Subfunctions and Second-Order Ordinary Differential Inequalities*

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1. Introduction

Subfunctions and solutions of differential inequalities have been used for many years in dealing with existence theorems and properties of solutions for both ordinary and partial differential equations. In 1915 in one of the earliest such applications Perron [1] used solutions of differential inequalities to establish the existence of a solution of the initial-value problem for the first-order equation $y' = f(x, y)$.

In a remarkable paper published in 1923, Perron [2] used subharmonic functions to study the Dirichlet problem for harmonic functions for bounded plane domains. By using properties of subharmonic functions and the fact that the Dirichlet problem for circles is solvable, a generalized solution of the Dirichlet problem for a bounded domain is obtained. The generalized solution is harmonic in the interior of the domain and the question of whether or not it assumes the specified boundary value at a boundary point can be dealt with separately.

In the next few years a more complete investigation of properties of subharmonic functions was carried out by F. Riesz [3]-[5]. It was

* This paper is based on a series of lectures given during the summer of 1967 at the Rocky Mountain Symposium in Ordinary Differential Equations sponsored by the Associated Rocky Mountain Universities.

observed by a number of authors that results in function theory and potential theory such as the Liouville Theorem and the Phragmen-Lindelof Theorems depend only on properties of subharmonic functions. Subsequently various authors, Tautz [6], Beckenbach and Jackson [7], Inoue [8], Jackson [9], and a number of others, examined the Perron method of attacking the Dirichlet problem to determine the properties of harmonic functions and subharmonic functions which are essential for the success of the method. This led to the successful application of the Perron method to the study of the Dirichlet problem for more general elliptic partial differential equations including certain types of nonlinear equations. These methods also led to Liouville Theorems and Phragmen-Lindelof Theorems for more general elliptic equations (see, for example, [10]-[13]).

If one is concerned with ordinary rather than partial differential equations, the program analogous to generalizing harmonic functions and subharmonic functions is to generalize linear functions and convex functions. There is an extensive literature in this area starting with a paper by Beckenbach [14] published in 1937. The papers [15]-[18] constitute a small sample of work on this theme. In most of these papers the authors deal with second-order ordinary differential equations for which it is assumed that all boundary-value problems are uniquely solvable. It is then shown that certain properties of convex functions carry over to subfunctions with respect to solutions of the differential equations.

There have been a number of papers in which solutions of differential inequalities have been employed in establishing existence theorems for boundary-value problems for ordinary differential equations. In particular, Caplygin and a number of later Soviet mathematicians using Caplygin's methods (for example, Babkin [19]) have obtained solutions of boundary-value problems as uniform limits of sequences of functions satisfying differential inequalities.

None of the papers referred to above appear to be concerned with the Perron method of attacking the boundary-value problem for second-order ordinary equations, that is, with the use of existence theorems in the small and the properties of subfunctions to establish the existence of solutions in the large. The present work is divided into two main divisions. The first sections will deal with differential equations $y'' = f(x, y, y')$ with the property that, when the boundary-value problem $y'' = f(x, y, y')$, $y(a) = A$, $y(b) = B$ has a solution $y(x) \in C^{(2)}[a, b]$, that solution is unique. In this case subfunctions can be defined in a meaningful way and the Perron method can be developed. Also the relationships between subfunctions and solutions of differential inequalities can be developed.

The second part of the work will be concerned with differential equations which are such that boundary-value problems are not necessarily uniquely solvable. In this part, results will be obtained by using solutions of differential inequalities and restrictions on the rate of growth of $f(x, y, y')$ with respect to y'.

2. Preliminary Results

In this section the local existence theorem and some related results will be given. We will always assume that $f(x, y, y')$ is continuous on $[a, b] \times R^2$ and for the moment we will assume that $[a, b]$ is a compact interval. Later we will consider cases where x ranges over an open interval or an infinite interval. In some of our results it would suffice to assume a type of piecewise continuity for $f(x, y, y')$, however, for simplicity we shall always assume f to be continuous on its domain.

Theorem 2.1. *Let $M > 0$ and $N > 0$ be given real numbers and let q be the maximum of $|f(x, y, y')|$ on the compact set $\{(x, y, y') : a \leqslant x \leqslant b, |y| \leqslant 2M, |y'| \leqslant 2N\}$. Then, if $\delta = \mathrm{Min}[(8M/q)^{1/2}, 2N/q]$, any boundary-value problem $y'' = f(x, y, y')$, $y(x_1) = y_1$, $y(x_2) = y_2$ with $[x_1, x_2] \subset [a, b]$, $x_2 - x_1 \leqslant \delta$, $|y_1| \leqslant M$, $|y_2| \leqslant M$, $|(y_1 - y_2)/(x_1 - x_2)| \leqslant N$ has a solution $y(x) \in C^{(2)}[x_1, x_2]$. Furthermore, given $\epsilon > 0$ there is a solution $y(x)$ such that $|y(x) - w(x)| < \epsilon$ and $|y'(x) - w'(x)| < \epsilon$ on $[x_1, x_2]$ provided $x_2 - x_1$ is sufficiently small where $w(x)$ is the linear function with $w(x_1) = y_1$, $w(x_2) = y_2$.* ([20], p. 1252).

Proof. The set

$$B[x_1, x_2] = \{z(x) \in C^{(1)}[x_1, x_2] : \| z \| \leqslant 2M, \| z' \| \leqslant 2N\}$$

is a closed convex subset of the Banach space $C^{(1)}[x_1, x_2]$. The mapping $T : C^{(1)}[x_1, x_2] \to C^{(1)}[x_1, x_2]$ defined by

$$(Tz)(x) = \int_{x_1}^{x_2} G(x, t) f(t, z(t), z'(t))\, dt + w(x),$$

where $G(x, t)$ is the Green's function for the boundary-value problem $y'' = 0, y(x_1) = y(x_2) = 0$, is completely continuous. For a $z \in B[x_1, x_2]$ we have

$$|(Tz)(x)| \leqslant \tfrac{1}{8}[q(x_2 - x_1)^2] + M,$$

and

$$|(Tz)'(x)| \leqslant \tfrac{1}{2}[q(x_2 - x_1)] + N$$

on $[x_1, x_2]$. Thus $x_2 - x_1 \leqslant \delta$ implies T maps $B[x_1, x_2]$ into itself. It then follows from the Schauder Fixed-Point Theorem that T has a fixed point in $B[x_1, x_2]$. The fixed point is a solution of the stated boundary-value problem. If $y(x)$ is a solution of the boundary-value problem with $y \in B[x_1, x_2]$, then

$$|y(x) - w(x)| \leqslant \tfrac{1}{8}[q(x_2 - x_1)^2]$$

and

$$|y'(x) - w'(x)| \leqslant \tfrac{1}{2}[q(x_2 - x_1)]$$

on $[x_1, x_2]$ and the last assertion of the theorem follows.

Corollary 2.2. *Assume that there exist constants $h > 0$ and $k > 0$ such that $|f(x, y, y')| \leqslant h + k(|y|)^{1/2}$ on $[a, b] \times R^2$. Then every boundary-value problem $y'' = f(x, y, y')$, $y(a) = \alpha$, $y(b) = \beta$ has a solution $y \in C^{(2)}[a, b]$* ([21], Lemma 2.2).

Proof. In this case we can choose $M > 0$ large enough that

$$|\alpha| \leqslant M, \qquad |\beta| \leqslant M, \qquad \left|\frac{\alpha - \beta}{a - b}\right| \leqslant M,$$

$$b - a \leqslant \left[\frac{8M}{h + k(2M)^{1/2}}\right]^{1/2}, \qquad \text{and} \qquad b - a \leqslant \frac{2M}{h + k(2M)^{1/2}}.$$

The result then follows from Theorem 2.1.

Using Corollary 2.2 and solutions of certain differential inequalities we can obtain solutions of some boundary-value problems for a modified form of the differential equation $y'' = f(x, y, y')$.

Definition 2.3. Let $\alpha(x), \beta(x) \in C^{(1)}[a, b]$ with $\alpha(x) \leqslant \beta(x)$ on $[a, b]$

and let $c > 0$ be such that $|\alpha'(x)| < c$ and $|\beta'(x)| < c$ on $[a, b]$. Then define

$$F^*(x, y, y') = \begin{cases} f(x, y, c) & \text{for} \quad y' \geqslant c, \\ f(x, y, y') & \text{for} \quad |y'| \leqslant c, \\ f(x, y, -c) & \text{for} \quad y' \leqslant -c, \end{cases}$$

and

$$F(x, y, y') = \begin{cases} F^*(x, \beta(x), y') + [y - \beta(x)]^{1/2} & \text{for} \quad y \geqslant \beta(x), \\ F^*(x, y, y') & \text{for} \quad \alpha(x) \leqslant y \leqslant \beta(x), \\ F^*(x, \alpha(x), y') - [\alpha(x) - y]^{1/2} & \text{for} \quad y \leqslant \alpha(x). \end{cases}$$

We will call $F(x, y, y')$ the modification of $f(x, y, y')$ associated with the triple $\alpha(x)$, $\beta(x)$, c. From the definition it follows that $F(x, y, y')$ is continuous on $[a, b] \times R^2$ and $|F(x, y, y')| \leqslant h + (|y|)^{1/2}$ on $[a, b] \times R^2$ where

$$h = \operatorname{Max}\{|f(x, y, y')| : a \leqslant x \leqslant b, \alpha(x) \leqslant y \leqslant \beta(x), |y'| \leqslant c\} + \operatorname*{Max}_{a \leqslant x \leqslant b} |\alpha(x)|^{\frac{1}{2}} + \operatorname*{Max}_{a \leqslant x \leqslant b} |\beta(x)|^{\frac{1}{2}}.$$

Next we define certain types of solutions of differential inequalities which will be used in the later work.

Definition 2.4. A function $\alpha(x)$ is called a $C^{(1)}$-lower solution of the differential equation $y'' = f(x, y, y')$ on an interval I in case $\alpha(x) \in C(I) \cap C^{(1)}(I^0)$, I^0 the interior of I, and

$$\underline{D}\alpha'(x) \equiv \liminf_{\delta \to 0} \frac{\alpha'(x + \delta) - \alpha'(x - \delta)}{2\delta} \geqslant f(x, \alpha(x), \alpha'(x))$$

on I^0. The function $\alpha(x)$ is called an $AC^{(1)}$-lower solution on I in case $\alpha(x) \in C(I) \cap C^{(1)}(I^0)$, $\alpha'(x)$ is absolutely continuous on each compact subinterval of I^0, and $\alpha''(x) \geqslant f(x, \alpha(x), \alpha'(x))$ almost everywhere on I. Similarly, $\beta(x)$ is a $C^{(1)}$-upper solution on I in case $\beta(x) \in C(I) \cap C^{(1)}(I^0)$ and

$$\bar{D}\beta'(x) \equiv \limsup_{\delta \to 0} \frac{\beta'(x + \delta) - \beta'(x - \delta)}{2\delta} \leqslant f(x, \beta(x), \beta'(x))$$

on I^0. $\beta(x)$ is an $AC^{(1)}$-upper solution on I in case $\beta(x) \in C(I) \cap C^{(1)}(I^0)$, $\beta'(x)$ is absolutely continuous on each compact subinterval of I^0, and $\beta''(x) \leqslant f(x, \beta(x), \beta'(x))$ almost everywhere on I.

When we say simply that $\alpha(x)$ is a lower solution or $\beta(x)$ is an upper solution we will mean that they can be of either type.

Theorem 2.5. *Let $\alpha(x)$, $\beta(x) \in C^{(1)}[a, b]$ be, respectively, lower and upper solutions of $y'' = f(x, y, y')$ on $[a, b]$ with $\alpha(x) \leqslant \beta(x)$ on $[a, b]$. Then, if $F(x, y, y')$ is the modification of $f(x, y, y')$ associated with the triple $\alpha(x)$, $\beta(x)$, c and if $\alpha(a) \leqslant \gamma \leqslant \beta(a)$, $\alpha(b) \leqslant \delta \leqslant \beta(b)$, the boundary-value problem*

$$y'' = F(x, y, y'), \qquad y(a) = \gamma, \qquad y(b) = \delta$$

has a solution $y \in C^{(2)}[a, b]$ satisfying $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, b]$.

Proof. By Corollary 2.2 the stated boundary-value problem has a solution $y \in C^{(2)}[a, b]$. Thus we only need show that $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, b]$. We will show only that $y(x) \leqslant \beta(x)$ since the arguments for $\alpha(x) \leqslant y(x)$ are essentially the same. Assume that $y(x) > \beta(x)$ at some points of $[a, b]$. Then $y(x) - \beta(x)$ has a positive maximum at a point $x_0 \in (a, b)$. It follows that $y'(x_0) = \beta'(x_0)$ and $| y'(x_0) | < c$; hence

$$y''(x_0) = F(x_0, y(x_0), y'(x_0)) = f(x_0, \beta(x_0), \beta'(x_0)) + [y(x_0) - \beta(x_0)]^{1/2}.$$

If $\beta(x)$ is a $C^{(1)}$-upper solution on $[a, b]$,

$$\bar{D}\beta'(x_0) \leqslant f(x_0, \beta(x_0), \beta'(x_0))$$

and

$$\underline{D}[y'(x_0) - \beta'(x_0)] = y''(x_0) - \bar{D}\beta'(x_0) \geqslant [y(x_0) - \beta(x_0)]^{1/2} > 0,$$

which is impossible at a maximum of $y(x) - \beta(x)$. If β is an $AC^{(1)}$-upper solution on $[a, b]$, then, since $F(x, y(x), y'(x))$ and $f(x, \beta(x), \beta'(x))$ are both continuous at $x = x_0$, there is a $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (a, b)$ and $[y'(x) - \beta'(x)]' > 0$ almost everywhere on $[x_0 - \delta, x_0 + \delta]$. This again is incompatible with $y(x) - \beta(x)$ having a maximum at x_0. We conclude that $y(x) \leqslant \beta(x)$ on $[a, b]$.

Theorem 2.6. *Assume that in addition to being continuous on $[a, b] \times R^2$ $f(x, y, y')$ is such that solutions of initial value problems for $y'' = f(x, y, y')$ are unique. Let $\alpha(x)$, $\beta(x) \in C^{(1)}[a, b]$ be lower and upper solutions on $[a, b]$ with $\alpha(x) \leqslant \beta(x)$ on $[a, b]$. Then, if $\alpha(x_0) = \beta(x_0)$ and $\alpha'(x_0) = \beta'(x_0)$ at some $x_0 \in [a, b]$, it follows that $\alpha(x) \equiv \beta(x)$ on $[a, b]$* ([21], Lemma 2.4).

Proof. Assume that the hypotheses of the Theorem are satisfied but that $\alpha(x) \not\equiv \beta(x)$ on $[a, b]$. We will consider the case where there is an $[x_0, x_1] \subset [a, b]$ such that $\alpha(x_0) = \beta(x_0)$, $\alpha'(x_0) = \beta'(x_0)$, and $\alpha(x) < \beta(x)$ on $(x_1, x_1]$. Let $F_1(x, y, y')$ be the modification of $f(x, y, y')$ as defined in Definition 2.3 for the interval $[x_0, x_1]$ and the triple $\alpha(x)$, $\beta(x)$, c_1. Then, if $\alpha(x_1) < \delta_1 < \beta(x_1)$, it follows from Theorem 2.5 that the boundary-value problem

$$y'' = F_1(x, y, y'), \qquad y(x_0) = \alpha(x_0), \qquad y(x_1) = \delta_1$$

has a solution $y_1 \in C^{(2)}[x_0, x_1]$ satisfying $\alpha(x) \leqslant y_1(x) \leqslant \beta(x)$ on $[x_0, x_1]$. Therefore, $y_1(x_0) = \alpha(x_0)$, $y_1'(x_0) = \alpha'(x_0)$; hence, by the way $F_1(x, y, y')$ is defined, there is a maximal interval $[x_0, x_2] \subset [x_0, x_1]$ on which $y_1(x)$ is a solution of $y'' = f(x, y, y')$. If $x_2 = x_1$, then

$$\alpha(x_1) < y_1(x_2) = y_1(x_1) = \delta_1 < \beta(x_1).$$

If $x_0 < x_2 < x_1$, it is still true that $\alpha(x_2) < y_1(x_2) < \beta(x_2)$ for, if either inequality were an equality, we would have $| y_1'(x_2) | < c_1$ and the interval $[x_0, x_2]$ would not be maximal.

