ELSEVIER
Theoretical Computer Science 168 (1996) 241–255
Theoretical Computer Science

Small deterministic Turing machines

Manfred Kudlek *

Universität Hamburg, Fachbereich Informatik, Vogt-Kölln-Straße 30, D-22527 Hamburg, Germany

1. Introduction

In this paper small deterministic Turing machines with one bi-infinite tape and one scanning head are considered. Let $DTM(m,n)$ denote the class of all such machines with m tape symbols (including the blank) and n states (excluding the halting state). It is known that there exist universal Turing machines in $DTM(5,6)$ [10] and in $DTM(4,7)$ [4]. It is also known that the halting problem for all machines from $DTM(1,n)$ and $DTM(m,1)$ is decidable. The first fact is trivial whereas the second one has been shown in [2]. In [4, 5] it is mentioned that the halting problem for $DTM(2,2)$ is decidable too; however without giving any proof and stating only [5] that this result is unpublished and unpublishable. In such a state it remained since 1961 or 1972, respectively, until 1988. By a reduction to few cases it was possible to solve the problem and bring it into a publishable form. A first version can be found in [1]. At that time the results from 1975 [6] stating the halting problem to be decidable for $DTM(2,2)$ and $DTM(2,3)$ have not been known to the authors. In that paper a completely different method was used. Neither were known the results by [7] stating that there are universal Turing machines in $DTM(2,24)$, $DTM(3,11)$, $DTM(5,5)$, $DTM(6,4)$, $DTM(10,3)$, and $DTM(21,2)$. All these results were only little known in Western countries before 1991.

It is also shown that the sets accepted by machines from $DTM(2,1)$ are regular, and that those accepted by machines from $DTM(3,1)$ are regular too, except one case giving a deterministic linear context-free language which is essentially the nonregular language $L = \{a^n b^n \mid 1 \leqslant n\}$.

The languages accepted by machines from $DTM(2,2)$ are also regular, except one case (up to symmetries) giving essentially the same language as in the exceptional case of $DTM(3,1)$, namely a deterministic linear context-free language. Thus, the halting problem is decidable for $DTM(2,2)$. To obtain this result several symmetries are used to reduce the number of machines to consider. Finally, also a machine from $DTM(4,4)$ is presented accepting a context-sensitive language, essentially $L = \{a^k \mid k = 2^n, 0 \leqslant n\}$.

* E-mail: kudlek@informatik.uni-hamburg.de.

PII S0304-3975(96)00078-3

Possibly, similar methods can be applied also to $DTM(3,2)$, $DTM(2,3)$, and perhaps to $DTM(3,3)$. But for the last case, probably the aid of an automatic system should be used.

2. Definitions

Let T be any Turing machine with one bi-infinite tape, one scanning head, an alphabet V of m symbols including the blank, and a set S of n states (excluding the halting state H). The set of movements is given by $\{L,M,R\}$, standing for movement to the left, no movement, and movement to the right, respectively. The set of instructions, or program P of T, is denoted by $P \subseteq S \times V \times V \times \{L,M,R\} \times S$, with its elements (p,x,y,I,q) written in the form $pxyIq$. For shortness, y,I,q if $y=x$, $I=M, q=p$, is omitted, respectively. If the halting state H appears, this is always understood as pxH, i.e. $pxxMH$. If T is a deterministic Turing machine P represents a (total) function.

The class of all such deterministic Turing machines will be denoted by $DTM(m,n)$. Only such machines will be considered here.

As usual, $V*$ denotes the free monoid generated by V, λ its neutral element (the empty word), $lg(w)$ the length of a word $w \in V*$, are $rev(w)$ its reversal (mirror image).

Let ${}^{\infty}V$ and V^{∞} stand for the set of all left-infinite and right-infinite words over V, respectively.

A configuration of T is any bi-infinite word $\alpha\binom{x}{p}\beta \in {}^{\infty}V\cdot(V\times S)\cdot V^{\infty}$ denoting the fact that the head is scanning x and T is in state p. As usually, $\alpha\binom{x}{p}\beta \xrightarrow{*} \gamma\binom{y}{q}\delta$ means that there exists a finite sequence of Turing steps from $\alpha\binom{x}{p}\beta$ to $\gamma\binom{y}{q}\delta$, including 0 steps and $q=H$. It is assumed that there is no continuation of any configuration $\gamma\binom{y}{H}\delta$.

Finally, let $\mathbb{N}_k$ denote the set $\mathbb{N}_k := \{i \in \mathbb{N} \mid 0 \leqslant i \leqslant k\}$.

