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Monodromy eigenvalues and zeta functions with differential forms

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Abstract

For a complex polynomial or analytic function f, there is a strong correspondence between poles of the so-called local zeta functions or complex powers $\int |f|^{2s}\omega$, where the ω are C^∞ differential forms with compact support, and eigenvalues of the local monodromy of f. In particular Barlet showed that each monodromy eigenvalue of f is of the form $\exp(2\pi\sqrt{-1}s_0)$, where s_0 is such a pole. We prove an analogous result for similar p-adic complex powers, called Igusa (local) zeta functions, but mainly for the related algebro-geometric topological and motivic zeta functions.

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Keywords: Monodromy; Topological zeta function; Differential form

0. Introduction

0.1. Let $f : X \to \mathbb{C}$ be a non-constant analytic function on an open part X of $\mathbb{C}^n$. We consider C^∞ functions φ with compact support on X and the corresponding differential forms $\omega = \varphi\, dx \wedge d\bar{x}$. Here and further $x = (x_1, \dots, x_n)$ and $dx = dx_1 \wedge \cdots \wedge dx_n$. For such ω the integral

$$Z(f, \omega; s) := \int_X \left|f(x)\right|^{2s} \omega,$$

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where $s \in \mathbb{C}$ with $\Re(s) > 0$, has been the object of intensive study. One verifies that $Z(f, \omega; s)$ is holomorphic in s. Either by resolution of singularities [7,16], or by the theory of Bernstein polynomials [15], one can show that it admits a meromorphic continuation to $\mathbb{C}$, and that all its poles are among the translates by $\mathbb{Z}_{<0}$ of a finite number of rational numbers. Combining results of Barlet [9,12], Kashiwara [27] and Malgrange [31], the poles of (the extended) $Z(f, \omega; s)$ are strongly linked to the eigenvalues of (local) monodromy at points of $\{f = 0\}$; see Section 1 for the concept of monodromy.

Theorem.

(1) *If s_0 is a pole of $Z(f, \omega; s)$ for some differential form ω, then* $\exp(2\pi\sqrt{-1}s_0)$ *is a monodromy eigenvalue of f at some point of $\{f = 0\}$.*
(2) *If λ is a monodromy eigenvalue of f at a point of $\{f = 0\}$, then there exist a differential form ω and a pole s_0 of $Z(f, \omega; s)$ such that $\lambda =$* $\exp(2\pi\sqrt{-1}s_0)$.

There are also more precise local versions in a neighbourhood of a point of $\{f = 0\}$. Similar results hold for a real analytic function $f : X (\subset \mathbb{R}^n) \to \mathbb{R}$ and integrals $\int_{X \cap \{f>0\}} f^s \varphi\, dx$; we refer to e.g. [10,11,13,14,26].

0.2. Let now $f : X \to \mathbb{Q}_p$ be a non-constant ($\mathbb{Q}_p$-)analytic function on a compact open $X \subset \mathbb{Q}_p^n$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Let $|\cdot|_p$ and $|dx|$ denote the p-adic norm and the Haar measure on $\mathbb{Q}_p^n$, normalized in the standard way. The p-adic integral

$$Z_p(f; s) := \int_X |f(x)|_p^s\, |dx|,$$

again defined for $s \in \mathbb{C}$ with $\Re(s) > 0$, is called the (p-adic) Igusa zeta function of f. Using resolution of singularities Igusa [24] showed that it is a rational function of p^{-s}; hence it also admits a meromorphic continuation to $\mathbb{C}$. In this context there is an intriguing conjecture of Igusa relating poles of (the extended) $Z_p(f; s)$ to eigenvalues of monodromy. More precisely, let f be a polynomial in n variables over $\mathbb{Q}$. Then we can consider $Z_p(f; s)$ for all prime numbers p (taking $X = \mathbb{Z}_p^n$).

Monodromy conjecture. (See [17].) *For all except a finite number of p, we have that, if s_0 is a pole of $Z_p(f; s)$, then* $\exp(2\pi\sqrt{-1}s_0)$ *is a monodromy eigenvalue of $f : \mathbb{C}^n \to \mathbb{C}$ at a point of $\{f = 0\}$.*

This conjecture was proved for $n = 2$ by Loeser [29]. There are by now various other partial results [5,6,30,34,35,40]. (We took $\mathbb{Q}_p$ for simplicity of notation; everything can be done over finite extensions of $\mathbb{Q}_p$.)

0.3. There are various 'algebro-geometric' zeta functions, related to the p-adic Igusa zeta functions: the motivic, Hodge and topological zeta functions, for which we refer to Section 1. Here we just mention that the motivic zeta function specializes to the various p-adic Igusa zeta functions (for almost all p). For those zeta functions a similar monodromy conjecture can be stated; and analogous partial results are valid.

0.4. We should note that for the complex (and real) integrals in 0.1 there are more precise results of Bernstein and Barlet, involving roots of the Bernstein polynomial of f (instead of monodromy eigenvalues). Similarly there is a finer conjecture for the poles of Igusa and related zeta functions, relating them to roots of the Bernstein polynomial [17,19]. However, the results of this paper do not involve Bernstein polynomials, so we just refer the interested reader to [8–10,15,25,28–30].

0.5. As in the complex (or real) case, one associates p-adic Igusa zeta functions, and also motivic, Hodge and topological zeta functions, to a function f *and* a differential form ω. In this 'algebro-geometric' context one considers algebraic differential forms ω; see Section 1.

To our knowledge a possible analogue of Theorem 0.1(2) in the context of p-adic and the related 'algebro-geometric' zeta functions was not studied before in the literature. For instance let f be a polynomial over $\mathbb{Q}$ satisfying $f(0)=0$, and let λ be a monodromy eigenvalue of f at 0. Does there exist a compact open neighbourhood X of 0 and an algebraic differential form ω such that (the meromorphic continuation of) $\int_X |f(x)|_p^s |\omega|_p$ has a pole s_0 satisfying $\lambda = \exp(2\pi\sqrt{-1}s_0)$? (If $\omega = g(x)\,dx$ for some polynomial g over $\mathbb{Q}$, the integral above is just $\int_X |f(x)|_p^s |g(x)|_p\,|dx|$.)

0.6. We will concentrate in this paper on the analogous question for the topological zeta function, since a positive answer in this context automatically yields a positive answer in the context of Hodge and motivic zeta functions (see Section 1), and also for Igusa zeta functions (by [19, Théorème 2.2]). We show for instance (Theorem 3.6):

Theorem. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a non-zero polynomial function (germ). Let λ be a monodromy eigenvalue of f at 0. Then there exist a differential n-form ω and a point $P\in\{f=0\}$, close to 0, such that the (local) topological zeta function at P, associated to f and ω, has a pole s_0 satisfying $\exp(2\pi\sqrt{-1}s_0)=\lambda$.*

If $f^{-1}\{0\}$ has an isolated singularity at 0, then we can take 0 itself as point P.