This being the case, we can construct another modification $F_2(x, y, y')$ of $f(x, y, y')$ on the interval $[x_0, x_2]$ with respect to the triple $\alpha(x)$, $y_1(x)$, c_2. Applying Theorem 2.5 again with $\alpha(x_2) < \delta_2 < y_1(x_2)$, we conclude that there is a solution $y_2 \in C^{(2)}[x_0, x_2]$ of the boundary-value problem

$$y'' = F_2(x, y, y'), \qquad y(x_0) = \alpha(x_0), \qquad y(x_2) = \delta_2$$

satisfying $\alpha(x) \leqslant y_2(x) \leqslant y_1(x)$ on $[x_0, x_2]$. As above, it follows that there is a maximal subinterval $[x_0, x_3] \subset [x_0, x_2]$ on which $y_2(x)$ is a solution of $y'' = f(x, y, y')$ and that $\alpha(x_3) < y_2(x_3) < y_1(x_3)$. This contradicts the assumption that solutions of initial value problems for $y'' = f(x, y, y')$ are unique. We conclude that $\alpha(x) \equiv \beta(x)$ on $[a, b]$.

3. The Relation between Subfunctions and Solutions of Differential Inequalities

In this section we define subfunctions and superfunctions and consider necessary and sufficient conditions for such functions, when sufficiently smooth, to be respectively lower and upper solutions of the differential equation.

Definition 3.1. A function $\varphi(x)$ is said to be a subfunction with respect to solutions of $y'' = f(x, y, y')$ on an interval I in case for any $[x_1, x_2] \subset I$ and any solution $y \in C^{(2)}[x_1, x_2]$ $y(x_i) \geqslant \varphi(x_i)$ for $i = 1, 2$ implies $y(x) \geqslant \varphi(x)$ on $[x_1, x_2]$. The function $\psi(x)$ is said to be a superfunction with respect to solutions of $y'' = f(x, y, y')$ on an interval I in case for any $[x_1, x_2] \subset I$ and any solution $y \in C^{(2)}[x_1, x_2]$ $y(x_i) \leqslant \psi(x_i)$ for $i = 1, 2$ implies $y(x) \leqslant \psi(x)$ on $[x_1, x_2]$.

Theorem 3.2. *Assume that $\varphi \in C(I) \cap C^{(1)}(I^0)$ is a subfunction on I with respect to solutions of $y'' = f(x, y, y')$. Then φ is a $C^{(1)}$-lower solution of the differential equation on I.*

Proof. Let $x_0 \in I^0$. Then, if $h \geqslant 0$, $k \geqslant 0$, $h + k > 0$ are sufficiently small, it follows from Theorem 2.1 that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_0 + h) = \varphi(x_0 + h), \qquad y(x_0 - k) = \varphi(x_0 - k)$$

has a solution $y \in C^{(2)}[x_0 - k, x_0 + h]$. Since φ is a subfunction on I, it follows that

$$\frac{\varphi'(x_0 + h) - \varphi'(x_0 - k)}{h + k} \geqslant \frac{y'(x_0 + h) - y'(x_0 - k)}{h + k} = y''(\xi) = f(\xi, y(\xi), y'(\xi))$$

for some $x_0 - k < \xi < x_0 + h$. Since f is continuous, it also follows from Theorem 2.1 that

$$f(\xi, y(\xi), y'(\xi)) \to f(x_0, \varphi(x_0), \varphi'(x_0)) \qquad \text{as} \qquad h + k \to 0.$$

Thus

$$\liminf_{\substack{h+k\to 0 \\ h\geqslant 0, k\geqslant 0, h+k>0}} \frac{\varphi'(x_0 + h) - \varphi'(x_0 - k)}{h + k} \geqslant f(x_0, \varphi(x_0), \varphi'(x_0))$$

and, in particular, $\underline{D}\varphi'(x_0) \geqslant f(x_0, \varphi(x_0), \varphi'(x_0))$ from which it follows that φ is a $C^{(1)}$-lower solution on I.

We also conclude from the proof of Theorem 3.2 that φ has a finite second derivative almost everywhere and $\varphi''(x) \geqslant f(x, \varphi(x), \varphi'(x))$ almost everywhere on I ([22], p. 128). It does not follow however that φ is necessarily an $AC^{(1)}$-lower solution on I. To see this let $h(x)$ be a continuous strictly increasing function on $[0, 1]$ with $h(0) = 0$, $h(1) = 1$ but with $h(x)$ not absolutely continuous on $[0, 1]$. Then $\varphi(x) \equiv \int_0^x h(t)\,dt$ is convex on $[0, 1]$, hence $\varphi \in C^{(1)}[0, 1]$ is a subfunction on $[0, 1]$ with respect to solutions of $y'' = 0$. However $\varphi' = h$ is not absolutely continuous on $[0, 1]$ and φ is not an $AC^{(1)}$-lower solution of $y'' = 0$ on $[0, 1]$.

Assuming only the continuity of $f(x, y, y')$ we conclude from Theorem 3.2 that a subfunction of class $C^{(1)}$ is necessarily a $C^{(1)}$-lower solution of the differential equation. Now we turn the question around and look at the possibility of a lower solution of the differential equation being a subfunction. For this to be the case it is obvious that more is required of $f(x, y, y')$ than continuity. This is apparent from the fact that a solution is also a lower solution, hence, if lower solutions are subfunctions, then solutions are subfunctions. From the definition of subfunctions it would then follow that, if a boundary-value problem on an interval $[x_1, x_2]$ has a solution $y \in C^{(2)}[x_1, x_2]$, that solution is unique. This will not be true in general and, therefore, we cannot expect lower solutions to be subfunctions unless stronger conditions are placed on $f(x, y, y')$. Also it is clear that a theorem which gives sufficient conditions for a lower solution to be a subfunction is automatically a theorem giving sufficient conditions for solutions of boundary-value problems to be unique.

Lemma 3.3. *Assume that $f(x, y, y')$ is nondecreasing in y for all fixed x, y'. Let φ be a $C^{(1)}$-lower solution and ψ a $C^{(1)}$-upper solution on a closed interval I and assume that at each point on I^0 at least one of the differential inequalities is a strict inequality. Further assume that $\varphi(x) \leqslant \psi(x) + M$ at the end points of I where $M \geqslant 0$. Then $\varphi(x) < \psi(x) + M$ on I.*

Proof. It suffices to consider only the case $M = 0$ since

$$\bar{D}\psi' = \bar{D}[\psi + M]' \leqslant f(x, \psi, \psi') \leqslant f(x, \psi + M, \psi')$$

from which it follows that $\psi^* = \psi + M$ is an upper solution. Hence, assume $\varphi(x) \leqslant \psi(x)$ at the endpoints of I but that also $\varphi(x) \geqslant \psi(x)$ at some points in I^0. Then $\varphi(x) - \psi(x)$ has a nonnegative maximum at some $x_0 \in I^0$ and at x_0

$$\begin{aligned}\underline{D}[\varphi'(x_0) - \psi'(x_0)] &\geqslant \underline{D}\varphi'(x_0) - \bar{D}\psi'(x_0)\\ &> f(x_0, \varphi(x_0), \varphi'(x_0)) - f(x_0, \psi(x_0), \psi'(x_0)).\end{aligned}$$

Since $\varphi'(x_0) = \psi'(x_0)$ and f is nondecreasing in y, we conclude that $\underline{D}[\varphi'(x_0) - \psi'(x_0)] > 0$ which is impossible with $\varphi(x) - \psi(x)$ having a maximum at x_0. We conclude that $\varphi(x) < \psi(x)$ on I.

Lemma 3.4. *Assume that on $[a, b] \times R^2$ $f(x, y, y')$ is nondecreasing in y for each fixed x, y' and that $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on each compact subset of $[a, b] \times R^2$. Let $[c, d] \subset [a, b]$ and assume that $\varphi \in C^{(1)}[c, d]$ is a $C^{(1)}$-lower solution on $[c, d]$. Then given $\epsilon > 0$ there is a function $\varphi_1 \in C^{(1)}[c, d]$ such that $\varphi(x) - \epsilon \leqslant \varphi_1(x) \leqslant \varphi(x)$ on $[c, d]$ and $\underline{D}\varphi_1'(x) > f(x, \varphi_1(x), \varphi_1'(x))$ on (c, d)* ([23], Lemma 2.2).

Proof. By assumption $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on the compact set

$$K = \{(x, y, y') : c \leqslant x \leqslant d, \mid \varphi(x) - y \mid \leqslant 1, \mid \varphi'(x) - y' \mid \leqslant 1\}.$$

Let $k > 0$ be an associated Lipschitz coefficient. For a given $\epsilon > 0$ let $\rho(x)$ satisfy the following conditions:

$$\rho''(x) = (k + 1)\,\rho'(x) \qquad \text{on} \qquad [c, d],$$

$$0 < \rho'(x) \leqslant 1 \qquad \text{on} \qquad (c, d),$$

and

$$-\operatorname{Min}[1, \epsilon] \leqslant \rho(x) \leqslant 0 \qquad \text{on} \qquad [c, d].$$

Then setting $\varphi_1 = \varphi + \rho$ we have that on $[c, d]$

$$\varphi(x) - \epsilon \leqslant \varphi_1(x) \leqslant \varphi(x)$$

and on (c, d)

$$\begin{aligned}
\underline{D}\varphi_1'(x) &= \underline{D}[\varphi'(x) + \rho'(x)] \\
&= \underline{D}\varphi'(x) + \rho''(x) \geqslant f(x, \varphi(x), \varphi'(x)) + (k + 1) \mid \rho'(x) \mid \\
&\qquad \geqslant f(x, \varphi(x), \varphi'(x)) + f(x, \varphi(x), \varphi'(x) + \rho'(x)) \\
&\qquad\qquad - f(x, \varphi(x), \varphi'(x)) + \mid \rho'(x) \mid \\
&\qquad \geqslant f(x, \varphi_1(x), \varphi_1'(x)) + \mid \rho'(x) \mid \\
&\qquad > f(x, \varphi_1(x), \varphi_1'(x)).
\end{aligned}$$

Theorem 3.5. *Let $f(x, y, y')$ be nondecreasing in y for fixed x, y' and satisfy a Lipschitz condition with respect to y' on each compact subset of $[a, b] \times R^2$. Let $[c, d] \subset [a, b]$ and assume that $\varphi, \psi \in C[c, d] \cap C^{(1)}(c, d)$ are respectively $C^{(1)}$-lower and $C^{(1)}$-upper solutions on $[c, d]$. Then, if $\varphi(x) \leqslant \psi(x) + M$, $M \geqslant 0$, at the endpoints of $[c, d]$, it follows that $\varphi(x) \leqslant \psi(x) + M$ on $[c, d]$* ([23], Theorem 2.2).

Proof. As in Lemma 3.3 it suffices to consider the case $M = 0$. Hence, assume $\varphi(c) \leqslant \psi(c)$ and $\varphi(d) \leqslant \psi(d)$ but $\varphi(x) > \psi(x)$ at some points in (c, d). Let $\epsilon = \text{Max}[\varphi(x) - \psi(x)]$ on $[c, d]$ and let $[c_1, d_1] \subset [c, d]$ such that $\varphi(c_1) - \psi(c_1) = \varphi(d_1) - \psi(d_1) = \frac{1}{2}\epsilon$, $\varphi(x) - \psi(x) \geqslant \frac{1}{2}\epsilon$ on $[c_1, d_1]$, and the maximum $\varphi(x) - \psi(x) = \epsilon$ is assumed on (c_1, d_1). By Lemma 3.4 there is a function $\varphi_1 \in C^{(1)}[c_1, d_1]$ such that $\varphi - \frac{1}{2}\epsilon \leqslant \varphi_1 \leqslant \varphi$ on $[c_1, d_1]$ and $\underline{D}\varphi_1' > f(x, \varphi_1, \varphi_1')$ on (c_1, d_1). Then $\varphi_1(x) \leqslant \psi(x) + \frac{1}{2}\epsilon$ at $x = c_1$ and $x = d_1$ and $\varphi_1(x) \geqslant \psi(x) + \frac{1}{2}\epsilon$ at some points in (c_1, d_1). This contradicts Lemma 3.3 and we conclude that $\varphi(x) \leqslant \psi(x)$ on $[c, d]$.

Remark. If in Theorem 3.5 we assume that φ is an $AC^{(1)}$-lower solution and ψ is an $AC^{(1)}$-upper solution, the conclusion of the Theorem is still valid. To see this we proceed as in the proof of the Theorem up to the point where the subinterval $[c_1, d_1] \subset [c, d]$ with the stated properties is obtained. Then we note that the proof of Lemma 3.4 leads to the conclusion that there is a function $\varphi_1 \in C^{(1)}[c_1, d_1]$ such that φ_1' is absolutely continuous on $[c_1, d_1]$, $\varphi(x) - \frac{1}{2}\epsilon \leqslant \varphi_1(x) \leqslant \varphi(x)$ on $[c_1, d_1]$, and $\varphi_1''(x) \geqslant f(x, \varphi_1(x), \varphi_1'(x)) + |\rho'(x)|$ almost everywhere on $[c_1, d_1]$. Since $\varphi_1(x) - \psi(x) \leqslant \frac{1}{2}\epsilon$ at $x = c_1$ and $x = d_1$ while $\varphi_1(x) - \psi(x) \geqslant \frac{1}{2}\epsilon$ at some points of (c_1, d_1), $\varphi_1(x) - \psi(x)$ has a positive maximum at some $x_0 \in (c_1, d_1)$. Furthermore $|\rho'(x)| > 0$ on (c_1, d_1), hence there is a $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (c_1, d_1)$ and $\varphi_1''(x) - \psi''(x) > 0$ almost everywhere on $[x_0 - \delta, x_0 + \delta]$. This is impossible with $\varphi_1(x) - \psi(x)$ having a maximum at x_0. Thus we conclude again that $\varphi(x) \leqslant \psi(x)$ on $[c, d]$.

Corollary 3.6. *If f satisfies the hypotheses of Theorem 3.5, then a lower solution φ on a subinterval $I \subset [a, b]$ is a subfunction on I.*

We have already observed that a $C^{(1)}$-lower solution is not necessarily an $AC^{(1)}$-lower solution. However, if $f(x, y, y')$ satisfies the hypotheses of Theorem 3.5, then it follows from Corollary 3.6 and the Remark preceding it that an $AC^{(1)}$-lower solution on $I \subset [a, b]$ is a subfunction on I. Then by Theorem 3.2 it will be a $C^{(1)}$-lower solution on I.