As for acceptance, there exist several possibilities to define sets of words accepted by Turing machines. The first one just gives the set of all such bi-infinite words being configurations which lead to acceptance.

Definition 1. $L_{\infty}(T,p) := \{\alpha\binom{x}{p}\beta \in {}^{\infty}V\cdot(V\times S)\cdot V^{\infty} \mid \exists\gamma\in{}^{\infty}V\,\exists\delta\in V^{\infty}\exists y\in V : \alpha\binom{x}{p}\beta \xrightarrow{*} \gamma\binom{y}{H}\delta\}$.

A second possibility is the use of the work space for acceptance. Note, that by this definition one gets only context-sensitive languages.

Definition 2.

$$\begin{aligned} &L_s(T,p) \\ &:= \left\{u\begin{pmatrix}x\\p\end{pmatrix}v \in V*\cdot(V\times S)\cdot V* \mid \exists k\in\mathbb{N}\,\forall i\in\mathbb{N}_k\,\exists u_i,v_i\in V*\exists x_i\in V\exists p_i\in S: \right.\\ &\qquad u_0=u \wedge v_0=v \wedge x_0=x \wedge p_0=p \wedge p_k=H \wedge lg(u_iv_i)=lg(uv) \end{aligned}$$

$$\wedge \forall i \in \mathbb{N}_k - \{k\} : \left(p_i \neq H \wedge u_i \binom{xi}{pi} v_i \rightarrow u_{i+1} \binom{x_{i+1}}{p_{i+1}} v_{i+1} \right)$$
$$\left. \wedge \exists j \in \mathbb{N}_k : u_j = \lambda \wedge \exists l \in \mathbb{N}_k : v_l = \lambda \right\}.$$

Thus, $lg(uv)+1$ is exactly the work space for accepting $u\binom{x}{p}v$, and this word is the relevant part of the initial configuration being scanned by the head.

From this definition trivially follows that

Lemma 1. $L_\infty(T,p) = {}^\infty V \cdot L_s(T,p) \cdot V^\infty$.

A third possibility is not to look onto the relevant part of one initial configuration only but to consider them in some global sense, namely that if w is a word of the set then no proper subword of it is also contained in the set.

Definition 3.

$$L_c(T,p) := \left\{ u\binom{x}{p} v \in V^* \cdot (V \times S) \cdot V^* \mid \forall \alpha \in {}^\infty V \, \forall \beta \in V^\infty : \right.$$
$$\alpha u \binom{x}{p} v \beta \in L_\infty(T,p) \wedge (u = yu' \Rightarrow \exists y' \in V : y'u' \notin L_c(T,p))$$
$$\left. \wedge (v = v'z \Rightarrow \exists z' \in V : v'z' \notin L_c(T,p)) \right\}.$$

This set is called the *core* of $L_\infty(T,p)$. Trivially again, it follows that

Lemma 2. $L_\infty(T,p) = {}^\infty V \cdot L_c(T,p) \cdot V^\infty$.

Finally, the normal definition of an accepted set can be defined by cutting off all 0's at left and right ends from words in $L_s(T,p)$.

Definition 4. $L_0(T,p) = \{0\}^* \backslash L_s(T,p) / \{0\}^*$

In the sequel it is assumed that $V = \mathbb{N}_k - \{k\}$, and that 0 represents the blank. It is easy to establish the following lemmas.

Lemma 3. *If there is no occurrence of pxH in the set of instructions of a Turing machine T, then $L_\infty(T,p) = L_s(T,p) = L_c(T,p) = L_0(T,p) = \emptyset$.*

Lemma 4. *For deterministic Turing machines it suffices that there is at most* 1 *occurrence of halting qxH in the set of instructions.*

Proof. Among all occurrences of halting in P choose one, e.g. qxH. Replace all other pyH by $pyxMq$. Then it is trivial to see that the accepted languages remain the same. □

Definition 5. Let $T = (S, V, P)$ be any deterministic Turing machine. Let $\pi : S \to S$ be any permutation of the set of states $S, \sigma : V \to V$ be any permutation of the alphabet V, and $\mu : \{L, M, R\} \to \{L, M, R\}$ be defined by $\mu(L) = R,\ \mu(M) = M,\ \mu(R) = L$.

Then define also $\pi(T) := (S, V, \pi(P))$ with $\pi(p)xyI\pi(q) \in \pi(P) \Leftrightarrow pxyIq \in P$, $\sigma(T) := (S, V, \sigma(PP)$ with $p\sigma(x)\sigma(y)Iq \in \sigma(P) \Leftrightarrow pxyIq \in P$, and $\mu(T) := (S, V, \mu(P))$ with $pxy\mu(I)q \in \mu(P) \Leftrightarrow pxyIq \in P$.