For $n=2$, we construct such ω in Section 2 using so-called curvettes. In arbitrary dimension we follow a similar approach, for which we first introduce a higher dimensional version of this notion (Proposition 3.2).

0.7. The zeta functions associated to f and the constructed ω in the theorem above can have *other* poles that do not induce monodromy eigenvalues of f. So for those zeta functions the analogue of Theorem 0.1(1) is (unfortunately) not true. It would be really interesting to have a complete analogue of Theorem 0.1, roughly saying that the monodromy eigenvalues of f correspond precisely to the poles of the zeta functions associated to f and some finite list of differential forms ω (including dx). Of course this would be a lot stronger than the (in arbitrary dimension) still wide open monodromy conjecture.

However, we indicate some examples where such a correspondence holds, for instance $f = y^a - x^b$ with $\gcd(a,b)=1$.

1. Monodromy and zeta functions

1.1. Let $f:\mathbb{C}^n\to\mathbb{C}$ be a non-constant polynomial function satisfying $f(b)=0$. Let $B\subset\mathbb{C}^n$ be a small enough ball with centre b; the restriction $f|_B$ is a topological fibration over a small enough pointed disc $D\subset\mathbb{C}\setminus\{0\}$ with centre 0. The fibre F_b of this fibration is called the (local) Milnor fibre of f at b; see e.g. [32]. The counterclockwise generator of the fundamental group of D induces an automorphism of the cohomologies $H^q(F_b, \mathbb{C})$, which is called the (local) monodromy of f at b. By a monodromy eigenvalue of f at b we mean an eigenvalue of the monodromy action on a least one of the $H^q(F_b, \mathbb{C})$. It is well known that $H^q(F_b, \mathbb{C}) = 0$ for $q \geqslant n$, and that all monodromy eigenvalues are roots of unity.

Let $P_q(t)$ denote the characteristic polynomial of the monodromy action on $H^q(F_b, \mathbb{C})$. If $f = \prod_j f_j^{N_j}$ is the decomposition of f in irreducible components and $d := \gcd_j N_j$, then $P_0(t) = t^d - 1$.

When b is an isolated singularity of $f^{-1}\{0\}$, then $H^q(F_b, \mathbb{C}) = 0$ for $q \neq 0, n-1$; and $P_0(t) = t - 1$.

1.2. Definition. The *monodromy zeta function* $\zeta_{f,b}(t)$ of f at b is the alternating product of all characteristic polynomials $P_q(t)$:

$$\zeta_{f,b}(t) := \prod_{q=0}^{n-1} P_q(t)^{(-1)^q}.$$

Note that there are also other conventions, see for example [2,3].

In particular for an isolated singularity the knowledge of $\zeta_{f,b}(t)$ and of $P_{n-1}(t)$ are equivalent.

1.3. We recall the following interesting and useful result, which is maybe not generally known.

Lemma. (See [18, Lemma 4.6].) *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial function. If λ is a monodromy eigenvalue of f at $b \in f^{-1}\{0\}$, then there exists $P \in f^{-1}\{0\}$ (arbitrarily close to b) such that λ is a zero or a pole of the monodromy zeta function of f at P.*

It is convenient to recall also the proof in order to see how the point P is obtained. Let $\Psi_{f,\lambda}$ be the sub-complex of the complex of nearby cycles of f corresponding to the eigenvalue λ, where both are viewed as (shifted) perverse sheaves. Let Σ be the largest analytic set given by the supports of the cohomology sheaves of $\Psi_{f,\lambda}$. Then, by perversity of $\Psi_{f,\lambda}$, at a generic point P of Σ the eigenvalue λ appears on exactly one cohomology group of the Milnor fibre of f at P.

1.4. A'Campo's formula. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial function satisfying $f(b) = 0$. Take an embedded resolution $\pi : X \to \mathbb{C}^n$ of $f^{-1}\{0\}$ (that is an isomorphism outside the inverse image of $f^{-1}\{0\}$). Denote by E_i, $i \in S$, the irreducible components of the inverse image $\pi^{-1}(f^{-1}\{0\})$, and by N_i the multiplicity of E_i in the divisor of $\pi^* f$. We put $E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j$ for $i \in S$.

Theorem. (See [2].) *Denoting by $\chi(\cdot)$ the topological Euler characteristic we have*

$$\zeta_{f,b}(t) = \prod_i \left(t^{N_i} - 1\right)^{\chi(E_i^\circ \cap \pi^{-1}\{b\})}.$$

1.5. Another kind of zeta functions are the topological, Hodge and motivic zeta functions, associated to a non-constant polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ and a regular differential n-form ω

on $\mathbb{C}^n$. (More generally one can consider an arbitrary smooth quasi-projective variety X_0 instead of $\mathbb{C}^n$ and a regular function f.) We will describe these zeta functions in terms of an embedded resolution of $f^{-1}\{0\} \cup \operatorname{div}\omega$. Now we denote by E_i, $i \in S$, the irreducible components of the inverse image $\pi^{-1}(f^{-1}\{0\} \cup \operatorname{div}\omega)$ and by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor of $\pi^* f$ and $\pi^*\omega$, respectively. We put $E_I^\circ := (\bigcap_{i\in I} E_i) \setminus (\bigcup_{j\notin I} E_j)$ for $I \subset S$, in particular $E_\emptyset^\circ = X \setminus (\bigcup_{j\in S} E_j)$. So the E_I° form a stratification of X in locally closed subsets.

Definition. The (global) *topological zeta function* of (f, ω) and its local version at $b \in \mathbb{C}^n$ are

$$Z_{\mathrm{top}}(f, \omega; s) := \sum_{I\subset S} \chi\big(E_I^\circ\big) \prod_{i\in I} \frac{1}{\nu_i + sN_i},$$

and

$$Z_{\mathrm{top},b}(f, \omega; s) := \sum_{I\subset S} \chi\big(E_I^\circ \cap \pi^{-1}\{b\}\big) \prod_{i\in I} \frac{1}{\nu_i + sN_i},$$

respectively, where s is a variable.

These invariants were introduced by Denef and Loeser in [19] for 'trivial ω,' i.e. for $\omega = dx_1 \wedge \cdots \wedge dx_n$. Their original proof that these expressions do not depend on the chosen resolution is by describing them as a kind of limit of p-adic Igusa zeta functions. Later they obtained them as a specialization of the intrinsically defined motivic zeta functions [20]. Another technique is applying the Weak Factorization Theorem [1,41] to compare two different resolutions. For arbitrary ω one can proceed analogously.

