|\mathfrak{b})\neq(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)$ then 1 is contained in $\mathfrak{a}$ and all of the numbers $2,3,\dots,n-k$ are contained in $\mathfrak{b}$. Therefore,

$(\mathfrak{a}|\mathfrak{b})r_{1,j}$ is standard and $x_{(\mathfrak{a}|\mathfrak{b})}T_{1,j} = x_{(\mathfrak{a}|\mathfrak{b})r_{1,j}}$, for $0 \le j \le n-k-1$. On the other hand, $x_{(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)}T_{1,j} = q^j x_{(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)}$, for $0 \le j \le n-k-1$. This completes the proof of the Lemma. □

This result has two useful corollaries.

Corollary 6.9. *Suppose that* $\mathfrak{t} \ne \mathfrak{t}^\lambda$ *is a standard* $(k|n-k)$*-tableau. Then*

$$\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}}\rangle = 0.$$

Proof. By Lemma 6.8, if $x_{(\mathfrak{a}|\mathfrak{b})}$ appears in $w'_{\mathfrak{t}^\lambda}$ then all but one of the entries in $\mathfrak{a}$ are contained in $\{1, n-k+1, \dots, n\}$. On the other hand, by Lemma 6.5, if $x_{(\mathfrak{a}|\mathfrak{b})}$ appears in $v'_{\mathfrak{t}}$ then all of the entries in $\mathfrak{a}$ are contained in the first column of $\mathfrak{t}$.

Suppose now that $\mathfrak{t} \ne \mathfrak{t}^\lambda$. Then, by the last paragraph, $x_{(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)}$ cannot appear in $v'_{\mathfrak{t}}$ and the only way that the inner product $\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}}\rangle$ can be non-zero is if the set of numbers in the first column of $\mathfrak{t}$ is of the form $T = \{1, j, n-k+1, \dots, n\}\setminus\{m\}$, for some integers j and m with $1 < j \le n-k$ and $n-k < m \le n$. Let $(\mathfrak{a}|\mathfrak{b})$ be the standard $(k|n-k)$-tableau whose first component contains exactly the numbers in $T\setminus\{j\}$ and let $(\mathfrak{a}'|\mathfrak{b}') = (\mathfrak{a}|\mathfrak{b})r_{1,j-1}$. Then $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}^\lambda$, $I_{\mathfrak{t}}(\mathfrak{a}|\mathfrak{b}) = 2$ and $I_{\mathfrak{t}}(\mathfrak{a}'|\mathfrak{b}') = 1$. Also $\ell(d(\mathfrak{a}'|\mathfrak{b}')) = \ell(d(\mathfrak{a}|\mathfrak{b})) + j - 1$, so $x_{(\mathfrak{a}|\mathfrak{b})}T_{1,j-1} = x_{(\mathfrak{a}'|\mathfrak{b}')}$. Therefore, by Lemma 6.5 and Lemma 6.8 and the remarks above,

$$v'_{\mathfrak{t}} = (-1)^k q^{\ell(d(\mathfrak{t}'))}\left(q^{-\ell(d(\mathfrak{a}'|\mathfrak{b}'))}x_{(\mathfrak{a}'|\mathfrak{b}')} - q^{-\ell(d(\mathfrak{a}|\mathfrak{b}))}x_{(\mathfrak{a}|\mathfrak{b})}\right) + \text{ other standard terms}$$

and

$$w'_{\mathfrak{t}^\lambda} = q^{\ell(w_{\lambda'})-\ell(d(\mathfrak{a}|\mathfrak{b}))}\left(x_{(\mathfrak{a}'|\mathfrak{b}')} + x_{(\mathfrak{a}|\mathfrak{b})}\right) + \text{ other standard terms,}$$

where none of the "other standard terms" appear both in $v'_{\mathfrak{t}}$ and in $w'_{\mathfrak{t}^\lambda}$. Consequently, $\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}}\rangle = 0$. Hence, $\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}}\rangle = 0$ whenever $\mathfrak{t} \ne \mathfrak{t}^\lambda$ as claimed. □

Corollary 6.10. *Suppose that* $\mathfrak{t}$ *is a standard* $(n-k, 1^k)$*-tableau and that* n *does not appear in the first row of* $\mathfrak{t}$*. Then* $\langle w'_{\mathfrak{t}}, v'_{\mathfrak{t}}\rangle = q^{k(n-k-2)}[n]_q$.