By the next lemma all Turing machines with movements of the head in one direction only can be eliminated from further consideration.

Lemma 5. *Let T be any DTM with movements only* $\{M, R\}$ *or* $\{L, M\}$ *in the set of instructions, respectively. Then* $L_s(T, p)$ *is regular.*

Proof. Let the movements be $\{M, R\}$, and the initial configuration $\alpha\binom{x}{p}\beta$. Then, from T, a finite automaton is constructed: $F_x := (V, S \times V; \{\binom{x}{p}\}, \{\binom{y}{q} \mid qyH \in P\}, R)$ with $R \subseteq (S \times V) \times (V \cup \{\lambda\}) \times (S \times V)$ given by $(\binom{y}{p}, \lambda, \binom{z}{q}) \in R$ if $pyzMq \in P$, and $(\binom{y}{p}, z, \binom{z}{q}) \in R$ if $pyzRq \in P$. By this construction it is obvious that $L_s(T, p) = \binom{x}{p}L(F_x)$.

Trivially, in the case of $\{L, M\}$ one gets $L_s(T, p) = rev(L(F_x)) \cdot \{\binom{x}{p}\}$. □

A stronger lemma is the following one.

Lemma 6. *If either*

$$\forall x \in V\, \forall p \in S\, \exists m, n \in \mathbb{N}\, \forall u_1 \in V^m\, \forall u_2 \in V^n\, \exists y \in V\, \exists q \in S \cup \{H\}\, \exists v \in V^{m+n}:$$

$$u_1 \begin{pmatrix} x \\ p \end{pmatrix} u_2 \xrightarrow{*} v \begin{pmatrix} y \\ q \end{pmatrix}$$

or

$$\forall x \in V\, \forall p \in S\, \exists m, n \in \mathbb{N}\, \forall u_1 \in V^m\, \forall u_2 \in V^n\, \exists y \in V\, \exists q \in S \cup \{H\} \exists v \in V^{m+n}:$$

$$u_1 \begin{pmatrix} x \\ p \end{pmatrix} u_2 \xrightarrow{*} \begin{pmatrix} y \\ q \end{pmatrix} v$$

where during the computation the head never leaves the workspace of length $m+n+1$*, then* $L_s(T, p)$ *is regular.*

Proof. By constructing a finite automaton with $(\binom{x}{p}, u_2, \binom{y}{q}) \in R$ or $(\binom{x}{p}, \overline{u_1}, \binom{y}{q}) \in R$ and final states $\binom{z}{H}$ where $\bar{v} = rev(v)$, respecting the initial part in front of or behind $\binom{x}{p}$ for the initial configuration. □

Although there exist deterministic Turing machines with all accepted sets $L_s(T, p)$, $L_c(T, p)$, $L_0(T, P)$ being different, only the first possibility will be considered in the sequel.

3. DTM(1, *n*)

Any Turing machine T from DTM$(1,n)$ can be considered as a finite directed graph with $n+1$ nodes from $S \cup \{H\}$. For each $p \in S$ it is decidable whether there exists a directed path from p to H, and this path has a length of at most n. Thus one gets

Theorem 1. *All sets accepted by Turing machines from* $DTM(1,n)$ *are finite, and the halting problem for* $DTM(1,n)$ *is decidable.*

4. DTM(2, 1)

Generally, it has been shown by Herman [2] that the halting problem of arbitrary 1 state DTMs with 1 head and 1 k-dimensional tape is decidable. He also proved that there exist accepted sets which are not regular.

Let $S = \{p\}$ and $V = \{0,1\}$. Then

Theorem 2. *All sets accepted by machines from* $DTM(2,1)$ *are regular, and the halting problem for* $DTM(2,1)$ *is decidable.*

Proof. Let σ be the symmetry defined by $\sigma(0)=1$, $\sigma(1)=0$, and μ that one defined by $\mu(L)=R$, $\mu(M)=M$, $\mu(R)=L$. By Lemma 4 it suffices to have at most 1 possibility of halting in the program. Let there be exactly one. It also suffices that this is $p1H$ for if it is $p0H$ then this may be treated using the symmetry σ. The case $p0L$ may be ruled out using the symmetry μ. Thus, only 2 cases remain:

	p
0	$\cdot R$
1	H

and

	p
0	$\cdot M$
1	H

The first one gives a regular set, the second one finite sets. □

5. DTM(3, 1)

Let $S=\{p\}$, $V=\{0,1,2\}$, and $f(p,x)\in V$, $g(p,x)\in\{L,M,R\}$, $h(p,x)\in S\cup\{H\}$ denote the new symbol, the head movement, and the new state, respectively, if T is in state p and x the scanned symbol.