It is natural and useful to study these invariants incorporating such a more general ω, see for example [5,6,38]. Note however that there one restricts to the situation where $\operatorname{supp}(\operatorname{div}\omega) \subset f^{-1}\{0\}$. In the original context of p-adic Igusa zeta functions, see e.g. [29, III 3.5].

1.6. There are finer variants of these zeta functions using, instead of Euler characteristics, Hodge polynomials or classes in the Grothendieck ring of varieties. We mention for instance the *Hodge zeta function*

$$Z_{\mathrm{Hod}}(f, \omega; T) = \sum_{I\subset S} H\big(E_I^\circ; u, v\big) \prod_{i\in I} \frac{(uv-1)T^{N_i}}{(uv)^{\nu_i} - T^{N_i}} \in \mathbb{Q}(u, v)(T),$$

where $H(\cdot; u, v) \in \mathbb{Z}[u, v]$ denotes the Hodge polynomial. Concerning Hodge and motivic zeta functions we refer to e.g. [20,21,33,39] for versions with 'trivial ω'; and to [5,6,38] involving more general ω. We just mention that, in contrast with topological zeta functions, Hodge and motivic zeta functions can be defined intrinsically as formal power series (in T) with coefficients determined by the behaviour of the arcs on $\mathbb{C}^n$ with respect to their intersection with $f^{-1}\{0\}$ and with $\operatorname{div}(\omega)$. Then one shows that they are rational functions (in T) by proving explicit formulae as above in terms of an embedded resolution. However, the fact that we allow (and need) differential forms ω with $\operatorname{supp}(\operatorname{div}\omega) \not\subset f^{-1}\{0\}$, causes some technical complications. In order to avoid these, one can *define* Hodge and motivic zeta functions by a formula as above in terms of an embedded resolution, and use the Weak Factorization Theorem to show independency of the choice of resolution. (In fact in this paper we will only use differential forms ω for which a given embedded resolution of $f^{-1}\{0\}$ is also an embedded resolution of $f^{-1}\{0\} \cup \operatorname{div} \omega$.)

1.7. We now explain why we choose not to give details here about Hodge and motivic zeta functions. The point is that, for a given f and ω, the motivic zeta function specializes to the Hodge zeta function, which in turn specializes to the topological zeta function. (Note for instance that $H(\cdot; 1, 1) = \chi(\cdot)$.) In particular, a pole of the topological zeta function will induce a pole of the other two. (The converse is not clear.) The problem that we want to treat here is, given a monodromy eigenvalue λ of f, find a form ω such that the zeta function associated to f and ω has a pole 'inducing λ.' Therefore in this paper we focus on the topological zeta function. We will succeed in proving the desired result for it, implying the analogous result for the 'finer' zeta functions.

2. Curves

2.1. We first prove our main result for curves. Ultimately Theorem 2.4 below will be essentially a special case of Theorems 3.5 and 3.6. It is however more precise in the case of non-isolated singularities. We also believe that it is useful to treat the case of curves first. The proof is easier and shorter, and we indicate a fact that is typical for curves.

2.2. In order to construct appropriate differential forms ω we use curve germs (sometimes called *curvettes*) that intersect the exceptional components of an embedded resolution transversely; we quickly recall this notion. Let

$$\mathbb{A}^2 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_i} X_i \xleftarrow{\pi_{i+1}} \cdots \xleftarrow{\pi_m} X_m$$

be a composition π of m blowing-ups with 0 the centre of π_1, and the centre of all other π_i belonging to the exceptional locus of $\pi_1 \circ \cdots \circ \pi_{i-1}$. In other words, all centres are points infinitely near to 0. Denote the exceptional curve of π_i, as well as its strict transform in X_m, by E_i. A *curvette* C_i of E_i is a smooth curve (germ) on X_m satisfying $C_i \cdot E_j = \delta_{ij}$ for all $j = 1, \ldots, m$. So C_i intersects E_i transversely in a point not belonging to other E_j. We denote $\bar{C}_i := \pi(C_i)$ the image curve (germ) of C_i in $(\mathbb{A}^2, 0)$.

We guess that the following should be known.

2.3. Proposition. *Let $\pi^*\bar{C}_i = \sum_{j=1}^m a_{ij} E_j + C_i$ for $i = 1, \ldots, m$. Then the determinant of the $(m \times m)$-matrix (a_{ij}) is equal to* 1*. In particular* $\gcd_{1 \leqslant i \leqslant m}\{a_{ij}\} = 1$ *for all j.*

Proof. For all $i, \ell \in \{1, \ldots, m\}$ we have

$$0 = \left(\pi^*\bar{C}_i\right) \cdot E_\ell = \sum_{j=1}^m a_{ij} E_j \cdot E_\ell + \delta_{i\ell}.$$

In other words, the matrix product $(a_{ij}) \cdot (-E_j \cdot E_\ell)$ is the identity matrix. Since minus the intersection matrix of the E_i has determinant 1, the same is true for (a_{ij}). $\square$

Note. It also follows that (a_{ij}) is symmetric, being the inverse of a symmetric matrix.

2.4. Theorem. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a non-zero polynomial function (germ). Let λ be a monodromy eigenvalue of f at 0, i.e. λ is an eigenvalue for the action of the local monodromy on $H^0(F_0, \mathbb{C})$ or $H^1(F_0, \mathbb{C})$. Then there exists a differential 2-form ω on $(\mathbb{C}^2, 0)$ such that $Z_{\mathrm{top},0}(f, \omega; s)$ has a pole s_0 satisfying $\exp(2\pi\sqrt{-1}s_0) = \lambda$.*

Proof. Take the minimal embedded resolution $\pi : X \to (\mathbb{C}^2, 0)$ of the curve (germ) $f^{-1}\{0\}$. Denote the irreducible components of $\pi^{-1}(f^{-1}\{0\})$, i.e. the exceptional curves and the components of the strict transform, by E_i and their multiplicities in the divisors of $\pi^* f$ and $\pi^*(dx \wedge dy)$ by N_i and $\nu_i - 1$, respectively. Take for each exceptional curve E_i a (generic) curvette $C_i \subset X$. Say $\bar{C}_i := \pi(C_i)$ is given by the (reduced) equation $g_i = 0$.

We first suppose that λ is a pole of the monodromy zeta function of f at 0. Say λ is a primitive dth root of unity. By A'Campo's formula there exists an exceptional curve E_{j_0} with $d \mid N_{j_0}$ and $\chi(E_j^\circ) < 0$. So E_{j_0} intersects at least three times other components E_j.