Proof. Recall that if $\mathfrak{t}$ is a standard λ-tableau then $d(\mathfrak{t}')d(\mathfrak{t})^{-1} = w_{\lambda'}$, with the lengths adding; this is well known and is easily proved by induction on the dominance order for tableaux. Therefore,

$$\begin{aligned}\langle w'_{\mathfrak{t}}, v'_{\mathfrak{t}}\rangle &= \langle w'_{\mathfrak{t}^\lambda}T_{d(\mathfrak{t})}, x_{(k|n-k)}y'_{k+1}T_{d(\mathfrak{t}')}\rangle = \langle w'_{\mathfrak{t}^\lambda}, x_{(k|n-k)}y'_{k+1}T_{d(\mathfrak{t}')}T^*_{d(\mathfrak{t})}\rangle\\ &= \langle w'_{\mathfrak{t}^\lambda}, x_{(k|n-k)}y'_{k+1}T_{d(\mathfrak{t}^{\lambda\prime})}\rangle = \langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}^\lambda}\rangle.\end{aligned}$$

Hence, it is enough to consider the case where $\mathfrak{t} = \mathfrak{t}^\lambda$.

Suppose that $\mathfrak{t} = \mathfrak{t}^\lambda$. Then, by Lemmas 6.5 and 6.8,

$$\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}^\lambda}\rangle = q^{2\ell(w_{\lambda'})}\left\{q^{-\ell(d(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda))}[n-k]_q + \sum_{\substack{(\mathfrak{a}|\mathfrak{b})\prec\mathfrak{t}^\lambda\\(\mathfrak{a}|\mathfrak{b})\ne(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)}} q^{-\ell(d(\mathfrak{a}|\mathfrak{b}))}\right\}.$$

Using the remarks before Lemma 6.4 it is not hard to see that $\ell(d(\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)) = k(n-k)$ and that $\ell(d(\mathfrak{a}|\mathfrak{b})) = (k-1)(n-k)+2-I_{\mathfrak{t}}(\mathfrak{a}|\mathfrak{b})$, whenever $(\mathfrak{a}|\mathfrak{b}) \prec \mathfrak{t}^\lambda$ and $(\mathfrak{a}|\mathfrak{b}) \ne (\mathfrak{a}^+_\lambda|\mathfrak{b}^+_\lambda)$.

Therefore,

$$\begin{aligned}\langle w'_{\mathfrak{t}^\lambda}, v'_{\mathfrak{t}^\lambda}\rangle &= q^{2\ell(w_{\lambda'})}\left\{q^{-k(n-k)}[n-k]_q + \sum_{i=2}^{k+1} q^{-(k-1)(n-k)-2+i}\right\}\\ &= q^{2\ell(w_{\lambda'})-k(n-k)}\left\{[n-k]_q + q^{n-k}\sum_{j=0}^{k-1} q^j\right\}\\ &= q^{2\ell(w_{\lambda'})-k(n-k)}[n]_q.\end{aligned}$$

As $\ell(w_{\lambda'}) = k(n-k-1)$, the result follows. □

Finally, we can prove the main result of this section.

Proposition 6.11. *Suppose that $\lambda = (n-k, 1^k)$, for some k with $0 \le k < n$. Then the Gram matrix $G(\lambda)$ of $S(\lambda)$ is divisibly diagonalizable over $\mathbb{Z}[q, q^{-1}]$ with $\binom{n-2}{k}$ elementary divisors equal to $[k]_q^!$ and with the remaining $\binom{n-2}{k-1}$ elementary divisors being equal to $[k]_q^![n]_q$.*

Proof. By Proposition 6.1 the Gram matrix $G(\lambda)$ of $S(\lambda)$ is equal to $[k]_q^!$ times the Gram matrix of $S'(\lambda)$. Therefore, by Lemma 4.1 it is enough to show that there is an invertible diagonal matrix D such that

$$G'(\lambda) = (\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle)_{\mathfrak{s},\,\mathfrak{t}\in \mathrm{Std}(k|n-k)} = D\cdot\begin{pmatrix} I & * \\ 0 & [n]_q U\end{pmatrix},$$

where I is a $\binom{n-2}{k}\times\binom{n-2}{k}$ identity matrix and U is a $\binom{n-2}{k-1}\times\binom{n-2}{k-1}$ upper triangular matrix with 1's down its diagonal. Here we order the rows and columns lexicographically with respect to the entries in the first columns of $\mathfrak{s}$ and $\mathfrak{t}$. Because D is invertible its non-zero entries must all be of the form $\pm q^m$, for some integer m.