By Lemma 5, and using the symmetry μ, similarly to DTM$(2,1)$, it suffices to consider only the case

	p
0	$\cdot L$
1	$\cdot R$
2	H

In the case $f(p,0)=f(p,1)=0$ one gets as accepted set

$$L_s(T,p)=2(0\cup 1)^*\binom{0}{p}\cup\binom{1}{p}1^*2\cup 2(0\cup 1)^*\binom{1}{p}1^*0\cup\binom{2}{p},$$

in the case $f(p,0)=0,\ f(p,1)=1$

$$L_s(T,p)=20^*\binom{0}{p}\cup\binom{1}{p}1^*2\cup\binom{2}{p},$$

in the case $f(p,0)=f(p,1)=1$

$$L_s(T,p)=20^*\binom{0}{p}\cup 10^*\binom{0}{p}(0\cup 1)^*2\cup\binom{1}{p}(0\cup 1)^*2\cup\binom{2}{p},$$

and in the last case $f(p,0)=1, f(p,1)=0$ the following accepted set:

$$\begin{aligned}L_s(T,p)=\Big\{ & 10^{m_k}1\cdots 10^{m_1}\binom{0}{p}1^{n_1}0\cdots 01^{n_k}2,\\ & 20^{m_{k+1}}10^{m_k}1\cdots 10^{m_1}\binom{0}{p}1^{n_1}0\cdots 01^{n_k}0,\\ & 20^{m_k}1\cdots 10^{m_1}\binom{1}{p}1^{n_1}0\cdots 01^{n_k}0,\\ & 10^{m_k}1\cdots 10^{m_1}\binom{1}{p}1^{n_1}0\cdots 01^{n_k}01^{n_{k+1}}2\,|\\ & k\geqslant 0, m_i\in\mathbb{N}, n_i\in\mathbb{N}\Big\}\\ & \cup\binom{2}{p}.\end{aligned}$$

Clearly, this is a linear deterministic context-free language not being regular. Thus:

Theorem 3. *All sets accepted by machines from* $DTM(3,1)$ *are regular except one case (up to symmetries) yielding a linear deterministic context-free language. The halting problem for* $DTM(3,1)$ *is decidable.*

6. DTM(2, 2)

At first some methods reducing the number of machines to be treated are considered. Let $S=\{p,q\}$ and $V=\{0,1\}$. In the program of a Turing machine T let $f(s,x)\in V$, $g(s,x)\in\{L,M,R\}$, $h(s,x)\in S\cup\{H\}$ denote the new symbol to be written, the movement of the head, and the new state, respectively, if T is in state s and the head is scanning x. By Lemma 4 let there be exactly 1 possibility of H. Thus, from originally $13^4=28561$ only $4\cdot 12^3=6912$ machines are left. Using Lemma 5, this number can be reduced to 3072.

Using the symmetries σ and μ from DTM(2, 1) and π defined by $\pi(p) = q, \pi(q) = p$, the following reductions decrease the number of Turing machines to be considered.

Lemma 7. *It suffices to have only* $h(q,1) = H$ *as halting.*

Proof. By using the symmetries σ, π, and $\sigma\pi = \pi\sigma$. □

By the next lemma $g(q,0) = L$ is eliminated.

Lemma 8. *It suffices to have only* $g(q,0) = M$ *or* $g(q,0) = R$.

Proof. By using the symmetry μ. □

The following one fixes p as initial state.

Lemma 9. *It suffices to have* p *as initial state.*

Proof. Let q be initial state. Then there are 2 cases to be considered, namely $h(q,0) = q$ and $h(q,0) = p$. The first case yields machines from DTM(2, 1) which have been considered already. Let therefore $h(q,0) = p$ with $f(q,0) = z$.