We associate to each curvette the following multiplicities:

$$\operatorname{div}(\pi^* g_\ell) = \sum_j a_{\ell j} E_j + C_\ell.$$

Note that $a_{\ell j} \neq 0$ only if E_j is exceptional. We take a differential form ω of the form $(\prod_\ell g_\ell^{m_\ell})\, dx \wedge dy$. The multiplicity of E_i in the divisor of $\pi^*\omega$ is $\nu_i - 1 + \sum_\ell a_{\ell i} m_\ell$. So in particular the candidate pole for $Z_{\mathrm{top},0}(f, w; s)$ associated to E_{j_0} is

$$s_0 = -\frac{\nu_{j_0} + \sum_\ell a_{\ell j_0} m_\ell}{N_{j_0}}.$$

We will find suitable m_ℓ such that (1) $\exp(2\pi\sqrt{-1}s_0) = \lambda$, and (2) s_0 is really a pole.

(1) Say $\lambda = \exp(-2\pi\sqrt{-1}\frac{b}{d})$ with $1 \leqslant b \leqslant d$ and $\gcd(b, d) = 1$. Since $\gcd_\ell\{a_{\ell j_0}\} = 1$ by Proposition 2.3, there exist $m_\ell \in \mathbb{Z}$ such that $\nu_{j_0} + \sum_\ell a_{\ell j_0} m_\ell = b\frac{N_{j_0}}{d}$. Consequently there exist $m_\ell \in \mathbb{Z}_{\geqslant 0}$ satisfying

$$\nu_{j_0} + \sum_\ell a_{\ell j_0} m_\ell = b\frac{N_{j_0}}{d} \bmod N_{j_0},$$

and we can choose such m_ℓ freely in their congruence class $\bmod N_{j_0}$. For all those m_ℓ clearly $\exp(2\pi\sqrt{-1}s_0) = \exp(-2\pi\sqrt{-1}\frac{b}{d}) = \lambda$.

(2) The candidate pole for $Z_{\mathrm{top},0}(f, \omega; s)$ associated to a component E_i of the strict transform is $-\frac{1}{N_i}$, and is thus different from s_0 for 'most' m_ℓ. The candidate pole associated to another exceptional E_i is $-\frac{\nu_i + \sum_\ell a_{\ell i} m_\ell}{N_i}$. Suppose that it is equal to s_0. Then

$$\frac{\nu_i}{N_i} - \frac{\nu_{j_0}}{N_{j_0}} + \sum_\ell \left(\frac{a_{\ell i}}{N_i} - \frac{a_{\ell j_0}}{N_{j_0}}\right) m_\ell = 0. \tag{$*$}$$

We know that $\det(a_{ij}) \neq 0$ by Proposition 2.3; in particular the vectors $(a_{\ell i})_\ell$ and $(a_{\ell j_0})_\ell$ cannot be dependent. So at least one of the coefficients of the m_ℓ in $(*)$ is non-zero, i.e. $(*)$ is never an

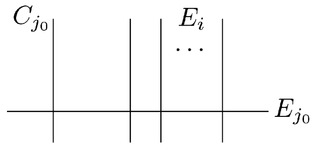

Fig. 1.

empty condition. Consequently 'most' sets $(m_\ell)_\ell$ in our allowed lattice satisfy $s_0 \neq -\frac{\nu_i+\sum_\ell a_{\ell i} m_\ell}{N_i}$ for all $i \neq j_0$.

The residue of s_0 (as candidate pole of order 1 for $Z_{\mathrm{top},0}(f,\omega;s)$) is

$$\frac{1}{N_{j_0}}\left(\chi\left(E_{j_0}^\circ\right) - 1 + \frac{1}{1+m_{j_0}} + \sum_i \frac{1}{\alpha_i}\right)$$

where $\alpha_i := \nu_i - \frac{\nu_{j_0}}{N_{j_0}} N_i + \sum_\ell (a_{\ell i} - a_{\ell j_0}\frac{N_i}{N_{j_0}}) m_\ell$ for an E_i intersecting E_{j_0}. (See Fig. 1.)

Since $\chi(E_{j_0}^\circ) - 1 \neq 0$, this expression is never identically zero as function in the m_ℓ, and hence non-zero for 'most' choices of $(m_\ell)_\ell$.

Secondly, if an eigenvalue λ of f at 0 is not a pole of the monodromy zeta function, it must be an eigenvalue on $H^0(F_0, \mathbb{C})$. By 1.1 the order d of λ as root of unity must divide all N_j associated to components of the strict transform. But then d divides all N_i.

Pick now any exceptional E_{j_0} with $\chi(E_{j_0}^\circ) < 0$ and proceed as in the previous case to construct a suitable ω and a pole s_0 of $Z_{\mathrm{top},0}(f,\omega;s)$ with $\exp(2\pi\sqrt{-1}s_0) = \lambda$. (There is always an exceptional E_i with $\chi(E_i^\circ) < 0$, except in the trivial case where $f^{-1}\{0\}$ is smooth or has normal crossings at 0. But then also the theorem is quite trivial, see Example 2.6.) □

2.5. Note. The zeta functions $Z_{\mathrm{top},0}(f,\omega;s)$ constructed in the proof above can in general have other poles that *do not* induce monodromy eigenvalues of f. It would be interesting to investigate the validity of the following statement, or its analogue for Hodge and motivic zeta functions.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *be a non-zero polynomial function* (*germ*). *There exist regular differential* 2*-forms* $\omega_1, \dots, \omega_r$ *on* $(\mathbb{C}^2, 0)$ *such that*

(1) *if* s_0 *is a pole of a zeta function* $Z_{\mathrm{top},0}(f,\omega_i;s)$*, then* $\exp(2\pi\sqrt{-1}s_0)$ *is a monodromy eigenvalue of* f *at* 0*, and*
(2) *for each monodromy eigenvalue* λ *of* f *at* 0*, there is a differential form* ω_i *and a pole* s_0 *of* $Z_{\mathrm{top},0}(f,\omega_i;s)$ *such that* $\exp(2\pi\sqrt{-1}s_0) = \lambda$.

We present some examples of this principle.

2.6. Baby example. (1) Let $f = x^N$ on $(\mathbb{C}^2, 0)$. The monodromy eigenvalues of f at 0 are $\exp(2\pi\sqrt{-1}\frac{b}{N})$ with $1 \leqslant b \leqslant N$. Take $\omega_b := x^{b-1}\,dx \wedge dy$ for $b = 1, \dots, N$. We have $Z_{\mathrm{top},0}(f,\omega_b;s) = \frac{1}{b+sN}$ with unique pole $s_0 = -\frac{b}{N}$.