Corollary 3.7. *If $f(x, y, y')$ satisfies the conditions of Theorem 3.5 and $y_1, y_2 \in C^{(2)}[x_1, x_2]$ are solutions on $[x_1, x_2] \subset [a, b]$ with $y_1(x_i) = y_2(x_i)$ for $i = 1, 2$, then $y_1(x) \equiv y_2(x)$ on $[x_1, x_2]$.*

Corollary 3.7 is no longer valid if the Lipschitz condition on $f(x, y, y')$ with respect to y' on compact sets is omitted, and, as a matter of fact, it cannot be weakened in any very significant way. For example the boundary-value problem

$$y'' = (\tfrac{3}{2})(\tfrac{5}{2})^{2/3} \mid y' \mid^{1/3}, \qquad y(-1) = y(+1) = 1$$

has solutions $y(x) \equiv 1$ and $y(x) = \mid x \mid^{5/2}$ of class $C^{(2)}[-1, +1]$.

Corollary 3.8. *If f, φ, ψ satisfy the conditions of Theorem 3.5 with $\varphi(c) = \psi(c)$ and $\varphi(x) \leqslant \psi(x)$ on $[c, d]$, then $\varphi(x) - \psi(x)$ is nondecreasing on $[c, d]$.*

Theorem 3.9. *Let $f(x, y, y')$ be nondecreasing in y for fixed x, y' on $[a, b] \times R^2$ and assume that solutions of initial-value problems for $y'' = f(x, y, y')$ are unique. Then, if φ and ψ are lower and upper solutions on $[c, d] \subset [a, b]$ with $\varphi(x) \leqslant \psi(x) + M$, $M \geqslant 0$, at $x = c$ and $x = d$, it follows that $\varphi(x) \leqslant \psi(x) + M$ on $[c, d]$.*

Proof. Assume that $\varphi(x) > \psi(x) + M$ at some points in (c, d) and let $N = \text{Max}[\varphi(x) - \psi(x)]$ on $[c, d]$. Then $\alpha(x) = \varphi(x)$ and $\beta(x) = \psi(x) + N$ satisfy the hypotheses of Theorem 2.6. It follows that $\alpha(x) \equiv \beta(x)$ on $[c, d]$. From this contradiction we conclude that $\varphi(x) \leqslant \psi(x) + M$ on $[c, d]$.

Corollary 3.10. *If $f(x, y, y')$ satisfies the hypotheses of Theorem 3.9, a lower solution on an interval $I \subset [a, b]$ is a subfunction on I.*

It will again be the case that, if φ is an $AC^{(1)}$-lower solution on I, then φ is a $C^{(1)}$-lower solution on I.

Corollary 3.11. *If $f(x, y, y')$ satisfies the hypotheses of Theorem 3.9, solutions of boundary-value problems of class $C^{(2)}[x_1, x_2]$, $[x_1, x_2] \subset [a, b]$, when they exist are unique.*

In the preceding results of this section conditions have been placed on $f(x, y, y')$ which are sufficient to imply that lower solutions are subfunctions and, consequently, that solutions of boundary-value problems, when they exist, are unique. In the next theorem we take the uniqueness of solutions of boundary-value problems as one of the hypotheses.

Theorem 3.12. *Assume that solutions of boundary-value problems for $y'' = f(x, y, y')$, when they exist, are unique in the sense of Corollary 3.7. Assume also that each initial-value problem for $y'' = f(x, y, y')$ has a solu-*

tion which extends throughout $[a, b]$. *Then, if* $I \subset [a, b]$ *and* $\varphi \in C^{(1)}(I)$ *is a lower solution on* I, φ *is a subfunction on* I [24].

Proof. Assume that φ is not a subfunction on I. Then there is an interval $[c, d] \subset I$ and a solution $y_0 \in C^{(2)}[c, d]$ such that $y_0(c) = \varphi(c)$, $y_0(d) = \varphi(d)$, and $y_0(x) < \varphi(x)$ on (c, d). Now define $F(x, y, y')$ on $[c, d] \times R^2$ by

$$F(x, y, y') = \begin{cases} f(x, y, y') & \text{for} \quad y \geqslant \varphi(x), \\ f(x, \varphi(x), y') - (\varphi(x) - y) & \text{for} \quad y \leqslant \varphi(x). \end{cases}$$

Since $F(x, y, y')$ is continuous on $[c, d] \times R^2$ and $\varphi \in C^{(1)}[c, d]$, it follows from Theorem 2.1 that there is a $\delta > 0$ such that $[x_1, x_2] \subset [c, d]$ and $x_2 - x_1 \leqslant \delta$ implies that the boundary-value problem

$$y'' = F(x, y, y'), \qquad y(x_1) = \varphi(x_1), \qquad y(x_2) = \varphi(x_2)$$

has a solution $y(x) \in C^{(2)}[x_1, x_2]$. Using the fact that φ is a lower solution we can show that $y(x) \geqslant \varphi(x)$ on $[x_1, x_2]$ with the same type of argument as used in the proof of Theorem 2.5. Consequently, for $[x_1, x_2] \subset [c, d]$ and $x_2 - x_1 \leqslant \delta$, the boundary value-problem

$$y'' = f(x, y, y'), \qquad y(x_1) = \varphi(x_1), \qquad y(x_2) = \varphi(x_2)$$

has a solution $y(x) \in C^{(2)}[x_1, x_2]$ with $\varphi(x) \leqslant y(x)$ on $[x_1, x_2]$. Thus φ is a subfunction "in the small".

Now $d - c > \delta$ since otherwise there would be a solution $y(x)$ with $y(c) = \varphi(c)$, $y(d) = \varphi(d)$, and $y(x) \geqslant \varphi(x)$ on $[c, d]$. This solution would be distinct from $y_0(x)$ contradicting our assumption concerning the uniqueness of solutions of boundary value problems. Now for each positive integer n let $P(n)$ be the proposition that there exists an interval $[c_n, d_n] \subset [c, d]$ with $0 < d_n - c_n \leqslant d - c - (n - 1)\delta$ and a solution $y_n(x) \in C^{(2)}[c_n, d_n]$ with $y_n(c_n) = \varphi(c_n)$, $y_n(d_n) = \varphi(d_n)$, and $y_n(x) < \varphi(x)$ on (c_n, d_n). Now $P(1)$ is true with $[c_1, d_1] = [c, d]$ and $y_1(x) = y_0(x)$. Assume $P(k)$ is true. Then $d_k - c_k > \delta$ otherwise we would obtain a contradiction of $y_k(x)$ being the distinct solution with boundary values $\varphi(c_k)$ and $\varphi(d_k)$. Let $z_1(x)$ be the solution of the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(c_k) = \varphi(c_k), \qquad y(c_k + \delta) = \varphi(c_k + \delta).$$

Since each initial-value problem has a solution extending throughout $[a, b]$, there is a solution $z_2(x)$ on $[c_k, d_k]$ such that $z_2(x) \equiv z_1(x)$ on

$[c_k, c_k + \delta]$. Now, if $P(k+1)$ is not true, we must have $z_2(x) \geqslant \varphi(x)$ on $[c_k + \delta, d_k]$. Also we must have $z_2(d_k) > \varphi(d_k)$. If $d_k - c_k - \delta \leqslant \delta$, the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(c_k + \delta) = \varphi(c_k + \delta), \qquad y(d_k) = \varphi(d_k)$$

has a solution $z_3 \in C^{(2)}[c_k + \delta, d_k]$ with $\varphi(x) \leqslant z_3(x)$. Then, since $z_3(d_k) < z_2(d_k)$ and solutions of boundary-value problems are unique,

$$\varphi(x) \leqslant z_3(x) \leqslant z_2(x)$$

on $[c_k + \delta, d_k]$. This implies

$$z_5(c_k + \delta) = z_2(c_k + \delta) \qquad \text{and} \qquad z_5'(c_k + \delta) = z_2'(c_k + \delta).$$

Consequently, $u(x)$ defined by

$$u(x) = \begin{cases} z_1(x) & \text{on} \quad [c_k, c_k + \delta] \\ z_5(x) & \text{on} \quad [c_k + \delta, d_k], \end{cases}$$

is of class $C^{(2)}[c_k, d_k]$ and is a solution on $[c_k, d_k]$ with $u(c_k) = y_k(c_k)$, $u(d_k) = y_k(d_k)$. However $u(x) \not\equiv y_k(x)$ on $[c_k, d_k]$ and this contradicts the uniqueness of solutions of boundary-value problems. We conclude that $d_k - c_k - \delta > \delta$. This being the case the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(c_k + \delta) = \varphi(c_k + \delta), \qquad y(c_k + 2\delta) = \varphi(c_k + 2\delta)$$

has a solution $z_4 \in C^{(2)}[c_k + \delta, c_k + 2\delta]$ with

$$\varphi(x) \leqslant z_4(x) \leqslant z_2(x)$$

on $[c_k + \delta, c_k + 2\delta]$. Again this implies $z_4(c_k + \delta) = z_2(c_k + \delta)$ and $z_4'(c_k + \delta) = z_2'(c_k + \delta)$. Hence

$$v(x) = \begin{cases} z_1(x) & \text{on} \quad [c_k, c_k + \delta], \\ z_4(x) & \text{on} \quad [c_k + \delta, c_k + 2\delta], \end{cases}$$

is such that $v \in C^{(2)}[c_k, c_k + 2\delta]$ and is a solution of the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(c_k) = \varphi(c_k), \qquad y(c_k + 2\delta) = \varphi(c_k + 2\delta).$$

This solution $v(x)$ has an extension $z_5(x)$ to all of $[a, b]$. Since $P(k+1)$ is assumed to be false we must have $z_5(x) \geqslant \varphi(x)$ on $[c_k + 2\delta, d_k]$ and

$z_5(d_k) > \varphi(d_k)$. Now the above arguments can be repeated and the assumption that $P(k+1)$ is false allows us to work our way across the interval $[c_k, d_k]$ by subintervals of length δ until we obtain a solution $w \in C^{(2)}[c_k, d_k]$ with $w(x) \geqslant \varphi(x)$ on $[c_k, d_k]$, $w(c_k) = \varphi(c_k) = y_k(c_k)$, and $w(d_k) = \varphi(d_k) = y_k(d_k)$. Then $w(x) \not\equiv y_k(x)$ on $[c_k, d_k]$ which again contradicts the uniqueness of solutions of boundary-value problems.

We conclude that, if $P(k)$ is true, it follows that $P(k+1)$ is true. Hence, $P(n)$ is true for all $n \geqslant 1$. This leads to the contradiction $0 < d - c - (n-1)\,\delta$ for all $n \geqslant 1$. Thus φ is a subfunction on I.

If one examines the induction argument used in the proof of Theorem 3.12, it can be seen that the assumption that each initial value problem has a solution extending throughout $[a, b]$ can be weakened. It would suffice to assume that for any $a \leqslant x_0 < b$ and any solution $y(x)$ of an initial-value problem at x_0 either $y(x)$ has an extension to $[x_0, b]$ or $y(x)$ is unbounded on $[x_0, c)$ where $[x_0, c)$ is a maximal interval of existence of $y(x)$. This will be the case if $f(x, y, y')$ satisfies a Nagumo condition. Nagumo conditions will be discussed in later sections.

The question of whether or not Theorem 3.12 remains valid if it is assumed only that $f(x, y, y')$ is continuous and that solutions of boundary-value problems when they exist are unique is still not answered.

In this Section most of the results have been stated in terms of lower solutions and subfunctions. There are corresponding results concerning upper solutions and superfunctions. When it becomes necessary to refer to such a result for superfunctions we shall simply refer to the subfunction statement of the result.

4. Properties of Subfunctions and the Study of Boundary-Value Problems by Subfunction Methods

We shall now use the Perron method in attempting to establish existence theorems for solutions of boundary value problems for second-order ordinary differential equations. First it will be necessary to make a more detailed examination of properties of subfunctions and superfunctions. Then these properties and Theorem 2.1, the existence "in the small" Theorem, will be used to establish existence "in the large". Again in this Section most results will be stated in terms of subfunctions and the obvious analogous results for superfunctions will not be stated.

As remarked earlier we will always assume that $f(x, y, y')$ is continuous on its domain $I \times R^2$. When additional hypotheses are required, they will be stated.

Theorem 4.1. *If φ is a subfunction on an interval $J \subset I$, then φ has right- and left-hand limits in the extended reals at each point in J^0 and has appropriate one-sided limits at finite endpoints of J.*

Proof. It will suffice to consider one case. Let $x_0 \in J^0$ and assume that $\varphi(x_0 - 0) = \lim_{x \to x_0 -} \varphi(x)$ does not exist in the extended reals. Then there exist real numbers α and β such that

$$\liminf_{x \to x_0 -} \varphi(x) < \alpha < \beta < \limsup_{x \to x_0 -} \varphi(x).$$

Let $\{t_n\}$, $\{s_n\}$ be strictly increasing sequences in J such that $t_n < s_n < t_{n+1}$ for $n \geqslant 1$, $\lim t_n = \lim s_n = x_0$, $\lim \varphi(t_n) = \lim\sup_{x \to x_0 -} \varphi(x)$, and $\lim \varphi(s_n) = \lim\inf_{x \to x_0 -} \varphi(x)$. With $\epsilon = \frac{1}{4}(\beta - \alpha)$ it follows from Theorem 2.1 that there is a $\delta > 0$ such that for any $[x_1, x_2] \subset [t_1, x_0]$ with $x_2 - x_1 \leqslant \delta$ the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_1) = y(x_2) = \tfrac{1}{2}(\alpha + \beta)$$

has a solution $y(x) \in C^{(2)}[x_1, x_2]$ with $| y(x) - \frac{1}{2}(\alpha + \beta) | < \epsilon$ on $[x_1, x_2]$. Let n be a fixed positive integer chosen large enough that $s_{n+1} - s_n \leqslant \delta$, $\varphi(s_n) < \alpha$, $\varphi(s_{n+1}) < \alpha$, and $\varphi(t_{n+1}) > \beta$. It follows that there is a solution of the above boundary-value problem with $[x_1, x_2] = [s_n, s_{n+1}]$ and $| y(x) - \frac{1}{2}(\alpha + \beta) | < \epsilon$ on $[s_n, s_{n+1}]$. Since φ is a subfunction on J and $\varphi(s_n) < y(s_n)$, $\varphi(s_{n+1}) < y(s_{n+1})$, it must be the case that $\varphi(t_{n+1}) \leqslant y(t_{n+1})$. However,

$$y(t_{n+1}) < \tfrac{1}{2}(\alpha + \beta) + \epsilon < \beta < \varphi(t_{n+1}).$$

From this contradiction we conclude that $\varphi(x_0 - 0)$ exists in the extended reals.

Corollary 4.2. *If φ is a bounded subfunction on $J \subset I$, then φ has at most a countable number of discontinuities on J. At each $x_0 \in J^0$*

$$\varphi(x_0) \leqslant \operatorname{Max}[\varphi(x_0 + 0), \varphi(x_0 - 0)].$$

Proof. The first assertion is a classical result that follows from the fact that $\varphi(x_0 + 0)$ and $\varphi(x_0 - 0)$ exist at each $x_0 \in J^0$. The second assertion follows readily from Theorem 2.1 and the fact that φ is a subfunction on J.

Next we consider differentiability of subfunctions. For a function $g(x)$ with a finite right-hand limit $g(x_0 + 0)$ at x_0, we define

$$Dg(x_0 +) = \lim_{x \to x_0+} \frac{g(x) - g(x_0 + 0)}{x - x_0}$$

provided the limit exists. Similarly, for a function $g(x)$ with a finite left-hand limit $g(x_0 - 0)$ at x_0, we define

$$Dg(x_0 -) = \lim_{x \to x_0-} \frac{g(x) - g(x_0 - 0)}{x - x_0}$$

provided the limit exists.