By Corollary 6.7, the rows of $G'(\lambda)$ which are indexed by those tableaux which have n in their first row have the required form. This accounts for the identity matrix in the top half of the Gram matrix $G'(\lambda)$.

Next, suppose that $\mathfrak{s}$ is a standard $(k|n-k)$-tableau and that n does not appear in the first row of $\mathfrak{s}$. If $\mathfrak{s} = \mathfrak{t}^\lambda$ then $\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle = 0$, for all $\mathfrak{t} \ne \mathfrak{s}$, by Corollary 6.9. If $\mathfrak{s} \ne \mathfrak{t}^\lambda$ then there exists an integer i, $1 < i < n-1$, such that $\ell(d(\mathfrak{s})r_i) < \ell(d(\mathfrak{s}))$. Therefore,

$$\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle = \langle w'_{\mathfrak{s}r_i} T_i, v'_{\mathfrak{t}}\rangle = \langle w'_{\mathfrak{s}r_i}, v'_{\mathfrak{t}}T_i\rangle.$$

By expanding $v'_{\mathfrak{t}}T_i$, and using induction, it follows that $\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle = 0$ if $\mathfrak{t}$ appears before $\mathfrak{s}$ in our chosen ordering of $\mathrm{Std}(\lambda)$. Similarly, if $\mathfrak{t}$ does not appear before $\mathfrak{s}$ then $[n]_q$ divides $\langle w'_{\mathfrak{s}}, v'_{\mathfrak{t}}\rangle$ by Corollary 6.10. □

Notice, in particular, that the Gram matrix calculation in the proof of the Proposition implies that $\{w'_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is indeed a basis of $S'(\lambda)$.

Proposition 6.11 also gives the decomposition numbers of $S(\lambda)$ (by inducing the corresponding principal indecomposable modules); however, as these are already known

we leave these as an exercise for the reader. We will, however, give one application of this result.

Let $\pi : S(\lambda) \longrightarrow S'(\lambda)$ be the isomorphism of Proposition 6.1 and for each standard λ-tableau $\mathfrak{t}$ let $w_\mathfrak{t} = \pi^{-1}(w'_\mathfrak{t})$. Then $\{w_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is a basis of $S(\lambda)$. Then, in the case where $S(\lambda)$ is not irreducible, the proof of Proposition 6.11 also gives a basis for the simple module $D(\lambda)$. More precisely, we have the following.

Corollary 6.12. *Suppose that R is a field, that $[k]_q^! \neq 0$ and that $[n]_q = 0$. Then $S(\lambda)$ is not irreducible and a basis of $D(\lambda) = S(\lambda)/(S(\lambda)^\perp \cap S(\lambda))$ is given by*

$$\{w_\mathfrak{t} + (S(\lambda)^\perp \cap S(\lambda)) \mid \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } n \text{ in first row of } \mathfrak{t}\},$$

and a basis of $S(\lambda)^\perp \cap S(\lambda)$ is given by $\{w_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)$ and n is in first row of $\mathfrak{t}\}$.

7. Some counterexamples

Let $R = \mathbb{Z}[q, q^{-1}]$. We write the mth cyclotomic polynomial in q as $\Phi_m = \Phi_m(q)$.

Andersen remarked that in general the Gram matrix $G(\lambda)$ is not diagonalizable [1, Remark 5.11]. We give two examples of this kind.

Note that $G(\lambda)$ is divisibly diagonalizable over $\mathbb{Z}_{(p)}[q, q^{-1}]$ for all but finitely many primes p. In fact, it suffices to exclude the primes occurring in the denominators of the entries of the matrices used to diagonalize $G(\lambda)$ over $\mathbb{Q}[q, q^{-1}]$.