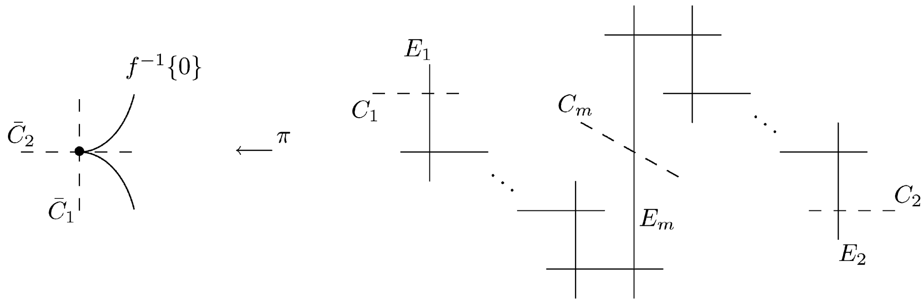

Fig. 2.

(2) Let $f = x^{dN} y^{dN'}$ on $(\mathbb{C}^2, 0)$ with $\gcd(N, N') = 1$. The monodromy eigenvalues of f at 0 are $\exp(2\pi\sqrt{-1}\frac{b}{d})$ with $1 \leqslant b \leqslant d$. Take $\omega_b := x^{bN-1} y^{bN'-1}\, dx \wedge dy$ for $b = 1, \ldots, d$. We have

$$Z_{\mathrm{top},0}(f, \omega_b; s) = \frac{1}{(bN + sdN)(bN' + sdN')} = \frac{1}{NN'(b + sd)^2}$$

with unique pole (of order 2) $s_0 = -b/d$.

2.7. Proposition. *Let $f = y^p - x^q$ on $(\mathbb{C}^2, 0)$ with $2 \leqslant p < q$ and* $\gcd(p, q) = 1$. *Take $\omega_{ij} := x^{i-1} y^{j-1} dx \wedge dy$ for $1 \leqslant i \leqslant q - 1$ and $1 \leqslant j \leqslant p - 1$.*

(1) *If s_0 is a pole of $Z_{\mathrm{top},0}(f, \omega_{ij}; s)$ for some ω_{ij}, then* $\exp(2\pi\sqrt{-1}s_0)$ *is a monodromy eigenvalue of f at* 0.
(2) *If λ is a monodromy eigenvalue of f at* 0, *then there is a form ω_{ij} and a pole s_0 of $Z_{\mathrm{top},0}(f, \omega_{ij}; s)$ such that* $\exp(2\pi\sqrt{-1}s_0) = \lambda$.

Proof. Let $\pi : X \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $f^{-1}\{0\}$, see Fig. 2. We can consider the strict transforms C_1 and C_2 of $\bar{C}_1 := \{x = 0\}$ and $\bar{C}_2 := \{y = 0\}$ as curvettes of the exceptional curves E_1 and E_2, respectively. Moreover, the strict transform of $f^{-1}\{0\}$ can be considered as a curvette C_m for E_m.

It is well known that the multiplicities of E_1, E_2 and E_m in $\mathrm{div}(\pi^* f)$ are p, q and pq, respectively, and that the multiplicity of E_m in $\mathrm{div}(\pi^*\, dx \wedge dy)$ is $p + q - 1$. Since the matrix (a_{ij}) is symmetric (where we use the notation of 2.3) we have $a_{1m} = a_{m1} = p$ and $a_{2m} = a_{m2} = q$. Consequently the multiplicity of E_m in $\mathrm{div}(\pi^*\omega_{ij})$ is $p + q - 1 + (i - 1)p + (j - 1)q = ip + jq - 1$.

The only monodromy eigenvalue on $H^0(F_0, \mathbb{C})$ is 1, and then by A'Campo's formula the eigenvalues on $H^1(F_0, \mathbb{C})$ are

$$\exp\left\{-2\pi\sqrt{-1}\left(\frac{i}{q} + \frac{j}{p}\right)\right\}$$

for $1 \leqslant i \leqslant q - 1$ and $1 \leqslant j \leqslant p - 1$.

On the other hand, for example by [36], we have already that -1 and $-\frac{ip+jq}{pq} = -(\frac{i}{q} + \frac{j}{p})$ are the only candidate poles of $Z_{\mathrm{top},0}(f, \omega_{ij}; s)$. In fact they really are poles which immediately implies (1) and (2).

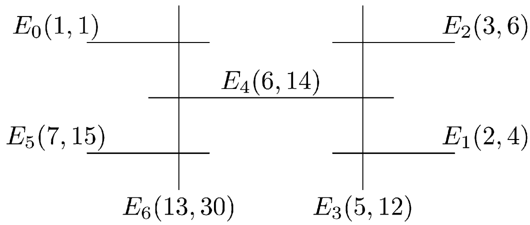

Fig. 3.

An elegant way to check this is the formula in [37, Theorem 3.3] which yields the following compact expression (this formula remains valid in the context of arbitrary differential forms ω):

$$\begin{aligned} Z_{\mathrm{top},0}(f,\omega_{ij};s) &= \frac{1}{ip+jq+spq}\left(-1+\frac{1}{1+s}+\frac{q}{i}+\frac{p}{j}\right) \\ &= \frac{jq+ip+(jq+ip-ij)s}{(ip+jq+spq)(1+s)}. \qquad \square \end{aligned}$$

2.8. Example. Let $f=(y^2-x^3)^2-x^6y$ on $(\mathbb{C}^2,0)$. This is one of the simplest irreducible singularities with two Puiseux pairs. The minimal embedded resolution of $f^{-1}\{0\}$ is described in Fig. 3. The numbers (ν_i,N_i) denote as usual 1+ (the multiplicity of E_i in $\operatorname{div}(\pi^* dx\wedge dy)$) and the multiplicity of E_i in $\operatorname{div}(\pi^* f)$, respectively.

By A'Campo's formula the monodromy eigenvalues of f at 0 are 1 and all primitive roots of unity of order 6, 10, 12 and 30. Take $\omega_{ij}=x^{i-1}y^{j-1}dx\wedge dy$ for $i,j\geqslant 1$. We checked that the statement in 2.5 is valid for example for the sets of differential forms $\{\omega_{ij}\mid 1\leqslant i\leqslant 5$ and $1\leqslant j\leqslant 3\}\cup\{\omega_{34}\}$ and $\{\omega_{ij}\mid 1\leqslant i\leqslant 3$ and $1\leqslant j\leqslant 5\}\setminus\{\omega_{24}\}$.

3. Arbitrary dimension

3.1. First we construct a higher dimensional generalization of the notion of curvette such that an analogue of Proposition 2.3 is still valid.