Theorem 4.3. *If φ is a bounded subfunction on $J \subset I$, then $D\varphi(x_0 +)$ and $D\varphi(x_0 -)$ exist in the extended reals for each $x_0 \in J^0$. The appropriate one-sided "derivatives" exist at finite end points of J.*

Proof. As in Theorem 4.1 it will suffice to consider just one case. Assume $x_0 \in J^0$ and that

$$\liminf_{x \to x_0+} \frac{\varphi(x) - \varphi(x_0 + 0)}{x - x_0} < \limsup_{x \to x_0+} \frac{\varphi(x) - \varphi(x_0 + 0)}{x - x_0}.$$

Let m be a real number strictly between these two limits. Then the initial-value problem

$$y'' = f(x, y, y'), \qquad y(x_0) = \varphi(x_0 + 0), \qquad y'(x_0) = m$$

has a solution $y(x) \in C^{(2)}[x_0, x_0 + \delta]$ for some $\delta > 0$. Since

$$\lim_{x \to x_0+} \frac{y(x) - y(x_0)}{x - x_0} = m$$

and m is strictly between the above lower and upper derivatives, there exist x_1, x_2, x_3 such that $x_0 < x_1 < x_2 < x_3 < x_0 + \delta$,

$$\frac{\varphi(x) - \varphi(x_0 + 0)}{x - x_0} < \frac{y(x) - y(x_0)}{x - x_0}$$

at $x = x_1$ and $x = x_3$, and

$$\frac{\varphi(x_2) - \varphi(x_0 + 0)}{x_2 - x_0} > \frac{y(x_2) - y(x_0)}{x_2 - x_0}.$$

It follows that $\varphi(x_1) < y(x_1)$, $\varphi(x_3) < y(x_3)$, and $\varphi(x_2) > y(x_2)$. This contradicts the fact that φ is a subfunction on J. We conclude that $D\varphi(x_0 +)$ exists in the extended reals.

Corollary 4.4. *If φ is a bounded subfunction on $J \subset I$, then φ has a finite derivative almost everywhere on J.*

Proof. This is an immediate consequence of Theorem 4.3 and classical results in the theory of functions of a real variable.

Lemma 4.5. *If $\varphi(x)$ is subfunction on the interval $J \subset I$ and is bounded above on each compact subinterval of J, then $\bar{\varphi}(x) = \limsup_{z \to x} \varphi(z)$ is a subfunction on J.*

Proof. Let $[x_1, x_2] \subset J$ and assume $y(x)$ is a solution of class $C^{(2)}[x_1, x_2]$ with $\bar{\varphi}(x_1) \leqslant y(x_1)$ and $\bar{\varphi}(x_2) \leqslant y(x_2)$. Then $\varphi(x_i) \leqslant \bar{\varphi}(x_i) \leqslant y(x_i)$ for $i = 1, 2$ and, since φ is a subfunction, $\varphi(x) \leqslant y(x)$ on $[x_1, x_2]$. It follows that for each $x_1 < x < x_2$ $\bar{\varphi}(x) = \limsup_{z \to x} \varphi(z) \leqslant y(x)$. Hence $\bar{\varphi}(x) \leqslant y(x)$ on $[x_1, x_2]$ and $\bar{\varphi}$ is a subfunction on J.

Next we consider some lattice properties of subfunctions.

Theorem 4.6. *Assume that the collection of subfunctions $\{\varphi_\alpha : \alpha \in A\}$ on the interval $J \subset I$ is bounded above at each point of J. Then $\varphi_0(x) = \sup_{\alpha \in A} \varphi_\alpha(x)$ is a subfunction on J.*

Proof. Assume $[x_1, x_2] \subset J$ and assume that $y(x) \in C^{(2)}[x_1, x_2]$ is a solution on $[x_1, x_2]$ with $\varphi_0(x) \leqslant y(x)$ at $x = x_1, x_2$. Then from the definition of $\varphi_0(x)$ it follows that $\varphi_\alpha(x) \leqslant y(x)$ at $x = x_1, x_2$ for each $\alpha \in A$. Since each φ_α is a subfunction on J, we conclude that $\varphi_\alpha(x) \leqslant y(x)$ on $[x_1, x_2]$ for each $\alpha \in A$. This implies $\varphi_0(x) \leqslant y(x)$ on $[x_1, x_2]$ and φ_0 is a subfunction on J.

Theorem 4.7. *Let φ be a subfunction on an interval $J \subset I$ and φ_1 a subfunction on an interval J_1 with $J_1 = \bar{J}_1 \cap J$. Furthermore, assume that*

$\varphi_1(x) \leqslant \varphi(x)$ at finite end points of J_1 which are contained in J. Then φ_2 defined by

$$\varphi_2(x) = \begin{cases} \mathrm{Max}[\varphi_1(x), \varphi(x)] & \text{for} \quad x \in J_1, \\ \varphi(x) & \text{for} \quad x \in J - J_1 \end{cases}$$

is a subfunction on J.

Proof. By hypothesis $\varphi_2(x) = \varphi(x)$ is a subfunction on $J - J_1$ and by Theorem 4.6 $\varphi_2(x)$ is a subfunction on J_1. Consequently, to complete the proof we need to show that we have the correct behavior on intervals $[x_1, x_2] \subset J$ which are not contained in either J_1 or $J - J_1$. We will consider just one case since the arguments in the other cases proceed in a similar way. Assume $x_1 \in J_1$, $x_2 \in J - J_1$, and x_3, $x_1 < x_3 < x_2$, is the right-hand end point of J_1. Assume $y(x) \in C^{(2)}[x_1, x_2]$ is a solution on $[x_1, x_2]$ with $\varphi_2(x_1) \leqslant y(x_1)$ and $\varphi_2(x_2) \leqslant y(x_2)$. Then $\varphi(x_1) \leqslant \varphi_2(x_1) \leqslant y(x_1)$ and $\varphi(x_2) = \varphi_2(x_2) \leqslant y(x_2)$, and since φ is a subfunction on J it follows that $\varphi(x) \leqslant y(x)$ on $[x_1, x_2]$. In particular, $\varphi_2(x) = \varphi(x) \leqslant y(x)$ on $(x_3, x_3]$. Also $\varphi_1(x_3) \leqslant \varphi(x_3) \leqslant y(x_3)$ and $\varphi_1(x_1) \leqslant \varphi_2(x_1) \leqslant y(x_1)$, hence, since φ_1 is a subfunction on J_1, $\varphi_1(x) \leqslant y(x)$ on $[x_1, x_3]$. It follows that $\varphi_2(x) = \mathrm{Max}[\varphi(x), \varphi_1(x)] \leqslant y(x)$ on $[x_1, x_3]$. Putting these things together we have $\varphi_2(x) \leqslant y(x)$ on $[x_1, x_2]$. The other possibilities are dealt with in a similar way and we conclude that φ_2 is a subfunction on J.

Theorem 4.8. *Assume that $f(x, y, y')$ is nondecreasing in y on $I \times R^2$ for each fixed x, y' and either satisfies a Lipschitz condition with respect to y' on each compact subset of $I \times R^2$ or is such that solutions of initial-value problems are unique. Then, if $\psi(x)$ is an upper solution on $[x_1, x_2] \subset I$ and $\varphi(x)$ is a bounded subfunction on $[x_1, x_2]$ with $\varphi(x_1 + 0) \leqslant \psi(x_1)$ and $\varphi(x_2 - 0) \leqslant \psi(x_2)$, it follows that $\varphi(x) \leqslant \psi(x)$ on (x_1, x_2).*

Proof. Assume that the stated conditions hold but that $\varphi(x) > \psi(x)$ at some points of (x_1, x_2). First we note that it suffices to consider the case where φ is upper semicontinuous on (x_1, x_2). To see this let $\bar\varphi(x) = \limsup_{z \to x} \varphi(z)$ on (x_1, x_2). Then by Lemma 4.5, $\bar\varphi(x)$ is a subfunction on (x_1, x_2). By Corollary 4.2 $\bar\varphi(x_1 + 0) = \varphi(x_1 + 0) \leqslant \psi(x_1)$ and $\bar\varphi(x_2 - 0) = \varphi(x_2 - 0) \leqslant \psi(x_2)$. Furthermore $\varphi(x) > \psi(x)$ at some points in (x_1, x_2) implies $\bar\varphi(x) > \psi(x)$ at some points in (x_1, x_2). Consequently, we may assume $\varphi(x)$ is upper semicontinuous on (x_1, x_2). This being the case $\varphi(x) > \psi(x)$ at some points in (x_1, x_2) and

$\varphi(x_1 + 0) \leqslant \psi(x_1)$, $\varphi(x_2 - 0) \leqslant \psi(x_2)$ implies $\varphi(x) - \psi(x)$ has a positive maximum M which is assumed on a compact set $E \subset (x_1, x_2)$.

Now assume $f(x, y, y')$ is nondecreasing in y and that solutions of initial-value problemas are unique. Let $x_0 = \text{lub } E$. Then there exists a $\delta > 0$ and $\epsilon > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subset (x_1, x_2), \qquad 0 < \epsilon < \psi(x_0 + \delta) + M - \varphi(x_0 + \delta),$$

and such that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_0 - \delta) = \psi(x_0 - \delta) + M,$$
$$y(x_0 + \delta) = \psi(x_0 + \delta) + M - \epsilon$$

has a solution $y(x) \in C^{(2)}[x_0 - \delta, x_0 + \delta]$. This follows from Theorem 2.1. Since $\psi(x) + M$ is an upper solution, it follows from Theorem 2.6 that $y(x) < \psi(x) + M$ on $(x_0 - \delta, x_0 + \delta)$. However,

$$\varphi(x_0 - \delta) \leqslant \psi(x_0 - \delta) + M = y(x_0 - \delta)$$

and

$$\varphi(x_0 + \delta) < \psi(x_0 + \delta) + M - \epsilon = y(x_0 + \delta),$$

which implies $\varphi(x_0) \leqslant y(x_0)$ since φ is a subfunction on (x_1, x_2). This contradicts the fact that $x_0 \in E$ and $\varphi(x_0) = \psi(x_0) + M$. We conclude that with these hypotheses $\varphi(x) \leqslant \psi(x)$ on (x_1, x_2).

Now assume that $f(x, y, y')$ is nondecreasing in y and satisfies a Lipschitz condition with respect to y' on each compact subset of $I \times R^2$. Let $[x_3, x_4] \subset (x_1, x_2)$ be such that $E \subset [x_3, x_4]$ and

$$\varphi(x_3) < \psi(x_3) + \frac{M}{2},$$

$$\varphi(x_4) < \psi(x_4) + \frac{M}{2}.$$

Then by Lemma 3.4 there is a $\psi_1 \in C^{(1)}[x_3, x_4]$ such that

$$\psi(x) \leqslant \psi_1(x) \leqslant \psi(x) + \frac{M}{2} \qquad \text{on} \qquad [x_3, x_4]$$

and

$$\bar{D}\psi_1' < f(x, \psi_1, \psi_1') \qquad \text{on} \qquad (x_3, x_4).$$

It follows that $\varphi(x_3) < \psi_1(x_3) + 1/2M$, $\varphi(x_4) < \psi_1(x_4) + 1/2M$, and $\varphi(x) = \psi(x) + M \geqslant \psi_1(x) + 1/2M$ on $E \subset (x_3, x_4)$. Hence, $\varphi(x) - \psi_1(x)$ has a positive maximum N on a compact subset $E_1 \subset (x_3, x_4)$. Let $x_5 = \text{lub } E_1$. Then there is, by Theorem 2.1, a $\delta > 0$ such that $[x_5 - \delta, x_5 + \delta] \subset (x_3, x_4)$ and such that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_5 \pm \delta) = \psi_1(x_5 \pm \delta) + N$$

has a solution $y(x) \in C^{(2)}[x_5 - \delta, x_5 + \delta]$. By Lemma 3.3

$$y(x) < \psi_1(x) + N \qquad \text{on} \qquad (x_5 - \delta, x_5 + \delta).$$

Since

$$\varphi(x_5 \pm \delta) \leqslant \psi_1(x_5 \pm \delta) + N = y(x_5 \pm \delta)$$

and φ is a subfunction, $\varphi(x_5) \leqslant y(x_5) < \psi_1(x_5) + N$, which contradicts $x_5 \in E_1$. We conclude again that $\varphi(x) \leqslant \psi(x)$ on (x_1, x_2).

We consider now properties of bounded functions which are simultaneously subfunctions and superfunctions. We will need a result concerning solutions of initial value problems which is well known. For the sake of completeness we include a proof of this result.

Lemma 4.9. *If $(x_0, y_0, y_0') \in I \times R^2$, there exist $\delta > 0$, $M_1 > 0$, and $M_2 > 0$ such that every solution of the initial-value problem $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0$ is defined on*

$$I_\delta = [x_0 - \delta, x_0 + \delta] \cap I.$$

Furthermore, $| y(x) | \leqslant M_1$ and $| y'(x) | \leqslant M_2$ on I_δ for all solutions.

Proof. Given $(x_0, y_0, y_0') \in I \times R^2$ first choose $\delta_1 > 0$ such that $I_{\delta_1} = [x_0 - \delta_1, x_0 + \delta_1] \cap I$ is a compact subinterval of I. Then let

$$M = \text{Max}\{| f(x, y, y') | : x \in I_1, | y - y_0 | \leqslant 1, | y' - y_0' | \leqslant 1\}.$$

Let $\delta = \text{Min}[\delta_1, 1/M, 1/(| y_0' | + 1)]$, $M_1 = | y_0 | + 1$, and $M_2 = | y_0' | + 1$. Then it is easy to see that every solution of the given initial value problem extends to I_δ and on I_δ all solutions satisfy $| y(x) | \leqslant M_1$, $| y'(x) | \leqslant M_2$.

Theorem 4.10. *Assume that $f(x, y, y')$ is such that $C^{(2)}$ solutions of boundary-value problems, when they exist, are unique. That is, assume that,* *if* $[x_1, x_2] \subset I$ *and* $y_1, y_2 \in C^{(2)}[x_1, x_2]$ *are solutions of* $y'' = f(x, y, y')$ *on* $[x_1, x_2]$ *with* $y_1(x_1) = y_2(x_1)$ *and* $y_1(x_2) = y_2(x_2)$, *then* $y_1(x) \equiv y_2(x)$ *on* $[x_1, x_2]$. *Assume that* $z(x)$ *is bounded on each compact subinterval of* $J \subset I$ *and that* $z(x)$ *is simultaneously a subfunction and a superfunction on* J. *Then* $z(x)$ *is a solution of* $y'' = f(x, y, y')$ *on an open subset of* J *the complement of which has measure zero. Furthermore, if* $x_0 \in J^0$ *is a point of continuity of* $z(x)$ *at which* $z(x)$ *does not have a finite derivative, then either* $Dz(x_0 +) = Dz(x_0 -) = + \infty$ *or* $Dz(x_0 +) = Dz(x_0 -) = - \infty$. *If* $z(x_0 + 0) > z(x_0 - 0)$, $Dz(x_0 +) = Dz(x_0 -) = + \infty$, *and if* $z(x_0 + 0) < z(x_0 - 0)$, $Dz(x_0 +) = Dz(x_0 -) = - \infty$.