We record the elementary divisors in "jump notation". That is, we write

$$\xrightarrow{f_1} m_1 \xrightarrow{f_2} m_2 \xrightarrow{f_3} m_3 \xrightarrow{f_4} \cdots \xrightarrow{f_s} m_s$$

to indicate that the matrix has the elementary divisor f_1 with multiplicity m_1, the elementary divisor $f_1 f_2$ with multiplicity $m_2, \ldots$, and the elementary divisor $f_1 \cdots f_s$ with multiplicity m_s.

Example 7.1. Let $\lambda = (3, 3, 2)$. The elementary divisors of $G(3, 3, 2)$ over $\mathbb{Q}[q, q^{-1}]$ are given by

$$\xrightarrow{\Phi_2^2} 1 \xrightarrow{\Phi_4} 20 \xrightarrow{\Phi_3\Phi_5} 20 \xrightarrow{\Phi_4} 1;$$

over $\mathbb{F}_2[q, q^{-1}]$ they are given by

$$\xrightarrow{\Phi_2^3} 1 \xrightarrow{\Phi_2} 20 \xrightarrow{\Phi_3\Phi_5} 20 \xrightarrow{\Phi_2} 1;$$

and, putting $q = 1$, over $\mathbb{Z}$ they are given by

$$\xrightarrow{2^3} 21 \xrightarrow{3\cdot 5} 21.$$

We claim that $G(3, 3, 2)$ is not diagonalizable over $\mathbb{Z}_{(2)}[q, q^{-1}]$. To see this suppose that it is diagonalizable. Then, considered as an element of $\mathbb{Z}_{(2)}[q, q^{-1}]$, any resulting diagonal entry must contain the factor $(q + 1)$ with exponent 2.

We give one application of Proposition 6.11.

We remark that the comparison of the elementary divisors over $\mathbb{Q}[q,q^{-1}]$ and over $\mathbb{Z}$ yields a contradiction to diagonalizability over $\mathbb{Z}_{(2)}[q,q^{-1}]$, too.

Example 7.2. Let $\lambda = (4,2,1,1)$. The elementary divisors of $G(4,2,1,1)$ over $\mathbb{Q}[q,q^{-1}]$ are given by

$$\xrightarrow{\Phi_2} 14 \xrightarrow{\Phi_2} 1 \xrightarrow{\Phi_4} 30 \xrightarrow{\Phi_7} 30 \xrightarrow{\Phi_4} 1 \xrightarrow{\Phi_2} 14;$$

over $\mathbb{F}_2[q,q^{-1}]$ they are given by

$$\xrightarrow{\Phi_2} 14 \xrightarrow{\Phi_2^2} 1 \xrightarrow{\Phi_2} 30 \xrightarrow{\Phi_7} 30 \xrightarrow{\Phi_2} 1 \xrightarrow{\Phi_2^2} 14;$$

over $\mathbb{F}_3[q,q^{-1}]$ they are given by

$$\xrightarrow{\Phi_2} 13 \xrightarrow{\Phi_2} 2 \xrightarrow{\Phi_4} 30 \xrightarrow{\Phi_7} 30 \xrightarrow{\Phi_4} 2 \xrightarrow{\Phi_2} 13;$$

and, putting $q=1$, over $\mathbb{Z}$ they are given by

$$\xrightarrow{2} 14 \xrightarrow{2^2} 31 \xrightarrow{7} 31 \xrightarrow{2^2} 14.$$

We claim that $G(4,2,1,1)$ is not diagonalizable over $\mathbb{Z}_{(2)}[q,q^{-1}]$. Again, by way of contradiction suppose that it is diagonalizable. In $\mathbb{F}_2[q,q^{-1}]$, 14 of the resulting diagonal entries contain the factor $(q+1)$ with exponent 1. Therefore, in $\mathbb{Z}_{(2)}[q,q^{-1}]$, 14 of them contain the factor $(q+1)$ with exponent 1 and the factor (q^2+1) with exponent 0. Similarly, in $\mathbb{F}_2[q,q^{-1}]$, 14 of the resulting diagonal entries contain the factor $(q+1)$ with exponent 7. Thus in $\mathbb{Z}_{(2)}[q,q^{-1}]$, 14 of them contain the factor $(q+1)$ with exponent 3 and the factor (q^2+1) with exponent 2. Hence in $\mathbb{F}_2[q,q^{-1}]$, no other diagonal entry can contain $(q+1)$ with odd exponent. But in $\mathbb{F}_2[q,q^{-1}]$, there is a diagonal entry containing $(q+1)$ to the power 3 and another containing it to the power 5 so, again, we have a contradiction.