Let X_0 be a smooth quasi-projective (complex) variety of dimension n and let

$$X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_i} X_i \xleftarrow{\pi_{i+1}} \cdots \xleftarrow{\pi_m} X_m$$

be a composition π of m blowing-ups π_i with smooth irreducible centre $Z_{i-1}(\subset X_{i-1})$ having normal crossings with the exceptional locus of $\pi_1\circ\cdots\circ\pi_{i-1}$ (see [23]). Denote the exceptional locus of π_i, as well as its consecutive strict transforms, by E_i.

Recall that, when created, E_i has the structure of a $\mathbb{P}^k$-bundle $E_i \xleftarrow{p_i} Z_{i-1}$, where $k=n-1-\dim Z_{i-1}$. We have $\operatorname{Pic} E_i\cong\mathbb{Z}L_i\oplus p_i^*\operatorname{Pic} Z_{i-1}$, where L_i is the divisor class corresponding to the canonical sheaf $\mathcal{O}_{E_i}(1)$ on E_i. The self-intersection E_i^2 of E_i on X_i, considered in $\operatorname{Pic} E_i$, is equal to $-L_i$ [22, Theorem II 8.24]. (When Z_{i-1} is a point, L_i is just the hyperplane class on $E_i\cong\mathbb{P}^{n-1}$.)

3.2. Proposition. *One can construct consecutively for $j=1,\dots,m$ a smooth hypersurface C_j on X_j such that*

(1) *C_j has normal crossings with $E_1 \cup E_2 \cup \cdots \cup E_j$, with (the strict transform of) previously created $C_1, \ldots, C_{j-1}$, and with the next centre of blowing-up Z_j (and such that $Z_j \not\subset C_j$);*
(2) *in $\operatorname{Pic} E_j$ we have $C_j \cap E_j = L_j + p_j^* B_j$ for some $B_j \in \operatorname{Pic} Z_{j-1}$;*
(3) *denoting $\tilde{C}_j := \pi_j(C_j) \subset X_{j-1}$, we have $\pi_j^* \tilde{C}_j = E_j + C_j$ in $\operatorname{Pic} X_j$. So the multiplicity of $\tilde{C}_j$ along Z_{j-1} is 1.*
Note that by (1) the strict transforms in X_m of all E_j and C_j form a normal crossings divisor.
(4) *Given another hypersurface H on X_m having normal crossings with $E_1 \cup \cdots \cup E_m$, we can choose $C_1, \ldots, C_m$ such that furthermore H and all E_j and C_j form a normal crossings divisor on X_m.*

Proof. Fix a $j \in \{1, \ldots, m\}$. Consider the sheaf $\mathcal{O}_{X_j}(1)$ on X_j, associated to the blowing-up map $\pi_j : X_j \to X_{j-1}$. We choose an ample invertible sheaf $\mathcal{L}$ on X_{j-1}. By [22, II Proposition 7.10] we have for some $k > 0$ that the sheaf $\mathcal{O}_{X_j}(1) \otimes \pi_j^* \mathcal{L}^k$ on X_j is very ample over X_{j-1}. So its global sections generate a base point free linear system on X_j; we take C_j as a general element of this linear system. By Bertini's theorem C_j satisfies (1). The inverse image on X_m of this linear system is still base point free. So a general element will also satisfy the extra condition in (4).

We now verify that the intersection product $C_j \cdot E_j$, considered in $\operatorname{Pic} E_j$, is of the form $L_j + p_j^* B_j$, which yields (2). Denote $\beta : E_j \hookrightarrow X_j$ and $\alpha : Z_{j-1} \hookrightarrow X_{j-1}$. Since the divisor class corresponding to $\mathcal{O}_{X_j}(1)$ on X_j is $-E_j$ we have

$$\begin{aligned} C_j \cdot E_j &= \bigl(-E_j + \pi_j^*(\cdots)\bigr) \cdot E_j = -E_j^2 + \beta^* \pi_j^*(\cdots) \\ &= L_j + p_j^* \alpha^*(\cdots). \end{aligned}$$

Finally we verify (3). Certainly $\pi_j^* \tilde{C}_j = \mu E_j + C_j$ where μ is the multiplicity of $\tilde{C}_j$ along Z_{j-1}. Intersecting with E_j yields

$$\beta^* \pi_j^* \tilde{C}_j = \mu(-L_j) + C_j \cdot E_j,$$

and hence by the previous calculation $p_j^*(\cdots) = -\mu L_j + L_j + p_j^*(\cdots)$. So indeed $\mu = 1$. $\square$

$$\begin{array}{ccc} E_j & \xrightarrow{\ \beta\ } & X_j \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle \pi_j} \\ Z_{j-1} & \xrightarrow{\ \alpha\ } & X_{j-1}. \end{array}$$

3.3. The C_j constructed above satisfy an analogous statement as Proposition 2.3 for curves. For the proof however we need another approach.

Proposition. *We use the notation of* 3.1 *and* 3.2. *Denote also $\bar{C}_i := \pi(C_i) \subset X_0$ and $\pi^* \bar{C}_i = \sum_{j=1}^m a_{ij} E_j + C_i$ for $i = 1, \ldots, m$. Then the determinant of the $(m \times m)$-matrix (a_{ij}) is equal to* 1. *In particular $\gcd_{1 \leqslant i \leqslant m}\{a_{ij}\} = 1$ for all j.*

Proof. We proceed by induction on m. When $m = 1$ we have $\pi^* \bar{C}_1 = E_1 + C_1$ by Proposition 3.2(3), and so indeed $a_{11} = 1$. Take now $m > 1$. By the same proposition we have

$\pi_m^* \tilde{C}_m = 1E_m + C_m$, where $\tilde{C}_m = \pi_m(C_m)$. Say Z_{m-1} is contained in precisely E_j, $j \in J(\subset \{1,\dots,m-1\})$. Then

$$\pi^* \bar{C}_m = \pi_m^* \left(\sum_{j=1}^{m-1} a_{mj} E_j + \tilde{C}_m \right) = \left(\sum_{j \in J} a_{mj} + 1 \right) E_m + (\cdots),$$

saying that $a_{mm} = \sum_{j\in J} a_{mj} + 1$. On the other hand, since Z_{m-1} is not contained in (the strict transform of) any $C_1, C_2, \dots, C_{m-1}$, we have $a_{im} = \sum_{j \in J} a_{ij}$ for $i = 1,\dots,m-1$. Hence

$$\det(a_{ij})_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant m}} = \begin{vmatrix} a_{11} & \cdots & a_{1,m-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & \cdots & a_{m-1,m-1} & 0 \\ a_{m1} & \cdots & a_{m,m-1} & 1 \end{vmatrix} = \det(a_{ij})_{\substack{1\leqslant i\leqslant m-1\\1\leqslant j\leqslant m-1}} = 1,$$

where the last equality is the induction hypothesis. □

Note. In contrast to the curve case, the matrix (a_{ij}) is in general not symmetric in higher dimensions, even when all π_i are point blowing-ups. Take for example $\dim X_0 = 3$, Z_1 a point on E_1, and Z_2 a point on $E_1 \cap E_2$. Then

$$(a_{ij}) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

3.4. We now present higher dimensional versions of Theorem 2.4. We first look at zeroes or poles of monodromy zeta functions. According to Lemma 1.3, this way we treat in fact all monodromy eigenvalues. For isolated singularities we present a finer result.