Proof. By Corollary 4.4 $z(x)$ has a finite derivative almost everywhere on J. If $x_0 \in J^0$ is a point at which $z(x)$ has a finite derivative, there is a $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset J$, $| z(x) | \leqslant | z(x_0) | + 1$ on $[x_0 - \delta, x_0 + \delta]$, and

$$\left| \frac{z(x_0 + \eta) - z(x_0 - \eta)}{2\eta} \right| \leqslant | z'(x_0) | + 1 \qquad \text{for} \qquad 0 < \eta \leqslant \delta.$$

It then follows from Theorem 2.1 that there is a δ_1, $0 < \delta_1 \leqslant \delta$, such that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_0 \pm \delta_1) = z(x_0 \pm \delta_1)$$

has a solution $y(x) \in C^{(2)}[x_0 - \delta_1, x_0 + \delta_1]$. Since $z(x)$ is simultaneously a subfunction and a superfunction on J, $z(x) \equiv y(x)$ on $[x_0 - \delta_1, x_0 + \delta_1]$. We conclude that $z(x)$ is a solution of $y'' = f(x, y, y')$ on an open subset of J the complement of which has measure zero.

Next let $x_0 \in J^0$ be a point of continuity of $z(x)$ at which $z(x)$ does not have a finite derivative. By Theorem 4.3, $Dz(x_0 +)$ and $Dz(x_0 -)$ both exist in the extended reals. If both are finite, then by the same argument as used above there is an interval around x_0 in which $z(x)$ is a solution. This contradicts the assumption that $z(x)$ does not have a finite derivative at x_0. Consequently, at least one of $Dz(x_0 +)$, $Dz(x_0 -)$ is not finite. To be specific assume $Dz(x_0 +) = + \infty$ and $Dz(x_0 -) \neq + \infty$. Then there exist numbers $\delta > 0$ and N such that

$$z(x) > w(x) = z(x_0) + N(x - x_0) \qquad \text{on} \qquad x_0 - \delta \leqslant x < x_0 .$$

By Theorem 2.1 there is a δ_1, $0 < \delta_1 \leqslant \delta$, such that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_0 - \delta_1) = w(x_0 - \delta_1), \qquad y(x_0) = z(x_0)$$

has a solution $y_1(x) \in C^{(2)}[x_0 - \delta_1, x_0]$. By Lemma 4.9 there is a $\delta_2 > 0$ such that $y_1(x)$ can be extended to be a solution on $[x_0 - \delta_1, x_0 + \delta_2]$ and such that all solutions of the initial-value problem

$$y'' = f(x, y, y'), \qquad y(x_0) = y_1(x_0), \qquad y'(x_0) = y_1'(x_0)$$

exist on $[x_0, x_0 + \delta_2]$ and satisfy $| y(x) - y(x_0) | \leqslant M(x - x_0)$ on $[x_0, x_0 + \delta_2]$ where $M = | y_1'(x_0) | + 1$. Again applying Theorem 2.1 we conclude that there is an η, $0 < \eta \leqslant \mathrm{Min}[\delta_1, \delta_2]$, such that for $0 < \delta_3 \leqslant \eta$ the boundary-value problem

$$y'' = f x, y, y'), \qquad y(x_0 - \delta_3) = y_1(x_0 - \delta_3),$$
$$y(x_0 + \delta_3) = y_1(x_0) + (M + \epsilon)\, \delta_3$$

has a solution $y_2(x) \in C^{(2)}[x_0 - \delta_3, x_0 + \delta_3]$ where $\epsilon > 0$ is fixed. Since $Dz(x_0 +) = +\infty$, we can assume that δ_3 is chosen so that $0 < \delta_3 \leqslant \eta$ and $y_1(x_0) + (M + \epsilon)\, \delta_3 < z(x_0 + \delta_3)$. Then

$$y_1(x_0 + \delta_3) < y_2(x_0 + \delta_3) < z(x_0 + \delta_3)$$

and

$$y_1(x_0 - \delta_3) = y_2(x_0 - \delta_3) \leqslant z(x_0 - \delta_3).$$

The last inequality follows from the fact that $z(x)$ is a superfunction and $y_1(x_0) = z(x_0)$, $y_1(x_0 - \delta_1) < z(x_0 - \delta_1)$. Since $z(x)$ is a superfunction, we conclude from the above inequalities that $y_2(x) \leqslant z(x)$ on $[x_0 - \delta_3, x_0 + \delta_3]$. From the same inequalities and the fact that solutions of boundary-value problems when they exist are unique, we conclude that $y_1(x) \leqslant y_2(x)$ on $[x_0 - \delta_3, x_0 + \delta_3]$. Thus $y_1(z_0) = z(x_0) = y_2(x_0)$ and $y_1'(x_0) = y_2'(x_0)$, hence, $y_2(x)$ is a solution of the initial-value problem with initial conditions $y(x_0) = y_1(x_0)$, $y'(x_0) = y_1'(x_0)$. But

$$| y_2(x_0 + \delta_3) - y_2(x_0) | = (M + \epsilon)\, \delta_3$$

which contradicts the fact that all solutions of this initial value problem satisfy $| y(x) - y_1(x_0) | \leqslant M(x - x_0)$ on $[x_0, x_0 + \delta_2]$. We are forced to conclude that $Dz(x_0 -) = +\infty$. By similar arguments, in some cases using the fact that $z(x)$ is also a subfunction, the other statements concerning the behavior of $z(x)$ at a point of continuity can be established.

We consider now the behavior of $z(x)$ at points of discontinuity. If $x_0 \in J^0$ is a point of discontinuity of $z(x)$, then by Theorem 4.1 $z(x_0 + 0)$ and $z(x_0 - 0)$ both exist and are finite since $z(x)$ is bounded on each compact interval of J. Furthermore, $z(x_0 + 0) \neq z(x_0 - 0)$ since, by Corollary 4.2,

$$\mathrm{Min}[z(x_0 + 0), z(x_0 - 0)] \leqslant z(x_0) \leqslant \mathrm{Max}[z(x_0 + 0), z(x_0 - 0)].$$

Assume that $z(x_0 + 0) > z(x_0 - 0)$ and that $Dz(x_0 +) \neq + \infty$. Then there is a $\delta > 0$ and an N such that $[x_0, x_0 + \delta] \subset J$ and $z(x) < w(x) = z(x_0 + 0) + N(x - x_0)$ on $x_0 < x \leqslant x_0 + \delta$. By Theorem 2.1 there is a δ_1, $0 < \delta_1 \leqslant \delta$, such that the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(x_0) = z(x_0 + 0), \qquad y(x_0 + \delta_1) = w(x_0 + \delta_1)$$

has a solution $y_1(x) \in C^{(2)}[x_0, x_0 + \delta_1]$. Since

$$y_1(x_0) = z(x_0 + 0) \geqslant z(x_0), \qquad y_1(x_0 + \delta_1) = w(x_0 + \delta_1) > z(x_0 + \delta_1)$$

and $z(x)$ is a subfunction, $y_1(x) \geqslant z(x)$ on $[x_0, x_0 + \delta_1]$. Now proceeding as in the paragraph above and using the fact that $z(x)$ is a subfunction, we can obtain a solution of the initial-value problem

$$y'' = f(x, y, y'), \qquad y(x_0) = y_1(x_0), \qquad y'(x_0) = y_1'(x_0),$$

the graph of which is not contained in the sector to the left of x_0 in which such solutions must be. From this contradiction we conclude that $Dz(x_0 +) = + \infty$. The other assertions about derivatives at points of discontinuity of $z(x)$ are dealt with in a similar way.

It should be noted that Theorem 4.10 is a combination of results concerning subfunctions and superfunctions. Let $\varphi(x)$ be a subfunction on $J \subset I$ and assume that $\varphi(x)$ is bounded on each compact subinterval of J. Then, if $\varphi(x)$ is continuous at $x_0 \in J^0$ and $D\varphi(x_0 -) = + \infty$, it follows that $D\varphi(x_0 +) = + \infty$. If $\varphi(x)$ is continuous at $x_0 \in J^0$ and $D\varphi(x_0 +) = - \infty$, then $D\varphi(x_0 -) = - \infty$. If $\varphi(x_0 + 0) > \varphi(x_0 - 0)$, $D\varphi(x_0 +) = + \infty$. If $\varphi(x_0 + 0) < \varphi(x_0 - 0)$, $D\varphi(x_0 -) = - \infty$. If $D\varphi(x_0 +)$ and $D\varphi(x_0 -)$ are both finite, $\varphi(x)$ is continuous at x_0. Similarly, let $\psi(x)$ be a superfunction on $J \subset I$ which is bounded on each compact subinterval of J. Then, if $\psi(x)$ is continuous at $x_0 \in J^0$ and $D\psi(x_0 +) = + \infty$, it follows that $D\psi(x_0 -) = + \infty$. If $\psi(x)$ is continuous at x_0 and $D\psi(x_0 -) = - \infty$, then $D\psi(x_0 +) = - \infty$. If $\psi(x_0 + 0) > \psi(x_0 - 0)$, $D\psi(x_0 -) = + \infty$. If $\psi(x_0 + 0) < \psi(x_0 - 0)$, $D\psi(x_0 +) = - \infty$. If $D\psi(x_0 +)$ and $D\psi(x_0 -)$ are both finite, $\psi(x)$ is continuous at x_0.

Now we begin the consideration of boundary value problems by the Perron method. We assume now that $f(x, y, y')$ is continuous on $[a, b] \times R^2$.

Definition 4.11. A bounded real-valued function φ defined on $[a, b]$ is said to be an underfunction with respect to the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(b) = B$$

in case $\varphi(a) \leqslant A$, $\varphi(b) \leqslant B$, and φ is a subfunction on $[a, b]$ with respect to solutions of $y'' = f(x, y, y')$. The bounded function $\psi(x)$ defined on $[a, b]$ is said to be an overfunction with respect to the boundary-value problem in case $\psi(a) \geqslant A$, $\psi(b) \geqslant B$, and ψ is a superfunction on $[a, b]$ with respect to solutions of $y'' = f(x, y, y')$.

Theorem 4.12. *Assume that $C^{(2)}$ solutions of boundary value problems for $y'' = f(x, y, y')$ on subintervals of $[a, b]$ are unique in the sense specified in the statement of Theorem 4.8. Assume that there exist both an overfunction ψ_0 and an underfunction φ_0 with respect to the boundary-value problem*

$$(*) \qquad y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(b) = B$$

and that $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$. Let Φ be the collection of all underfunctions φ such that $\varphi(x) \leqslant \psi_0(x)$ on $[a, b]$. Then $z(x) = \sup_{\varphi \in \Phi} \varphi(x)$ is simultaneously a subfunction and a superfunction on $[a, b]$.

Proof. It follows from Theorem 4.6 that z is a subfunction on $[a, b]$. Now assume that $z(x)$ is not a superfunction on $[a, b]$. Then there is a subinterval $[x_1, x_2] \subset [a, b]$ and a solution $y(x) \in C^{(2)}[x_1, x_2]$ such that $y(x_1) \leqslant z(x_1)$, $y(x_2) \leqslant z(x_2)$, but $y(x) > z(x)$ at some points of (x_1, x_2). Define $z_1(x)$ on $[a, b]$ by

$$z_1(x) = \begin{cases} \operatorname{Max}[y(x), z(x)] & \text{on} \quad [x_1, x_2] \\ z(x) & \text{on} \quad [a, b] - [x_1, x_2]. \end{cases}$$

Then, by Theorem 4.7, $z_1(x)$ is a subfunction on $[a, b]$ also $z_1(a) = z(a) \leqslant A$, $z_1(b) = z(b) \leqslant B$. Furthermore,

$$y(x_1) \leqslant z(x_1) \leqslant \psi_0(x_1), \qquad y(x_2) \leqslant z(x_2) \leqslant \psi_0(x_2),$$

and ψ_0 a superfunction implies $y(x) \leqslant \psi_0(x)$ on $[x_1, x_2]$. Consequently,

$z_1(x) \leqslant \psi_0(x)$ on $[a, b]$. Hence, $z_1 \in \Phi$ and $z_1(x) \leqslant z(x)$ but $z_1(x) = y(x) > z(x)$ at some points in (x_1, x_2). From this contradiction we conclude that $z(x)$ is a superfunction on $[a, b]$.

Definition 4.13. The function $z(x)$ obtained in Theorem 4.12 depends on the boundary-value problem (*) and on the overfunction $\psi_0(x)$. It will be designated by $z(x; \psi_0)$ and will be called a generalized solution of the boundary-value problem (*).

Since $z(x; \psi_0)$ is simultaneously a subfunction and a superfunction on $[a, b]$, the assertions made in Theorem 4.10 apply to $z(x; \psi_0)$. We now consider the behavior of $z(x; \psi_0)$ at the end points on $[a, b]$.

Theorem 4.14. *Assume that the hypotheses of Theorem 4.12 are satisfied and let $z(x; \psi_0) = z(x)$ be the corresponding generalized solution of (*). Then $z(a) = A$. If $Dz(a+) \neq +\infty$, $z(a+0) \leqslant z(a)$. If $z(a+0) < A$, $Dz(a+) = -\infty$. Hence, if $Dz(a+)$ is finite, $z(a+0) = z(a) = A$. Similar statements apply at $x = b$.*

Proof. It is clear that, if $\varphi(x)$ is a subfunction on $[a. b]$, then $\varphi_c(x)$ defined by $\varphi_c(x) = \varphi(x)$ on $(a, b]$ and $\varphi_c(a) = \varphi(a) + c$, $c > 0$, is a subfunction on $[a, b]$. From this observation and the definition of $z(x)$ it is clear that $z(a) = A$.

Now assume that $Dz(a+) \neq +\infty$ and $z(a+0) > z(a)$. Then there is a δ, $0 < \delta < b - a$, and an N such that

$$z(x) < w(x) = z(a+0) + N(x-a) \quad \text{on} \quad a < x \leqslant a + \delta.$$

It follows from Theorem 2.1 that for $0 < \epsilon < z(a+0) - z(a)$ and $0 < \delta_1 \leqslant \delta$ sufficiently small, the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(a) = z(a+0) - \epsilon, \qquad y(a+\delta_1) = w(a+\delta_1)$$

has a solution $y(x) \in C^{(2)}[a, a+\delta_1]$. Since $z(x)$ is a subfunction, $z(x) \leqslant y(x)$ on $[a, a+\delta_1]$ which implies

$$z(a+0) \leqslant y(a+0) = y(a) = z(a+0) - \epsilon.$$

From this contradiction it follows that $Dz(a+) \neq +\infty$ implies $z(a+0) \leqslant z(a)$.

Finally, assume $z(a+0) < A$ and $Dz(a+) \neq -\infty$. Then using the same type of argument as above and the fact that $z(x)$ is also a superfunction on $[a, b]$ we obtain a contradiction. Hence, $z(a+0) < A$ implies $Dz(a+) = -\infty$.

If $Dz(a+)$ is finite, then combining the assertions of the Theorem we have $A = z(a) \geqslant z(a+0) \geqslant A$.

From the preceding results we see that the Perron method of studying the boundary value problem (*) can be separated into two parts. The first part of the problem is to establish the existence of an overfunction ψ_0 and an underfunction φ_0 such that $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$. The second part is to establish conditions under which the generalized solution $z(x; \psi_0)$ is of class $C^{(2)}[a, b]$ and is a solution on $[a, b]$. In view of Theorems 4.10, 4.12, and 4.14, to accomplish this it suffices to show that $Dz(x+)$ is finite on $[a, b)$ and $Dz(x-)$ is finite on $(a, b]$.