We claim that $G(4,2,1,1)$ is not diagonalizable over $\mathbb{Z}_{(3)}[q,q^{-1}]$. Assume it to be diagonalizable. In $\mathbb{Z}_{(3)}[q,q^{-1}]$, 14 of the resulting diagonal entries contain $(q+1)$ with exponent 1. This contradicts the fact that in $\mathbb{F}_3[q,q^{-1}]$, only 13 of them contain $(q+1)$ with exponent 1.

The two claims independently imply that $G(4,2,1,1)$ is not diagonalizable over $\mathbb{Z}[q,q^{-1}]$.

We remark that the comparison of the elementary divisors over $\mathbb{Q}[q,q^{-1}]$ and over $\mathbb{Z}$ yields a contradiction to diagonalizability over $\mathbb{Z}_{(2)}[q,q^{-1}]$, too.

Finally, we give a (non-exhaustive) list of elementary divisors of some divisibly diagonalizable Gram matrices for non-hooks, calculated using GAP 3 [12] and MAGMA [9]. We omit the respective conjugate partition; compare Proposition 5.9.

n	λ	Elementary divisors of $G(\lambda)$
4	(2, 2)	$\xrightarrow{\Phi_2} 1 \xrightarrow{\Phi_3} 1$
5	(3, 2)	$\xrightarrow{1} 1 \xrightarrow{\Phi_3} 3 \xrightarrow{\Phi_4} 1$
6	(4, 2)	$\xrightarrow{1} 4 \xrightarrow{\Phi_4} 1 \xrightarrow{\Phi_2} 3 \xrightarrow{\Phi_5} 1$
	(3, 3)	$\xrightarrow{\Phi_2} 1 \xrightarrow{\Phi_3} 3 \xrightarrow{\Phi_4} 1$
	(3, 2, 1)	$\xrightarrow{1} 4 \xrightarrow{\Phi_3} 4 \xrightarrow{\Phi_5} 4 \xrightarrow{\Phi_3} 4$
7	(5, 2)	$\xrightarrow{1} 8 \xrightarrow{\Phi_5} 5 \xrightarrow{\Phi_3\Phi_6} 1$
	(4, 3)	$\xrightarrow{1} 1 \xrightarrow{\Phi_3} 7 \xrightarrow{\Phi_4} 5 \xrightarrow{\Phi_4} 1$
	(3, 3, 1)	$\xrightarrow{\Phi_2} 6 \xrightarrow{\Phi_3} 2 \xrightarrow{\Phi_5} 12 \xrightarrow{\Phi_4} 1$
8	(6, 2)	$\xrightarrow{1} 13 \xrightarrow{\Phi_3\Phi_6} 1 \xrightarrow{\Phi_2} 5 \xrightarrow{\Phi_7} 1$
	(5, 3)	$\xrightarrow{1} 8 \xrightarrow{\Phi_4} 6 \xrightarrow{\Phi_2} 7 \xrightarrow{\Phi_5} 6 \xrightarrow{\Phi_6} 1$
	(4, 4)	$\xrightarrow{\Phi_2} 1 \xrightarrow{\Phi_3} 7 \xrightarrow{\Phi_4} 5 \xrightarrow{\Phi_5} 1$
9	(7, 2)	$\xrightarrow{1} 19 \xrightarrow{\Phi_7} 7 \xrightarrow{\Phi_4\Phi_8} 1$
	(6, 3)	$\xrightarrow{1} 21 \xrightarrow{\Phi_5} 19 \xrightarrow{\Phi_6} 1 \xrightarrow{\Phi_3} 6 \xrightarrow{\Phi_7} 1$
	(5, 4)	$\xrightarrow{1} 1 \xrightarrow{\Phi_3} 15 \xrightarrow{\Phi_4} 18 \xrightarrow{\Phi_5} 7 \xrightarrow{\Phi_6} 1$

We do not know of an example of a Gram matrix $G(\lambda)$ that is diagonalizable over $\mathbb{Z}[q, q^{-1}]$, but not divisibly diagonalizable.

For a general partition λ, we cannot decide whether $G(\lambda)$ is diagonalizable over $\mathbb{Z}[q, q^{-1}]$.

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