3.5. Theorem. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a non-zero polynomial function (germ). Let λ be a zero or a pole of the monodromy zeta function of f at 0. Then there exists a differential n-form ω on $(\mathbb{C}^n, 0)$ such that $Z_{\mathrm{top},0}(f, \omega; s)$ has a pole s_0 satisfying* $\exp(2\pi\sqrt{-1}s_0) = \lambda$.

Proof. Let $f : X_0(\subset \mathbb{C}^n) \to \mathbb{C}$ be a relevant representative of f in the sense that some embedded resolution of $f^{-1}\{0\} \subset X_0$ only has exceptional components that intersect the inverse image of 0. Take such an embedded resolution $\pi : X_m \to X_0$, which is a composition of m blowing-ups as in 3.1. Slightly abusing notation, E_i can now denote an exceptional component of π or an irreducible component of the strict transform. As usual N_i and $\nu_i - 1$ are the multiplicities of E_i in the divisor of $\pi^* f$ and $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$, respectively.

Say λ is a dth root of unity. By A'Campo's formula there exists an exceptional component E_{j_0} with $d \mid N_{j_0}$ and $\chi(E_{j_0}^\circ \cap \pi^{-1}\{0\}) \neq 0$.

We take for $i = 1,\dots,m$ smooth hypersurfaces C_i as in Proposition 3.2 (considered in X_m); as extra hypersurface in 3.2(4) we take the strict transform of $f^{-1}\{0\}$. Say the images $\bar{C}_i$ in X_0 of the C_i have (reduced) equation $g_i = 0$. As before we denote $\pi^* \bar{C}_i = \sum_{j=1}^m a_{ij} E_j + C_i$; the (a_{ij}) satisfy Proposition 3.3. We take for the moment a differential form ω of the form $(\prod_\ell g_\ell^{m_\ell})\, dx_1 \wedge \cdots \wedge dx_n$. The multiplicity of E_i in the divisor of $\pi^*\omega$ is $\nu_i - 1 + \sum_\ell a_{\ell i} m_\ell$.

The candidate pole for $Z_{\mathrm{top},0}(f,\omega;s)$ associated to E_{j_0} is $s_0 = -\frac{\nu_{j_0}+\sum_\ell a_{\ell j_0} m_\ell}{N_{j_0}}$. As in the proof of Theorem 2.4 we want to find suitable m_ℓ. Completely analogously as in that proof, this time using Proposition 3.3,

(1) we find a lattice of non-negative m_ℓ 'mod N_{j_0}' such that s_0 satisfies $\exp(2\pi\sqrt{-1}s_0)=\lambda$, and
(2) for 'most' such m_ℓ the candidate poles associated to other E_i are different from s_0.

The argument showing that s_0 is really a pole for suitable such m_ℓ is more subtle now. We introduce some notation to describe the residue of s_0.

Let C_ℓ, $\ell \in J_0$, be the hypersurfaces C_i that intersect $E_{j_0} \cap \pi^{-1}\{0\}$ (in X_m). Denote

$$C_J^0 := \left(\left(\bigcap_{j\in J} C_j\right) \Big\backslash \left(\bigcup_{i\in J_0\setminus J} C_i\right)\right) \cap \left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) \quad \text{for } J \subset J_0;$$

in particular

$$C_\emptyset^\circ = \left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) \setminus \bigcup_{i\in J_0} C_i.$$

These C_J° form a locally closed stratification of $E_{j_0}^\circ \cap \pi^{-1}\{0\}$. The residue of s_0 is of the form

$$\frac{1}{N_{j_0}}\left(\chi\left(C_\emptyset^\circ\right) + \sum_{\emptyset\neq J\subset J_0} \chi\left(C_J^\circ\right) \prod_{j\in J} \frac{1}{1+m_j} + \text{contribution of } \left(E_{j_0}\setminus E_{j_0}^\circ\right)\cap\pi^{-1}\{0\}\right),$$

the last contribution also being a rational function in the m_ℓ of negative degree. As in the proof of Theorem 2.4, if $\chi(C_\emptyset^\circ)\neq 0$, this expression is never identically zero as function in the m_ℓ, and so non-zero for 'most' choices of $(m_\ell)_\ell$.

We do not see how to exclude the theoretical possibility that this expression *is* identically zero (with then necessarily $\chi(C_\emptyset^\circ)=0$). In this case we will adapt our choice of ω to be sure to have the desired pole.

For each ℓ in J_0 we construct, as in Proposition 3.2, not just one hypersurface C_ℓ, but several ones $C_{\ell 1}, C_{\ell 2}, \dots, C_{\ell t}$, all general enough elements in the linear system that was considered there. Then still all E_i, $C_{\ell j}$, and other C_ℓ will form a normal crossings divisor on X_m. Say $g_{\ell k}=0$ is the equation of the image of $C_{\ell k}$ in X_0. Now we take ω of the form $\prod_{\ell\in J_0}(\prod_{k=1}^t g_{\ell k}^{m_{\ell k}}) \prod_{\ell\notin J_0} g_\ell^{m_\ell}\, dx_1\wedge\cdots\wedge dx_n$ such that for $\ell\in J_0$ the sum $\sum_{k=1}^t m_{\ell k}$ is an allowed m_ℓ 'mod N_{j_0}' as before. The candidate pole s_0 for $Z_{\mathrm{top},0}(f,\omega;s)$ associated to E_{j_0} is as above, it still satisfies $\exp(2\pi\sqrt{-1}s_0)=\lambda$, and for 'most' such $m_{\ell k}$ and m_ℓ the candidate poles associated to other E_i are different from s_0. We now verify that for some t the expression for the residue of s_0 is not identically zero as function in the $m_{\ell k}$ and m_ℓ.