Lemma 4.15. *Let $f(x, y, y')$ be nondecreasing in y on $[a, b] \times R^2$ for fixed x, y'. Assume that $f(x, y, y')$ is such that lower and upper solutions of the differential equation are subfunctions and superfunctions, respectively. Then, if $u(x) \in C^{(2)}[a, b]$ is a solution of $y'' = f(x, y, y')$ on $[a, b]$, there exist overfunctions and underfunctions with respect to any boundary-value problem on $[a, b]$.*

Proof. For a sufficiently large $M > 0$ $\psi_0(x) = u(x) + M$ is an overfunction, $\varphi_0(x) = u(x) - M$ is an underfunction, and $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$.

Lemma 4.16. *Assume that $f(x, y, y')$ is nondecreasing in y on $[a, b] \times R^2$ for fixed x, y' and is such that lower and upper solutions of the differential equation are subfunctions and superfunctions. Further assume that there is a $k > 0$ such that $|f(x, 0, y') - f(x, 0, 0)| \leqslant k |y'|$ on $a \leqslant x \leqslant b$ for all y'. Then again there exist overfunctions and underfunctions with respect to every boundary-value problem on $[a, b]$* ([23], Theorem 6.1).

Proof. Let $M = \operatorname{Max} |f(x, 0, 0)|$ on $[a, b]$ and let $w(x)$ be the solution of the boundary-value problem

$$w'' = -kw' - M, \qquad w(a) = 0, \qquad w'(b) = 0.$$

Then $w(x) \geqslant 0$ and $w'(x) \geqslant 0$ on $[a, b]$; hence,

$$w'' = -kw' - M = -k|w'| - M \leqslant f(x, 0, w') - f(x, 0, 0) - M.$$

Thus $w'' \leqslant f(x, 0, w') \leqslant f(x, w, w')$ on $[a, b]$ and we conclude that, for a given boundary-value problem, $\psi_0(x) = w(x) + M$ will be an overfunction provided $M > 0$ is sufficiently large.

Similarly, let $v(x)$ be the solution of the boundary-value problem

$$v'' = -kv' + M, \qquad v(a) = 0, \qquad v'(b) = 0.$$

Then $v(x) \leqslant 0$ and $v'(x) \leqslant 0$ on $[a, b]$, from which it follows that

$$v'' = k \mid v' \mid + M \geqslant f(x, 0, v') - f(x, 0, 0) + M \geqslant f(x, 0, v')$$

on $[a, b]$. Then $v'' \geqslant f(x, v, v')$ on $[a, b]$ and, for $M > 0$ sufficiently large, $\varphi_0(x) = v(x) - M$ is an underfunction with respect to a given boundary-value problem. Obviously for such functions we will have $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$.

Theorem 4.17. *Assume that $f(x, y, y')$ is nondecreasing in y on $[a, b] \times R^2$ for fixed x, y' and assume either that $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on each compact subset of $[a, b] \times R^2$ or that solutions of initial-value problems are unique. In addition assume that there is a $k > 0$ such that $|f(x, 0, y') - f(x, 0, 0)| \leqslant k \mid y' \mid$ on $[a, b]$ for all y'. Then for any boundary-value problem on $[a, b]$ with an associated overfunction $\psi_0(x)$ the generalized solution $z(x) = z(x; \psi_0)$ belongs to $C^{(2)}(a, b)$ and $z'' = f(x, z, z')$ on (a, b).* ([23], Corollary 6.1).

Proof. First it follows from Lemma 4.16 that, with respect to a given boundary-value problem on $[a, b]$, there is an overfunction $\psi_0(x)$ and an underfunction $\varphi_0(x)$ with $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$. Consequently, the generalized solution $z(x) = z(x; \psi_0)$ is defined. Furthermore, the hypotheses imply that solutions of boundary-value problems when they exist are unique, hence, the conclusions of Theorem 4.10 apply to $z(x)$. Thus it suffices to show that $Dz(x_0 +)$ and $Dz(x_0 -)$ are finite at every point of (a, b). Let $x_0 \in (a, b)$ and assume that $z(x_0 + 0) \geqslant z(x_0 - 0)$. The alternative case can be dealt with in a similar way and will not be discussed.

We break the discussion up into two cases. First assume that $z(x_0 + 0) \geqslant 0$. Let $\psi(x)$ be a solution of $\psi'' = -k\psi' - M$, $M = \text{Max} \mid f(x, 0, 0) \mid$ on $[a, b]$, with $\psi(x_0) = 0$, $\psi'(x) \geqslant 0$ on $[x_0, b]$, and $\psi(b) \geqslant z(b - 0)$. Such a solution can be determined by elementary calculations. Then, as in Lemma 4.16, $\psi_1(x) = \psi(x) + z(x_0 + 0)$ is an upper solution on $[x_0, b]$ with $z(x_0 + 0) \leqslant \psi_1(x_0)$ and $z(b - 0) \leqslant \psi_1(b)$. It follows from Theorem 4.8 that $z(x) \leqslant \psi_1(x)$ on (x_0, b), which implies that $Dz(x_0 +) \leqslant \psi_1'(x_0) < +\infty$. Applying Theorem 4.10 we conclude that $z(x)$ is continuous at x_0. Now let $\psi(x)$ be a solution of $\psi'' = k\psi' - M$ on $[a, x_0]$ such that $\psi(x_0) = 0$, $\psi(a) \geqslant z(a + 0)$, and $\psi'(x) \leqslant 0$ on $[a, x_0]$. Then, again by Theorem 4.8, $z(x) \leqslant \psi(x) + z(x_0)$, which implies $Dz(x_0 -) \geqslant \psi'(x_0) > -\infty$. We conclude that in this case $z(x)$ has a finite derivative at x_0.

Finally assume $z(x_0 + 0) < 0$ and let $\varphi_1(x)$ be a solution of $\varphi'' = k\varphi' + M$ on $[a, x_0]$ such that $\varphi(a) \leqslant z(a + 0)$, $\varphi(x_0) = 0$, and $\varphi'(x) \geqslant 0$ on $[a, x_0]$. Then, by Theorem 4.8, $z(x) \geqslant \varphi_1(x) - z(x_0 - 0)$ on (a, x_0), which implies $Dz(x_0 -) \leqslant \varphi_1'(x_0) < +\infty$. It follows from Theorem 4.10 that $z(x)$ is continuous at x_0. In a similar way we can show that $Dz(x_0 +) > -\infty$. Thus we again conclude that $z(x)$ has a finite derivative at x_0.

Since $z(x)$ has a finite derivative at each point of (a, b), it follows from Theorem 4.10 that $z(x) \in C^{(2)}(a, b)$ and is a solution of the differential equation on (a, b).

Theorem 4.18. *Assume that $f(x, y, y')$ satisfies the hypotheses of Theorem 4.17 on $[a, b] \times R^2$. Then the boundary-value problem*

$$y'' = f(x, y, y'), \qquad y(a) = y(b) = 0$$

has a unique solution $y(x) \in C^{(2)}[a, b]$. Furthermore, on $[a, b]$

$$|\, y(x) \,| \leqslant \frac{M}{k^2} [e^{k(b-a)} - e^{1/2k(b-a)} - \tfrac{1}{2} k(b - a)]$$

and

$$|\, y'(x) \,| \leqslant \frac{M}{k} [e^{k(b-a)} - 1],$$

where M and k are as in Theorem 4.17 ([23], Theorem 6.3).

Proof. Let $\psi_1(x)$ be the solution of the boundary-value problem $\psi'' = -k\psi' - M$, $\psi(a) = 0$, $\psi'(b) = 0$. Then $\psi_1'(x) \geqslant 0$ on $[a, b]$ and $\psi_1(x)$ is an overfunction with respect to the boundary-value problem. Similarly, the solution $\psi_2(x)$ of the boundary-value problem $\psi'' = k\psi' - M$, $\psi(b) = 0$, $\psi'(a) = 0$ satisfies $\psi_2'(x) \leqslant 0$ on $[a, b]$ and is also an overfunction. Consequently, since $\psi_1(b) > 0$ and $\psi_2(a) > 0$, $\psi_0(x) = \mathrm{Min}[\psi_1(x), \psi_2(x)]$ is also an overfunction.

The solution $\varphi_1(x)$ of the boundary-value problem $\varphi'' = -k\varphi' + M$, $\varphi(a) = 0$, $\varphi'(b) = 0$ and the solution $\varphi_2(x)$ of the boundary-value problem

$\varphi'' = k\varphi' + M$, $\varphi(b) = 0$, $\varphi'(a) = 0$ satisfy $\varphi_1'(x) \leqslant 0$, $\varphi_2'(x) \geqslant 0$, and are both underfunctions. Therefore, $\varphi_0(x) = \mathrm{Max}[\varphi_1(x), \varphi_2(x)]$ is an underfunction. Also $\varphi_0(x) \leqslant \psi_0(x)$. It follows from Theorem 4.17 that $z(x; \psi_0)$ is a solution on (a, b). Now $\varphi_0(x) \leqslant z(x; \psi_0) \leqslant \psi_0(x)$ on $[a, b]$, $\varphi_0(a) = \psi_0(a) = \varphi_0(b) = \psi_0(b) = 0$, and $\varphi_0'(a)$, $\varphi_0'(b)$, $\psi_0'(a)$, $\psi_0'(b)$ are finite. It follows that $Dz(a+)$ and $Dz(b-)$ are finite which by Theorem 4.14 implies $z(x; \psi_0) \in C^{(2)}[a, b]$ and is a solution of the boundary-value problem. The fact that the solution is unique is a consequence of either Corollary 3.7 or Corollary 3.11.

The functions $\varphi_0(x)$ and $\psi_0(x)$ can be computed and it can be shown that

$$\max_{a\leqslant x\leqslant b} \psi_0(x) = -\min_{a\leqslant x\leqslant b} \varphi_0(x) = \frac{M}{k^2}\left[e^{k(b-a)} - e^{1/2k(b-a)} - \tfrac{1}{2}k(b-a)\right].$$

This establishes the desired bound on $|y(x)|$ for the solution $y(x)$. If $x_0 \in (a, b)$ and $y(x_0) \geqslant 0$, let $\psi_1(x; x_0)$ and $\psi_2(x; x_0)$ be the respective solutions of the boundary-value problems

$$\psi'' = -k\psi' - M, \qquad \psi(x_0) = 0, \qquad \psi'(b) = 0,$$

and

$$\psi'' = k\psi' - M, \qquad \psi(x_0) = 0, \qquad \psi'(a) = 0.$$

Then $y(x) \leqslant \psi_1(x; x_0) + y(x_0)$ on $[x_0, b]$ and $y(x) \leqslant \psi_2(x; x_0) + y(x_0)$ on $[a, x_0]$. It follows that $\psi_2'(x_0; x_0) \leqslant y'(x_0) \leqslant \psi_1'(x_0; x_0)$. Similarly, if $y(x_0) < 0$ and $\varphi_1(x; x_0)$, $\varphi_2(x; x_0)$ are the respective solutions of the boundary-value problems

$$\varphi'' = -k\varphi' + M, \qquad \varphi(x_0) = 0, \qquad \varphi'(b) = 0$$

and

$$\varphi'' = k\varphi' + M, \qquad \varphi(x_0) = 0, \qquad \varphi'(a) = 0,$$

then $y(x) \geqslant \varphi_1(x; x_0) + y(x_0)$ on $[x_0, b]$, $y(x) \geqslant \varphi_2(x; x_0) + y(x_0)$ on $[a, x_0]$, and $\varphi_1'(x; x) \leqslant y'(x_0) \leqslant \varphi_2'(x_0; x_0)$. These functions and derivatives can be computed and we obtain

$$|y'(x)| \leqslant \frac{M}{k}\left(e^{k(b-a)} - 1\right) \qquad \text{on} \qquad [a, b].$$

Corollary 4.19. *Assume $f(x, y, y')$ is nondecreasing in y on $[a, b] \times R^2$ for fixed x, y' and satisfies a uniform Lipschitz condition with* *respect to y' on $[a, b] \times R^2$. Then for any A, B the boundary-value problem*

$$(*) \qquad y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(b) = B$$

has a unique solution $y(x) \in C^{(2)}[a, b]$. ([23], Corollary 6.4; [25]).

Proof. Let $w(x)$ be the linear function with $w(a) = A$, $w(b) = B$. Then the function $g(x, y, y') = f(x, y + w(x), y' + w'(x))$ satisfies the hypotheses of Theorem 4.18, hence, the boundary-value problem

$$y'' = g(x, y, y'), \qquad y(a) = y(b) = 0$$

has a solution $\omega(x)$. Then $y(x) = w(x) + \omega(x)$ is a solution of (*). That the solution is unique follows from Corollary 3.7.

Corollary 4.20. *If $f(x, y)$ is continuous on $[a, b] \times R$ and is non-decreasing in y for fixed x, then for any A, B the boundary-value problem*

$$y'' = f(x, y), \qquad y(a) = A, \qquad y(b) = B$$

has a unique solution $y \in C^{(2)}[a, b]$.

We will now consider an application of Theorem 4.18 to obtain a disconjugacy condition for a linear third order differential equation. The equation

$$y''' + p_0(x)\, y'' + p_1(x)\, y' + p_2(x)\, y = 0 \tag{4.1}$$

with $p_j(x) \in C[a, b]$ is said to be disconjugate on $[a, b]$ in case no non-trivial solution has more than two zeros on $[a, b]$ counting multiplicities of zeros.

A number of disconjugacy conditions have been given which relate bounds on the coefficients in Eq. (4.1) to the interval length $b - a$. Nehari [26] has shown that, if $h = b - a$ and

$$\frac{1}{2}\int_a^b |p_0| + \frac{h}{4}\int_a^b |p_1| + \frac{h^2}{8}\int_a^b |p_2| \leqslant 1,$$

then Eq. (4.1) is disconjugate on $[a, b]$. In 1963 Lasota [27] proved that (4.1) is disconjugate on $[a, b]$ if

$$\frac{1}{4} P_0 h + \frac{1}{\pi^2} P_1 h^2 + \frac{1}{2\pi^2} P_2 h^3 \leqslant 1,$$

where $P_j = \text{Max} \mid p_j(x) \mid$ on $[a, b]$. If $p_1(x) \leqslant 0$ on $[a, b]$, Lasota's condition reduces to

$$\frac{1}{4} P_0 h + \frac{1}{2\pi^2} P_2 h^3 \leqslant 1.$$

The result obtained by the use of Theorem 4.18 will be based on the assumption $p_1(x) \leqslant 0$ on $[a, b]$ and will involve P_0, P_2, and h. We first establish an existence theorem for certain types of boundary value problems for nonlinear third-order equations.

Theorem 4.21. *Assume that $f(x, y, y', y'')$ is continuous on $[a, b] \times R^3$ and satisfies the following conditions:* (i) *f is nondecreasing in y' for each fixed x, y, y'',* (ii) *f satisfies a Lipschitz condition with respect to y'' on each compact subset of $[a, b] \times R^3$, and* (iii) *for each $M > 0$ there is a $k_M > 0$ such that $| f(x, y, 0, y'') - f(x, y, 0, 0) | \leqslant k_M | y'' |$ for all $a \leqslant x \leqslant b$, $| y | \leqslant M$, and all y''. Then there is a $\delta > 0$ such that for any $[x_1, x_2] \subset [a, b]$ with $x_2 - x_1 \leqslant \delta$ there is a solution $y_1(x)$ of $y''' = f(x, y, y', y'')$ on $[x_1, x_2]$ with $y_1(x_1) = y_1'(x_1) = y_1'(x_2) = 0$ and a solution $y_2(x)$ on $[x_1, x_2]$ with $y_2'(x_1) = y_2'(x_2) = y_2(x_2) = 0$.* ([28], Theorem 1).