If $g(q,0) = M$ then let γ be the deterministic *gsm*-mapping defined by the finite automaton

$$\left(V \cup (V \times S), \{A,B\}, \{A\}, \{B\}, \left\{AxxA, A\binom{z}{p}\binom{0}{q}B, BxxB \mid x \in V\right\}\right)$$

with initial state A and final state B. Then

$$L_s(T,q) = \gamma\left(L_s(T,p) \cap V^*\binom{z}{p}V^*\right) \cup \binom{1}{q}.$$

If $g(q,0) = R$ then let δ and ε be deterministic *gsm*-mappings defined by the finite automata

$$\left(V \cup (V \times S), \{A,B,C\}, \{A\}, \{C\}, \left\{AxxA, A\binom{x}{p}xB, Bz\binom{0}{q}C, CxxC \mid x \in V\right\}\right)$$

with initial state A and final state C, and

$$\left(V \cup (V \times \{p\}), \{A,B\}, \{A\}, \{B\}, \left\{A\binom{x}{p}xB, BxxB \mid x \in V\right\}\right)$$

with initial state A and final state B. Then

$$L_s(T,q) = \binom{0}{q} \in (L_s(T,p) \cap (V \times \{p\}) \cdot V^*)$$

$$\cup\, rev(\delta(rev(L_s(T,p) \cap V^*z(V \times \{p\})V^*))) \cup \binom{1}{q}$$

where *rev* denotes the mirror image. □

Thus, the following cases remain to be considered: $h(q,0) = p$ with $g(q,0) = M$ or $g(q,0) = R$, and with $h(p,0) = h(p,1) = q$ or $h(p,0) = p, h(p,1) = q$ or $h(p,0) = q$, $h(p,1) = p$. $h(p,0) = h(p,1) = p$ trivially can be ruled out.

The next reduction eliminates $h(q,0) = q$.

Lemma 10. *It suffices to have $h(q,0) = p$ only.*

Proof. If $h(q,0) = q$ then, starting with p, all configurations ending in q yield regular sets since this is effectively a machine from DTM(2, 1). After reaching q this is again such a machine yielding regular sets only. □

Thus, after eliminating the case $g(p,0) = R, g(p,1) = L, g(q,0) = M$ by the symmetry μ, the following cases remain (omitting f and h):

	p	q
0	$\cdot L \cdot$	$\cdot Mp$
1	$\cdot R \cdot$	H

	p	q
0	$\cdot L \cdot$	$\cdot Rp$
1	$\cdot L \cdot$	H

	p	q
0	$\cdot L \cdot$	$\cdot Rp$
1	$\cdot M \cdot$	H

	p	q
0	$\cdot L \cdot$	$\cdot Rp$
1	$\cdot R \cdot$	H

	p	q
0	$\cdot M \cdot$	$\cdot Rp$
1	$\cdot L \cdot$	H

	p	q
0	$\cdot R \cdot$	$\cdot Rp$
1	$\cdot L \cdot$	H

with the possibilities for $h(p,0)$ and $h(p,1)$ mentioned above. This leaves $6 \cdot 3 \cdot 4 \cdot 2 = 144$ machines from originally $4 \cdot 12^3 = 6912$.

The first case to consider is

Case 1: $g(q,0) = M$ by the possibilities for $h(p,0)$ and $h(p,1)$.

Case 1.1: $h(p,0) = h(p,1) = q$. Then

$$0\binom{0}{p} \to \binom{0}{q} f(p,0) \to \binom{f(q,0)}{p} f(p,0),$$

$$1\binom{0}{p} \to \binom{1}{q} f(p,0) \to H,$$

$$\binom{1}{p} 0 \to f(p,1)\binom{0}{q} \to f(p,1)\binom{f(q,0)}{p},$$

$$\binom{1}{p} 1 \to f(p,1)\binom{1}{q} \to H.$$

Case 1.1.1: $f(q,0) = 0$. Then

$$0\binom{0}{p} \xrightarrow{*} \binom{0}{p} f(p,0),$$

$$\binom{1}{p} 0 \xrightarrow{*} f(p,1)\binom{0}{p} \xrightarrow{*} \binom{0}{p} f(p,0) \quad \text{if } f(p,1) = 0,$$

$$\binom{1}{p}0 \xrightarrow{*} f(p,1)\binom{0}{p} \xrightarrow{*} H \quad \text{if } f(p,1)=1.$$

which shows no effective movement to the right, thus yielding a regular set.

Case 1.1.2: $f(q,0)=1$. Then

$$0\binom{0}{p} \xrightarrow{*} \binom{1}{p}f(p,0) \xrightarrow{*} f(p,1)\binom{1}{p} \quad \text{if } f(p,0)=0,$$

$$0\binom{0}{p} \xrightarrow{*} \binom{1}{p}f(p,0) \xrightarrow{*} H \quad \text{if } f(p,0)=1,$$

$$\binom{1}{p}0 \xrightarrow{*} f(p,1)\binom{1}{p}.$$

which shows no effective movement to the left, thus also yielding a regular set.