Denote $L_k := (\bigcup_{\ell\in J_0} C_{\ell k}) \cap (E_{j_0}^\circ \cap \pi^{-1}\{0\})$ for $k=1,\dots,t$. Since all $C_{\ell k}$ all *general* elements we have that all $\chi(L_k)$ are equal, that also the $\chi(L_k\cap L_{k'})$ are equal for all $k<k'$, and

more generally that the $\chi(L_{k_1} \cap L_{k_2} \cap \cdots \cap L_{k_s})$ are equal for all $1 \leqslant k_1 < k_2 < \cdots < k_s \leqslant t$. The residue of s_0 is of the form

$$\frac{1}{N_{j_0}}\left(\chi\left(\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) \setminus \bigcup_{k=1}^{t} L_k\right) + \cdots\right),$$

where the other terms form a rational function of negative degree in the $m_{\ell k}$ and the m_ℓ. If $\chi((E_{j_0}^\circ \cap \pi^{-1}\{0\}) \setminus \bigcup_{k=1}^t L_k) \neq 0$, then this residue is not identically zero and we are done. Finally we show that this must be the case for some t.

Because of the normal crossings property we have $\bigcap_{k=1}^T L_k = \emptyset$ for some $T (\leqslant n)$. Suppose that $\chi((E_{j_0}^\circ \cap \pi^{-1}\{0\}) \setminus \bigcup_{k=1}^t L_k) = 0$ for *all* $t = 1, \ldots, T$. These T conditions can be rewritten as

$$\begin{cases} \chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) - \chi(L_1) = 0, \\ \chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) - 2\chi(L_1) + \chi(L_1 \cap L_2) = 0, \\ \chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) - 3\chi(L_1) + 3\chi(L_1 \cap L_2) - \chi(L_1 \cap L_2 \cap L_3) = 0, \\ \cdots \\ \chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) - (T-1)\chi(L_1) + \cdots + (-1)^{T-1}\chi\left(\bigcap_{k=1}^{T-1} L_k\right) = 0, \\ \chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) - T\chi(L_1) + \cdots + (-1)^{T-1}T\chi\left(\bigcap_{k=1}^{T-1} L_k\right) + (-1)^T \cdot 0 = 0. \end{cases}$$

One easily verifies that the $(T \times T)$-determinant of coefficients of this homogeneous linear system of equations in the $\chi(\cdots)$ is non-zero. Hence in particular we should have

$$\chi\left(E_{j_0}^\circ \cap \pi^{-1}\{0\}\right) = 0,$$

contradicting our choice of E_{j_0}. $\square$

3.6. Theorem. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a non-zero polynomial function (germ).*

(a) *Let λ be a monodromy eigenvalue of f at* 0*. Then there exist a differential n-form ω and a point P in a neighbourhood of* 0 *such that $Z_{\mathrm{top},P}(f, \omega; s)$ has a pole s_0 satisfying* $\exp(2\pi\sqrt{-1}s_0) = \lambda$. *Moreover, P can be chosen as a generic point in the set Σ that was introduced after Lemma* 1.3.

(a′) *If the eigenvalue λ appears* only *at* 0*, then there exists a differential n-form ω such that $Z_{\mathrm{top},0}(f, \omega; s)$ has a pole s_0 satisfying* $\exp(2\pi\sqrt{-1}s_0) = \lambda$.

(b) *Suppose that $f^{-1}\{0\}$ has an isolated singularity at* 0*, and let λ be a monodromy eigenvalue of f at* 0*. Then there exists a differential n-form ω such that $Z_{\mathrm{top},0}(f, \omega; s)$ has a pole s_0 satisfying* $\exp(2\pi\sqrt{-1}s_0) = \lambda$.

Proof. Parts (a) and (a′) follow immediately from Theorem 3.5 and Lemma 1.3.

Part (b) is a special case of (a′) for $\lambda \neq 1$. It could however happen in (b) that $\lambda = 1$ is not a zero or a pole of the monodromy zeta function of f at 0 (when n is even). In that case we pick any exceptional E_{j_0} with $\chi(E_{j_0}^\circ) \neq 0$, and we proceed as in the proof of Theorem 3.5 to construct a suitable ω and a pole s_0 of $Z_{\mathrm{top},0}(f,\omega;s)$ with $\exp(2\pi\sqrt{-1}s_0) = 1$. (Note that the constructed s_0 is in this case indeed an integer.) By A'Campo's formula there is always such an exceptional E_{j_0}, except when n is even and the characteristic polynomial $P_{n-1}(t) = t-1$. E.g. by [4, p. 70] this implies that f has a so-called non-degenerate or Morse singularity at 0 (i.e., f is in local analytic coordinates of the form $y_1^2 + y_2^2 + \cdots + y_n^2$). In this easy special case one has

$$Z_{\mathrm{top},0}(f, dx_1 \wedge \cdots \wedge dx_n; s) = \frac{n}{(1+s)(n+2s)}. \qquad \square$$

3.7. As in the curve case the zeta functions constructed in the proof of Theorem 3.5 can in general have other poles that do not induce monodromy eigenvalues of f, and it would be interesting to study in arbitrary dimension the validity of the 'principle' in Note 2.5. We present an example below.

3.8. Example. Let $f = x^d + y^d + z^d$ on $(\mathbb{C}^3, 0)$ with $d \geqslant 3$. Blowing up the origin yields an embedded resolution π of $f^{-1}\{0\}$. The exceptional surface $E \cong \mathbb{P}^2$ has multiplicities 2 and d in $\mathrm{div}(\pi^* dx \wedge dy \wedge dz)$ and $\pi^* f$, respectively. It intersects the strict transform in a smooth curve D of degree d and hence with Euler characteristic $3d - d^2$. By A'Campo's formula the monodromy zeta function of f at 0 is

$$\zeta_{f,0}(t) = \left(t^d - 1\right)^{d^2-3d+3};$$

and the monodromy eigenvalues of f at 0 are precisely all dth roots of unity. Take $\omega_i := x^{i-1}\, dx \wedge dy \wedge dz$ for $1 \leqslant i \leqslant d$. The strict transform of $\{x = 0\}$ intersects E in a line; this line intersects D transversely in d points. Hence

$$\begin{aligned} Z_{\mathrm{top},0}(f,\omega_i;s) &= \frac{1}{(2+i)+sd}\left((d-1)^2 + \frac{2-d}{i} + \frac{2d-d^2}{1+s} + \frac{d}{i(1+s)}\right) \\ &= \frac{(i(d-1)^2 + 2 - d)s + (2+i)}{i((2+i)+sd)(1+s)}. \end{aligned}$$

When $i \neq d-2$, one easily verifies that the two candidate poles -1 and $-\frac{2+i}{d}$ are really poles. When $i = d-2$ we have

$$Z_{\mathrm{top},0}(f,\omega_{d-2};s) = \frac{1+(d-2)^2 s}{(d-2)(1+s)^2}$$

and -1 is a pole (of order 2 if $d > 3$ and of order 1 if $d = 3$).

So the set of differential forms $\{\omega_i \mid 1 \leqslant i \leqslant d\}$ satisfies the analogous principle as in 2.5. We can even delete ω_{d-2} from this set.

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