Proof. Let $M > 0$ be fixed and let

$$B[x_1, x_2] \equiv \{z(x) \in C[x_1, x_2] : \| z \| = \text{Max} \mid z(x) \mid \leqslant M\},$$

where $[x_1, x_2] \subset [a, b]$.

It follows from Theorem 4.18 that for each $z \in B[x_1, x_2]$ the boundary-value problem

$$y'' = f(x, z(x), y, y'), \qquad y(x_1) = y(x_2) = 0$$

has a unique solution $u_z(x) \in C^{(2)}[x_1, x_2]$. Let

$$q_M = \text{Max} \mid f(x, z, 0, 0) \mid \quad \text{for} \quad a \leqslant x \leqslant b, \mid z \mid \leqslant M$$

and

$$E(t) = [e^{k_M t} - e^{1/2 k_M t} - \tfrac{1}{2} k_M t].$$

Then it also follows from Theorem 4.18 that

$$\mid u_z(x) \mid \leqslant \frac{q_M}{{k_M}^2} E(x_2 - x_1)$$

on $[x_1, x_2]$ for each $z \in B[x_1, x_2]$. Now define the mapping $T: B[x_1, x_2] \to C[x_1, x_2]$ by $Tz = w$, where $w(x) = \int_{x_1}^{x} u_z(t)\, dt$. Then $\| w \| \leqslant (q_M/k_M^2)(x_2 - x_1) E(x_2 - x_1)$, consequently, $T(B) \subset B$ if $(q_M/k_M^2)(x_2 - x_1) E(x_2 - x_1) \leqslant M$. $B[x_1, x_2]$ is a closed convex subset of the Banach space $C[x_1, x_2]$ and it is not difficult to show that T is continuous and completely continuous on $B[x_1, x_2]$. Since $E(t)$ is increasing in t, it follows from the Schauder Fixed-Point Theorem that T has a fixed point in $B[x_1, x_2]$ if $x_2 - x_1 \leqslant \delta$ where $(q_M/k_M^2)\, \delta E(\delta) = M$. If $y(x)$ is the fixed point, $y''' = f(x, y, y', y'')$ on $[x_1, x_2]$ and $y(x_1) = y'(x_1) = y'(x_2) = 0$.

If the mapping T is defined by $Tz = w$ with $w(x) = \int_{x_2}^{x} u_z(t)\, dt$, the same estimates hold and in this case for $x_2 - x_1 \leqslant \delta$ with $(q_M/k_M^2)\, \delta E(\delta) = M$ there is a solution $y(x)$ on $[x_1, x_2]$ with $y(x_2) = y'(x_2) = y'(x_1) = 0$.

Theorem 4.22. *Let $h = b - a$. Then, if $P_0 \neq 0$ and*

$$hP_2 E(P_0 h) \leqslant P_0^2,$$

Eq. (4.1) is disconjugate on $[a, b]$ ([28], Theorem 2).

Proof. Consider the equation

$$L[y] = y''' + p_0(x)\, y'' + p_1(x)\, y' + p_2(x)\, y = g(x),$$

where $p_j(x)$, $g(x) \in C[a, b]$ and $p_1(x) \leqslant 0$ on $[a, b]$. It is clear that the hypotheses of Theorem 4.21 are satisfied. In this case

$$q_M \leqslant \| g \| + \| p_2 \| M = \| g \| + P_2 M$$

and

$$k_M = \| p_0 \| = P_0 .$$

Thus given $M > 0$ and $[x_1, x_2] \subset [a, b]$, the boundary-value problems

$$L[y] = g(x), \qquad y(x_1) = y'(x_1) = y'(x_2) = 0 \tag{4.2}$$

and

$$L[y] = g(x), \qquad y'(x_1) = y(x_2) = y'(x_2) = 0 \tag{4.3}$$

have solutions if

$$\frac{(x_2 - x_1)\,[\| g \| + P_2 M]\, E[P_0(x_2 - x_1)]}{P_0^2} \leqslant M.$$

This will be the case if

$$\frac{(x_2 - x_1)\, P_2 E[P_0(x_2 - x_1)]}{P_0^2} \leqslant 1 - \frac{(x_2 - x_1)\, \| g \|\, E[P_0(x_2 - x_1)]}{P_0^2 M}. \tag{4.4}$$

Let $M > 0$ be chosen and fixed. It is known ([28], p. 630) that $L[y] = 0$ is disconjugate on $[a, b]$ if and only if for any $x_0 \in [a, b]$ the solution $u(x; x_0)$ of the initial-value problem $L[y] = 0$, $y(x_0) = y'(x_0) = 0$, $y''(x_0) = 1$ satisfies $u(x; x_0) > 0$ on $[a, b]$ for $x \neq x_0$. Assume $L[y] = 0$ is not disconjugate on $[a, b]$. Then there is an $x_1 \in [a, b]$ such that $u(x; x_1)$ has another zero in $[a, b]$. It follows that there is an $x_2 \in (a, b)$ at which $u'(x_2; x_1) = 0$, to be specific assume $x_1 < x_2$. Then $x_2 - x_1 < h$ and it follows that

$$\frac{(x_2 - x_1)\, P_2 E[P_0(x_2 - x_1)]}{P_0^2} < \frac{h P_2 E[P_0 h]}{P_0^2} \leqslant 1.$$

Let

$$g(x) = \epsilon[6 + 6p_0(x)\,(x - x_1) + 3p_1(x)\,(x - x_1)^2 + p_2(x)\,(x - x_1)^3].$$

Then for $\epsilon > 0$ sufficiently small, inequality (4.4) will be satisfied; consequently, problem (4.2) has a solution. It is easy to see that this solution must be of the form $y(x) = cu(x; x_1) + \epsilon(x - x_1)^3$ where c is a constant. Then $y'(x_2) = cu'(x_2; x_1) + 3\epsilon(x_2 - x_1)^2 = 3\epsilon(x_2 - x_1)^2 \neq 0$, which contradicts $y(x)$'s being a solution of (4.2). If $x_2 < x_1$, we contradict (4.3)'s having a solution. Thus the assumption $u'(x_2; x_1) = 0$ for $x_2 \neq x_1$ leads to a contradiction and we conclude that $u(x; x_0) > 0$ for all x, $x_0 \in [a, b]$ with $x \neq x_0$. Thus $L[y] = 0$ is disconjugate on $[a, b]$.

The inequality of Theorem 4.22 is an improvement over that of Lasota for the $p_1(x) \leqslant 0$ case in that P_0 and h can be large provided P_2 is sufficiently small. By examining the proof of Theorem 4.22 and noting that

$$\lim_{P_0 \to 0} \frac{E(P_0 h)}{P_0^2} = \frac{3}{8} h^2,$$

we see that when $P_0 = 0$ $L[y] = 0$ is disconjugate on $[a, b]$ provided $\frac{3}{8} h^3 P_2 \leqslant 1$. In this case the result is inferior to that of Lasota.

5. Boundary-Value Problems on Infinite Intervals

The subfunction technique can also be used to advantage in dealing with certain types of boundary-value problems on infinite intervals.

In this Section we will consider some illustrations of problems of this type.

Theorem 5.1. *Let f be continuous on $[a, \infty) \times R$, be nondecreasing in y for fixed x, and satisfy $f(x, 0) \equiv 0$ on $[a, \infty)$. Then for any real A the boundary-value problem*

$$y'' = f(x, y), \qquad y(a) = A \tag{5.1}$$

has a unique bounded solution $y(x) \in C^{(2)}[a, \infty)$ ([29], Theorem 3.1).

Proof. We consider only the case $A \geqslant 0$ and, in particular, the case $A > 0$ since $y(x) \equiv 0$ is a bounded solution when $A = 0$. The case $A < 0$ can be dealt with in a similar way.

If $A > 0$, it follows from Corollary 3.6 that $\psi_0(x) \equiv A$ is a superfunction on $[a, \infty)$. Let Φ be the collection of all subfunctions $\varphi(x)$ on $[a, \infty)$ such that $\varphi(x) \leqslant \psi_0(x)$ on $[a, \infty)$. Then Φ is not empty since $\varphi(x) \equiv 0$ belongs to Φ. Let $z(x) = \sup_{\varphi \in \Phi} \varphi(x)$. Then $0 \leqslant z(x) \leqslant \psi_0(x)$ on $[a, \infty)$ and it follows from Theorem 4.17 that $z(x) \in C^{(2)}(a, \infty)$ and is a solution of the differential equation on (a, ∞). By Corollary 4.20 the boundary-value problem

$$y'' = f(x, y), \qquad y(a) = A, \qquad y(a + 1) = 0$$

has a solution $y_1(x)$. By Theorem 4.7 $\varphi_0(x)$ defined by $\varphi_0(x) = y_1(x)$ on $[a, a + 1]$, $\varphi_0(x) = 0$ on $(a + 1, \infty)$ is a subfunction on $[a, \infty)$. Also, since $\psi_0(x) \equiv A$ is a superfunction, $y_1(x) \leqslant \psi_0(x)$ on $[a, a + 1]$. It follows that $\varphi_0(x) \in \Phi$ and $y_1(x) \leqslant z(x) \leqslant \psi_0(x)$ on $[a, a + 1]$. Then by Theorem 4.14, $z(x) \in C^{(2)}[a, \infty)$ and is a solution of the boundary-value problem (5.1).

Now assume that problem (5.1) has two distinct bounded solutions $z_1(x)$, $z_2(x) \in C^{(2)}[a, \infty)$. Then, since by Corollary 3.7 solutions of boundary-value problems on finite intervals are unique, there is an $x_0 \geqslant a$ such that $z_1(x) \equiv z_2(x)$ on $[a, x_0]$ and $z_1(x) \neq z_2(x)$ for all $x > x_0$. To be specific, assume $z_1(x) > z_2(x)$ on (x_0, ∞). Then $z_1''(x) - z_2''(x) = f(x, z_1(x)) - f(x, z_2(x)) \geqslant 0$ on $[x_0, \infty)$, $z_1(x_0) - z_2(x_0) = 0$ and $z_1(x) - z_2(x) > 0$ for $x > x_0$. This obviously implies $z_1(x) - z_2(x)$ is unbounded on $[x_0, \infty)$. From this contradiction we conclude the uniqueness of bounded solutions of (5.1).

The bounded solution $z(x)$ of (5.1) with $A > 0$ satisfies $z'(x) \leqslant 0$ on $[a, \infty)$. Other questions of interest arise concerning the solution $z(x)$.

Is $z(x) > 0$ on $[a, \infty)$? This of course will be the case if initial-value problems for $y'' = f(x, y)$ have unique solutions. Another condition on $f(x, y)$ which implies $z(x) > 0$ is given in [29]. If $z(x) > 0$, is $\lim_{x\to\infty} z(x) > 0$? Does (5.1) also have unbounded solutions? These questions have been considered by a number of authors. Subfunction methods can also be applied to the study of these questions.

The following Theorem with slightly different hypotheses has been proven by other methods by Schuur [30].

Theorem 5.2. *Let $f(x, y, y')$ be continuous on $[a, \infty) \times R^2$, be nondecreasing in y for fixed x, y', be nondecreasing in y' for fixed x, y, and satisfy $f(x, ,0, 0) \equiv 0$ on $[a, \infty)$. Assume either that $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on each compact subset of $[a, \infty) \times R^2$ or that solutions of initial-value problems for $y'' = f(x, y, y')$ are unique. Then for any real A the boundary-value problem*

$$y'' = f(x, y, y'), \qquad y(a) = A \tag{5.2}$$

has a unique bounded solution on $[a, \infty)$.

Proof. As in Theorem 5.1 it suffices to consider only the case $A > 0$. Since $f(x, 0, 0) \equiv 0, f(x, A, 0) \geqslant 0$ and it follows from either Corollary 3.6 or Corollary 3.10 that $\psi_0(x) \equiv A$ is a superfunction on $[a, \infty)$. Again let $z(x) = \sup_{\varphi\in\Phi} \varphi(x)$, where Φ is the collection of all subfunctions $\varphi(x)$ on $[a, \infty)$ such that $\varphi(x) \leqslant \psi_0(x)$ on $[a, \infty)$. Φ is not empty since it contains $\varphi(x) \equiv 0$. Then $0 \leqslant z(x) \leqslant \psi_0(x) \equiv A$ on $[a, \infty)$ and it follows from Theorem 4.12 and either Corollary 3.7 or Corollary 3.11 that $z(x)$ is simultaneously a subfunction and a superfunction on $[a, \infty)$.

It follows from Theorem 4.10 that $z(x)$ has a finite derivative almost everywhere. Let $x_0 > a$ be a point at which $z(x)$ has a finite derivative. Then by Theorem 4.10 there is an open interval around x_0 on which $z(x)$ is a solution. Let (c, d) be a maximal such interval. Now suppose that $z'(x_1) > 0$ for some $x_1 \in (c, d)$ and that $z'(x) < 0$ at some points of (x_1, d). Then let $x_2 \in (x_1, d)$ be such that $z'(x_2) = 0$ and $z'(x) > 0$ on $[x_1, x_2)$. It follows that

$$0 > -z'(x_1) = z'(x_2) - z'(x_1) = \int_{x_1}^{x_2} f(s, z(s), z'(s))\, ds \geqslant 0.$$

From this contradiction it follows that, if $z'(x_1) > 0$ for $x_1 \in (c, d)$, then $z'(x) > 0$ on $[x_1, d)$. Now assume $c > a$; then, since (c, d) is maximal, it follows from Theorem 4.10 that $z(x)$ cannot have a finite derivative at c and that $Dz(c+) = \pm\infty$. Assume first that $Dz(c+) = +\infty$. Then it follows from the mean-value theorem that $z'(x) > 0$ at points arbitrarily close to c, hence, by the above observation $z'(x) > 0$ on (c, d). However, in this case $z''(x) = f(x, z(x), z'(x)) \geqslant 0$ on (c, d), which is incompatible with $Dz(c+) = +\infty$. Now assume $Dz(c+) = -\infty$. Then by applying the mean-value theorem and the above observation again we conclude that there is a $\delta > 0$ such that $z'(x) < 0$ on $(c, c+\delta]$. Then $z''(x) = f(x, z(x), z'(x)) \leqslant f(x, A, 0)$ on $(c, c+\delta]$ but $Dz(c+) = -\infty$ implies that $z''(x)$ is not bounded above on $(c, c+\delta]$. Thus $Dz(c+)$ must be finite and from this contradiction we conclude that $c = a$. It also follows by the same argument that $Dz(a+)$ is finite. Since $x_0 > a$ was an arbitrary point at which $z(x)$ has a finite derivative, we conclude from Theorems 4.10 and 4.14 that $z(x) \in C^{(2)}[a, \infty)$ and is a solution of (5.2). Furthermore, we conclude that $z'(x) \leqslant 0$ on $[a, \infty)$ for, if $z'(x_0) > 0$, then $z'(x) > 0$ and $z''(x) \geqslant 0$ on $[x_0, \infty)$ and $z(x)$ would be unbounded on $[x_0, \infty)$.

Now assume (5.2) has two distinct bounded solutions $z_1(x)$ and $z_2(x)$. Since it is again the case that with the hypotheses of the Theorem solutions of boundary-value problems on finite intervals are unique, there is an $x_0 \geqslant a$ such that $z_1(x) \equiv z_2(x)$ on $[a, x_0]$ and $z_1(x) > z_2(x)$ on (x_0, ∞). Then there is an $x_1 \geqslant x_0$ at which $z_1'(x_1) - z_2'(x_1) > 0$. Arguing as above and using the monotoneity of f, it follows that $z_1'(x) - z_2'(x) > 0$ on $[x_1, \infty)$ which implies $z_1''(x) - z_2''(x) \geqslant 0$ on $[x_1, \infty)$. This implies $z_1(x) - z_2(x)$ is unbounded on $[x_1, \infty)$ and we conclude that bounded solutions of (5.2) are unique.