Case 1.2: $h(p,0)=q, h(p,1)=p$. Then

$$0\binom{0}{p} \to \binom{0}{q}f(p,0) \to \binom{f(q,0)}{p}f(p,0),$$

$$1\binom{0}{p} \to \binom{1}{q}f(p,0) \to H,$$

$$\binom{1}{p}0 \to f(p,1)\binom{0}{p},$$

$$\binom{1}{p}1 \to f(p,1)\binom{1}{p}.$$

Case 1.2.1: $f(q,0)=0$. Then

$$0\binom{0}{p} \xrightarrow{*} \binom{0}{p}f(p,0)$$

and

$$L_s(T,p) = 10^*\binom{0}{p} \cup 10^*\binom{1}{p}1^*0 \quad \text{if } f(p,1)=0,$$

$$L_s(T,p) = 10^*\binom{0}{p} \cup \binom{1}{p}1^*0 \quad \text{if } f(p,1)=1.$$

Case 1.2.2: $f(q,0)=1$. Then

$$0\binom{0}{p} \xrightarrow{*} \binom{1}{p}f(p,0) \to f(p,1)\binom{f(p,0)}{p}$$

and the machine has no effective movement to the left, thus yielding a regular set.

Case 1.3: $h(p,0) = p$, $h(p,1) = q$. Then

$$0\binom{0}{p} \to \binom{0}{p} f(p,0),$$

$$1\binom{0}{p} \to \binom{1}{p} f(p,0),$$

$$\binom{1}{p} 0 \to f(p,1)\binom{0}{q} \to f(p,1)\binom{f(q,0)}{p},$$

$$\binom{1}{p} 1 \to f(p,1)\binom{1}{q} \to H.$$

Case 1.3.1: $f(q,0) = 0$. Then

$$\binom{1}{p} 0 \xrightarrow{*} f(p,1)\binom{0}{p} \to \binom{f(p,1)}{p} f(p,0).$$

This gives no effective movement to the right, thus yielding regular sets.

Case 1.3.2: $f(q,0) = 1$. Then

$$\binom{1}{p} 0 \xrightarrow{*} f(p,1)\binom{1}{p}$$

and

$$L_s(T,p) = 10^*\binom{0}{p}0^*1 \cup \binom{1}{p}0^*1 \quad \text{if } f(p,0) = 0,$$

$$L_s(T,p) = 10^*\binom{0}{p} \cup \binom{1}{p}0^*1 \quad \text{if } f(p,0) = 1.$$

Now the 120 remaining machines are considered with

Case 2: $g(q,0) = R$.

Case 2.1: $g(p,0) = g(p,1) = L$.

Case 2.1.1: $h(p,0) = h(p,1) = p$:

$$x\binom{y}{p} \to \binom{x}{p} f(p,y)$$

yielding no halt at all.

Case 2.1.2: $h(p,0) = p$, $h(p,1) = q$ or $h(p,0) = q$, $h(p,1) = p$. Let $m \in \{0,1\}$ and $n = 1 - m$ such that $h(p,m) = p$ and $h(p,n) = q$. Then

$$x\binom{m}{p} \to \binom{x}{p} f(p,m),$$

$$0\binom{n}{p} \to \binom{0}{q} f(p,n) \to f(q,0)\binom{f(p,n)}{p},$$

$$1\binom{n}{p} \to \binom{1}{q} f(p,n) \to H$$

giving no effective movement to the right, and thus only regular sets.

Case 2.1.3: $h(p,0) = h(p,1) = q$:

$$0\binom{y}{p} \to \binom{0}{q} f(p,y) \to f(q,0)\binom{f(p,y)}{p},$$

$$1\binom{y}{p} \to \binom{1}{q} f(p,y) \to H$$

giving no effective movement at all, and thus only regular sets.

For the remaining 2 subcases the following considerations give a further reduction of the number of cases.

Let $j \in \{0,1\}$ be such that $g(p,j) = L$ and $k = 1-j$. If $h(p,j) = q$ then there is no effective movement to the left since

$$x\binom{j}{p} \to \binom{x}{q} f(p,j) \to f(q,x)\binom{f(p,j)}{p},$$

$$\binom{k}{p} \to \binom{f(p,k)}{h(p,k)} \quad \text{if } g(p,k) = M,$$

$$\binom{k}{p} y \to f(p,k)\binom{y}{h(p,k)} \quad \text{if } g(p,k) = R,$$

thus giving only regular sets. Therefore, $h(p,j) = p$ has to be considered only.

If $h(p,k) = p$ then the machine will never halt. Thus assume $h(p,k) = q$.

Case 2.2: $g(p,k) = M$. Then

$$j\binom{j}{p} \to \binom{j}{p} f(p,j),$$

$$k\binom{j}{p} \to \binom{k}{p} f(p,j) \to \binom{0}{q} f(p,j) \to f(q,0)\binom{f(p,j)}{p} \quad \text{if } f(p,k) = 0,$$

$$\binom{k}{p} y \to \binom{0}{q} y \to f(q,0)\binom{y}{p},$$

$$k\binom{j}{p} \to \binom{k}{p} f(p,j) \to \binom{1}{q} f(p,j) \to H \quad \text{if } f(p,k) = 1,$$

$$\binom{k}{p} y \to \binom{1}{q} y \to H$$

yielding effectively a machine from $DTM(2,1)$ with only regular sets.

Thus

Case 2.3: $g(p,k)=R$ may be assumed such that the following cases remain: $g(p,j)=L, h(p,j)=p,\ g(p,k)=R, h(p,k)=q, g(q,0)=R, h(q,0)=p$ with $j, f(p,j), f(p,k), f(q,0) \in \{0,1\}$ and $k \neq j$. This leaves 16 Turing machines to consider.

At first

Case 2.3.1: $f(p,j)=1$ one has

$$j\binom{j}{p} \to \binom{j}{p}1,$$

$$k\binom{j}{p} \to \binom{k}{p}1 \to f(p,k)\binom{1}{q} \to H,$$

$$\binom{k}{p}0j \to f(p,k)\binom{0}{q}j \to f(p,k)f(q,0)\binom{j}{p},$$

$$\binom{k}{p}0k \to f(p,k)\binom{0}{q}k \to f(p,k)f(q,0)\binom{k}{p},$$

$$\binom{k}{p}1 \to f(p,k)\binom{1}{q} \to H$$

yielding as accepted sets

$$L_s(T,p) = kj^*\binom{j}{p} \cup \binom{k}{p}(0k)^*1 \cup L_s'(T,p)$$

with

$$L_s'(T,p) = \binom{k}{p}(0k)^*0j \qquad \text{if } f(q,0)=k \text{ or } (f(q,0)=j \text{ and } f(p,k)=k),$$

$$L_s'(T,p) = kj^*\binom{k}{p}(0k)^*0j \quad \text{if } f(q,0)=f(p,k)=j$$

being all regular.

Case 2.3.2: $f(p,j)=0$. Here one has

$$j\binom{j}{p} \to \binom{j}{p}0,$$

$$k\binom{j}{p}j \to \binom{k}{p}0j \to f(p,k)\binom{0}{q}j \to f(p,k)f(q,0)\binom{j}{p},$$

$$k\binom{j}{p}k \to \binom{k}{p}0k \to f(p,k)\binom{0}{q}k \to f(p,k)f(q,0)\binom{k}{p},$$

$$\binom{k}{p}0j \to f(p,k)\binom{0}{q}j \to f(p,k)f(q,0)\binom{j}{p},$$

$$\binom{k}{p}0k \to f(p,k)\binom{0}{q}k \to f(p,k)f(q,0)\binom{k}{p},$$

$$\binom{k}{p}1 \to f(p,k)\binom{1}{q} \to H.$$

Case 2.3.2.1: $f(q,0) = k$. This yields as accepted sets

$$L_s(T,p) = \left(10^*\binom{0}{p}0^*1 \cup \binom{1}{p}\right)(0^+1)^*1 \quad \text{if } k = 1,$$

$$L_s(T,p) = \left(\binom{0}{p} \cup 0\binom{1}{p}1^*0 \cup 01^+\binom{1}{p}\right)(01^*0)^*1 \quad \text{if } k = 0$$

being all regular.

Case 2.3.2.2: $f(q,0) = j$. Here all 4 remaining machines have to be considered.

$$j = 0 : \ L_s(T,p) = \left(1\binom{0}{p}1 \cup \binom{1}{p}\right)(01)^*1$$

$j = 1, f(p,0) = 0$:

$$L_s(T,p) = \left(\binom{0}{p} \cup 0(11)^*\left(\binom{1}{p}(0 \cup 1) \cup 1\binom{1}{p}\right)\right)(0(0 \cup 1))^*1$$

all being regular.

The only remaining machine, with $j = 1$, $f(p,0) = 1$ however, does not yield a regular set but

$$\begin{aligned} L_s(T,p) = \Big\{ & 0(11)^{m_k}u_{k-1}\cdots u_1(11)^{m_1}\binom{0}{p}0(00)^{n_0}1(00)^{n_1}v_1\cdots v_{k-1}(00)^{n_k}01, \\ & 0(11)^{m_k}u_{k-1}\cdots u_1(11)^{m_1}\binom{1}{p}(00)^{n_1}v_1\cdots v_{k-1}(00)^{n_k}01, \\ & 01(11)^{m_k}u_{k-1}\cdots u_1(11)^{m_1}\binom{0}{p}0(00)^{n_0}1(00)^{n_1}v_1\cdots v_{k-1}(00)^{n_k}1, \\ & 01(11)^{m_k}u_{k-1}\cdots u_1(11)^{m_1}\binom{0}{p}(00)^{n_1}v_1\cdots v_{k-1}(00)^{n_k}1 \mid \\ & k \geqslant 0, m_i \geqslant 0, n_i \geqslant 0, (u_i = 0, v_i = 1) \text{ or } (u_i = v_i = 01)\Big\}. \end{aligned}$$

It is easy to see that this set is not regular but a linear deterministic context-free language being similar to that one from $DTM(3,1)$.

Summarizing, one obtains

Theorem 4. *All sets accepted by Turing machines from $DTM(2,2)$ are regular except one case only (up to symmetries) giving a linear deterministic context-free language. The halting problem for $DTM(2,2)$ is decidable.*

It should be noted that the machine last mentioned is not the beasy beaver machine for $DTM(2,2)$ since that one is the following which can be shown easily:

	p	q
0	$1Lq$	$1Rp$
1	Rq	H

It should also be mentioned that in most cases the sets $L_s(T,p)$ and $L_c(T,p)$ are identical, but that there are some exceptions.

7. DTM(4, 4)

Here only a machine accepting a language being not context-free is presented. This machine is

	p	q	r	s
0	R	R	L	L
1	Rq	$0Rp$	L	Lq
2	R	H	Rp	$\cdots$
3	Lr	$2Ls$	$\cdots$	$\cdots$

where the open positions are irrelevant and may be filled by M.

For the accepted language the following fact holds:

$$L_s(T,p) \cap \binom{2}{p} 1^*33 = \left\{ \binom{2}{p} 1^{2^n} 33 | n \geqslant 0 \right\}.$$

showing that $L_s(T,p)$ is not context-free.

Recently, it has been shown [3] that there exist Turing machines in $DTM(2,5)$ and $DTM(3,6)$ closely related to Collatz problems with unknown solutions.

Finally, in 1992 the existence of a universal Turing machine in $DTM(3,10)$ was shown [8], and in 1995 the existence of another one in $DTM(18,2)$ [9].

There exists also an unpublished proof by Pavlockaya on the decidability of the halting problem for $DTM(3,2)$.

Acknowledgements

The author thanks the unknown reviewer for careful reading and useful hints.

References

[1] V. Diekert and M. Kudlek, Small deterministic Turing machines, Papers on Automata and Languages, Dept. of Math., Karl Marx Univ. of Economics, Budapest, 1988–4, (1989) 77–87.

[2] G.T. Herman, The Halting problem of one state Turing machines with n-dimensional tape, *Z. Math. Logik Grundl. Math.* **14** (1968) 185–191.

[3] P. Michel, Busy beaver competition and Collatz-like problems, *Arch. Math. Logic* **32** (1993) 351–367.

[4] M.I. Minsky, Size and structure of universal Turing machines using tag systems, in: *Recursive Function Theory, Symposia in Pure Mathematics* Vol. 5 (AMS, Providence, RI, 1962) 229–238.

[5] M.I. Minsky, *Computation. Finite and Infinite Machines* (Prentice-Hall, Englewood Cliffs, NJ, 1967–1972).

[6] L.M. Pavlockaya, Dostatočnye usloviya razrešimosti problemy ostanovki dlya mašin Tyuringa, *Problemy Kibernetiki* **33** (1978) 91–118.

[7] Yu.V. Rogozhin, Sem' universal'nyh mašin Tyuringa, *Matematičeskie Issledovaniya* **69** (1982) 76–90.

[8] Yu.V. Rogozhin, Universal'naya mašina Tyuringa s 10 sostayaniyah i 3 simbolami, *Izvestiya Akademii Nauk Respubliki Moldova Matematika* (10) (1992) 80–82.

[9] Yu.V. Rogozhin, On the notion of universality and Shannon's problem for Turing machines, Presented at MCU/UMC'95, Paris.

[10] S. Watanabe, 5-symbol 8-state and 5-symbol 6-state universal Turing machines, *J. ACM* **8** (1961) 476–483.