Theorem 5.3. *Assume that $f(x, y, y')$ satisfies the hypotheses of Theorem 5.2 except that $f(x, y, y')$ is non-increasing in y' for fixed x, y. Then there is a $\delta > 0$ such that for any A with $|A| < \delta$ the boundary-value problem (5.2) has a bounded solution $y(x) \in C^{(2)}[a, \infty)$.*

Proof. As in Theorem 5.2 the generalized solution can be defined for any A and, if $A > 0$, $0 \leqslant z(x) \leqslant A$ on $[a, \infty)$. We will consider only the case $A > 0$. If $(c, d) \subset [a, \infty)$ is an interval on which $z(x)$ is a solution and if $z'(x_0) < 0$ for some $c < x_0 < d$, it follows in this case that $z'(x) < 0$ on $(c, x_0]$. This leads as in the proof of Theorem 5.2 to the conclusion that $Dz(d-)$ is finite. It follows that $z(x)$ is a solution on (a, ∞).

It follows from Theorem 2.1 that for suitable sufficiently small $A > 0$ and $\delta > 0$ the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(a + \delta) = 0$$

has a solution $y(x) \in C^{(2)}[a, a + \delta]$. Then $\varphi_0(x)$ defined by $\varphi_0(x) = y(x)$ on $[a, a + \delta]$ and $\varphi_0(x) \equiv 0$ on $(a + \delta, \infty)$ is a subfunction on $[a, \infty)$. Also, since $\psi_0(x) \equiv A$ is a superfunction, $y(x) \leqslant A$ on $[a, a + \delta]$. Therefore $\varphi_0(x) \leqslant z(x) \leqslant A$ on $[a, \infty)$ and it follows from Theorem 4.14 that for such an A $z(x) \in C^{(2)}[a, \infty)$ and is a solution of (5.2).

If $A_0 > 0$ is such that (5.2) has a solution $z_0(x) \in C^{(2)}[a, \infty)$ with $z_0(a) = A_0$, then for any $0 < A < A_0$, $\varphi_0(x) = z_0(x) - (A_0 - A)$ is an underfunction for the boundary-value problem (5.2) with boundary value A at $x = a$. Thus for any $0 \leqslant A \leqslant A_0$ there is a bounded $C^{(2)}[a, \infty)$ solution of (5.2). This completes the proof of the Theorem.

In this case, bounded solutions are not necessarily unique as evidenced by the problem $y'' = -y'$, $y(0) = A > 0$, which has solutions $y(x) \equiv A$ and $y(x) = Ae^{-x}$.

The arguments used in the proofs of Theorems 5.2 and 5.3 can be applied to boundary-value problems on finite intervals. If $f(x, y, y')$ satisfies the hypotheses of Theorem 5.2, a generalized solution $z(x)$ of the boundary-value problem

$$y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(b) = B$$

is of class $C^{(2)}[a, b)$ and $z(a) = A$. If f satisfies the hypotheses of Theorem 5.3, the generalized solution $z(x) \in C^{(2)}(a, b]$ and $z(b) = B$.

6. A Relation between the Global Existence of Solutions of Initial-Value Problems and the Existence of Solutions of Boundary-Value Problems

As is well known in the case of linear differential equations, the uniqueness of solutions of boundary-value problems implies their existence. In this section we shall see that this is also the case for nonlinear equations provided all solutions of initial value problems exist globally.

Theorem 6.1. *Assume that $I \subset R$ is an interval and that $f(x, y, y')$ is continuous on $I \times R^2$. Assume that for every $(x_0, y_0, y_0') \in I \times R^2$ the initial-value problem*

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0' \tag{6.1}$$

has a unique solution $y(x) \in C^{(2)}(I)$. *Further, assume that, if for any* $[x_1, x_2] \subset I$ *and any* A, B *the boundary-value problem*

$$y'' = f(x, y, y'), \qquad y(x_1) = A, \qquad y(x_2) = B \tag{6.2}$$

has a solution $y(x) \in C^{(2)}[x_1, x_2]$, *then that solution is unique. Then for any proper subinterval* $[x_1, x_2] \subset I$ *and any* A, B *boundary value problem (6.2) has a solution.*

Proof. Let $[x_1, x_2] \subset I$ be a proper subinterval. Then either x_1 or x_2 is an interior point of I and to be specific we assume x_1 is. Let $I_1 = I \cap (-\infty, x_1]$. Let A, B be given and consider the corresponding boundary-value problem (6.2). Let $y(x; m)$ be the solution of the initial-value problem (6.1) with initial conditions $y(x_1) = A$, $y'(x_1) = m$. By assumption, $y(x; m)$ is unique and is a solution on all of I. It follows that $y(x_2; m)$ is a continuous function of m so that its range is an interval. To complete the proof it suffices to show that the range of $y(x_2; m)$ is neither bounded above nor bounded below. We will prove only that the range is not bounded above since an exactly analogous argument is used to show that it is not bounded below.

Assume that, on the contrary, the range of $y(x_2; m)$ is bounded above and let $\beta = \text{lub}\{y(x_2; m) : -\infty < m < +\infty\}$. Let $z(x)$ be the solution of the initial-value problem (6.1) with initial conditions $z(x_2) = \beta$, $z'(x_2) = 0$. Then $z(x)$ is a solution on I and $z(x_1) \neq A$ since, if $z(x_1) = A$, $z(x) = y(x; m)$ for some m, but $z(x_2) > y(x_2; m)$. This follows from the fact that uniqueness of solutions of boundary-value problems implies $y(x; m_1) < y(x; m_2)$ for $x > x_1$ and $m_2 > m_1$. Assume first that $z(x_1) < A$. Then for all m, $y(x_1; m) > z(x_1)$ and $y(x_2; m) < z(x_2)$, and it follows again from the uniqueness of solutions of boundary-value problems that $y(x; m) > z(x)$ on I_1 for all m. Let $u(x) = \text{glb}\{y(x; m) \colon -\infty < m < +\infty\}$ for all $x \in I_1$. Then $z(x) \leqslant u(x) \leqslant y(x; 1)$ on I_1. By Theorem 3.12 solutions are superfunctions, consequently, by Theorem 4.6 $u(x)$ is a superfunction on I_1. Now suppose that $u(x)$ is not a subfunction on I_1. Then there is a subinterval $[c, d] \subset I_1$ and a solution $v(x)$ on $[c, d]$ such that $u(c) \leqslant v(c)$, $u(d) \leqslant v(d)$, but $u(x) > v(x)$ at some points in (c, d). With the hypotheses of the theorem solutions of initial-value problems are continuous with respect to initial conditions. Consequently, if $\epsilon > 0$ is sufficiently small the solution $v_1(x)$ with initial conditions $v_1(c) = v(c)$, $v_1'(c) = v(c) + \epsilon$ will satisfy $v(x) < v_1(x)$ on $(c, d]$ but $v_1(x) < u(x)$ at some points of (c, d). Then, for $\eta > 0$ sufficiently small, the solution $v_2(x)$ satisfying initial conditions $v_2(d) = v_1(d)$, $v_2'(d) = v_1'(d) - \eta$ will satisfy $v_2(x) > v_1(x)$ on $[c, d)$ but $v_2(x) < u(x)$ at some points of (c, d). Thus $v_2(x)$ is a solution on $[c, d]$ with $u(c) < v_2(c)$, $u(d) < v_2(d)$, but $u(x) > v_2(x)$ at some points in (c, d). By the definition of $u(x)$ there is an m_1 and an m_2 such that $u(c) < y(c; m_1) < v_2(c)$ and $u(d) < y(d; m_2) < v_2(d)$. Then, if $m_0 = \text{Max}[m_1, m_2]$, $y(c; m_0) \leqslant y(c; m_1) < v_2(c)$ and $y(d; m_0) \leqslant y(d; m_2) < v_2(d)$, but $v_2(x) < u(x) \leqslant y(x; m_0)$ at some points in (c, d). This contradicts the uniqueness of the solutions of boundary-value problems since $v_2(x)$ and $y(x; m_0)$ will agree at two distinct points in (c, d) and differ between the two points. We conclude that $u(x)$ is a subfunction on I_1. It follows from Theorem 4.10 that $u(x)$ is a solution on a nonnull open subset of I_1. Let (a, b) be a maximal open subinterval contained in I_1 on which $u(x)$ is a solution. Then it follows from Theorem 4.10 that, if a is not the left end of I_1, $Du(a+) = \pm\infty$, and, if $b \neq x_1$, $Du(b-) = \pm\infty$. However, there is a solution $y(x)$ on I such that $u(x) \equiv y(x)$ on (a, b) since solutions of initial-value problems are unique and exist globally. Thus in the above cases, neither $Du(a+) = \pm\infty$ nor $Du(b-) = \pm\infty$ is possible and we conclude that $u(x)$ is a solution on $I_1 - \{x_1\}$. It follows that, if $y(x)$ is the solution on I with $y(x) \equiv u(x)$ on $I_1 - \{x_1\}$, then $u(x_1 - 0) = y(x_1) < A$. However, it then follows that, for $x_0 \in I_1$, $x_0 \neq x_1$ and for $\epsilon > 0$ sufficiently small, the solution $y_1(x)$ of the initial-value problem with initial conditions

$$y_1(x_0) = y(x_0) = u(x_0), y_1'(x_0) = y'(x_0) + \epsilon = u'(x_0) + \epsilon$$

satisfies $y_1(x) > y(x)$ for $x > x_0$ and $y(x_1) < y_1(x_1) < A$. But then, for m sufficiently large, $y(x; m)$ and $y_1(x)$ will constitute distinct solutions for some boundary-value problem on some subinterval of $[x_0, x_1]$. From this contradiction we conclude that $z(x_1) < A$ is not possible.

From $z(x_1) > A$ and $z(x_2) = \beta > y(x_2; m)$ for each m, it follows that $y(x; m) < z(x)$ on $[x_1, x_2]$ for all m. In this case we define $w(x) = \text{lub}\{y(x; m) : -\infty < m < +\infty\}$ on $[x_1, x_2]$ and we can argue as above that $w(x)$ is simultaneously a subfunction and a superfunction on $[x_1, x_2]$. This leads to the conclusion that $w(x)$ is a solution on (x_1, x_2) and $w(x_1 + 0) > A$, which leads to a contradiction as before. We conclude that the range of $y(x_2; m)$ is not bounded above. In a similar manner it can be proven that the range is not bounded below. It follows that (6.2) has a solution.

The assumption that solutions of initial value problems are unique can be omitted in Theorem 6.1 if it is assumed that all solutions of all initial-value problems exist on I.

The conclusion of Theorem 6.1 cannot be strengthened to assert that boundary-value problems on I itself have solutions when I is a compact interval. The following example which shows that this is the case was communicated to the author by Keith Schrader.

Consider $y'' = -y + \arctan y$, $-\frac{1}{2}\pi < \arctan y < \frac{1}{2}\pi$, with $I = [0, \pi]$. Since f is independent of y' and $|f_y| \leqslant 1$ on $[0, \pi] \times R$, it follows that all initial-value problems have unique solutions defined on $[0, \pi]$. Now assume that on $[x_1, x_2] \subset [0, \pi]$ there are solutions $y_1(x)$ and $y_2(x)$ with $y_1(x_1) = y_2(x_1)$, $y_1(x_2) = y_2(x_2)$, and $y_1(x) > y_2(x)$ on (x_1, x_2). Then

$$\begin{aligned} w''(x) &= y_1''(x) - y_2''(x) \\ &= -w(x) + \arctan y_1(x) - \arctan y_2(x) \geqslant -w(x) \qquad \text{on} \qquad [x_1, x_2]. \end{aligned}$$

The equation $y'' = -y$ is disconjugate on an interval of length less than π, hence, it follows from Theorem 3.12 that on such an interval a lower solution is a subfunction. Thus, if $[x_1, x_2]$ is a proper subinterval of $[0, \pi]$, $w''(x) \geqslant -w(x)$ on $[x_1, x_2]$ and $w(x_1) = w(x_2) = 0$ implies $w(x) \leqslant 0$ on $[x_1, x_2]$. We conclude that solutions of boundary-value problems on proper subintervals of $[0, \pi]$ are unique. Now assume $[x_1, x_2] = [0, \pi]$. Let $u(x)$ be the solution of the initial-value problem $u'' = -u$, $u(\frac{1}{2}\pi) = w(\frac{1}{2}\pi)$, $u'(\frac{1}{2}\pi) = w'(\frac{1}{2}\pi)$. Then, since $w'' > -w$ on $(0, \pi)$, it follows that $w(x) > u(x)$ on intervals on each side of $x = \frac{1}{2}\pi$. Again using Theorem 3.12 and the fact that $y'' = -y$ is disconjugate on intervals of length less than π, we conclude that $w(0) > u(0)$ and $w(\pi) > u(\pi)$. Since $w(0) = w(\pi) = 0$ and $u(\frac{1}{2}\pi) = w(\frac{1}{2}\pi) > 0$, this would imply that $u(x)$ has two zeros on $(0, \pi)$ which is impossible. Thus solutions of boundary-value problems on $[0, \pi]$, when they exist, are unique. Therefore, the hypotheses of Theorem 6.1 are satisfied. Now suppose that the boundary-value problem

$$y'' = -y + \arctan y, \qquad y(0) = 0, \qquad y(\pi) = 3\pi$$

has a solution $y(x)$ with $y'(0) = m$. Let $v(x)$ be the solution of the initial-value problem

$$v'' = -v + \pi, \qquad v(0) = 0, \qquad v'(0) = m + 1.$$

Then $v'' = -v + \pi > -v + \arctan v$ and it follows from Theorem

3.12 that $v(x)$ is a subfunction on $[0, \pi]$ with respect to solutions of $y'' = -y + \arctan y$. Since $v(x) > y(x)$ on an interval to the right of $x = 0$, it follows that $v(x) > y(x)$ on $(0, \pi]$. Computing $v(x)$ we get $v(x) = (m + 1) \sin x - \pi \cos x + \pi$ and $y(\pi) < v(\pi) = 2\pi$. From this contradiction we conclude that the stated boundary value problem has no solution.

Now we apply Theorem 6.1 to obtain results concerning existence of solutions of boundary value problems when $f(x, y, y')$ satisfies a Lipschitz condition with respect to y and y'. These results have been dealt with in References [31]-[35], but our methods will be quite different.

Assume that $f(x, y, y')$ is continuous on $I \times R^2$ and assume that there exist continuous functions $\ell_1(x)$, $\ell_2(x)$, $k_1(x)$, $k_2(x)$ on I such that

$$\begin{aligned} G_1(x, y_1 - y_2, y_1' - y_2') &\leqslant f(x, y_1, y_1') - f(x, y_2, y_2') \\ &\leqslant G_2(x, y_1 - y_2, y_1' - y_2') \end{aligned} \tag{6.3}$$

on $I \times R^2$ where

$$G_1(x, y, y') = \begin{cases} k_1(x)\, y + \ell_1(x)\, y' & \text{for} \quad y \geqslant 0, \quad y' \geqslant 0, \\ k_1(x)\, y + \ell_2(x)\, y' & \text{for} \quad y \geqslant 0, \quad y' \leqslant 0, \\ k_2(x)\, y + \ell_2(x)\, y' & \text{for} \quad y \leqslant 0, \quad y' \leqslant 0, \\ k_2(x)\, y + \ell_1(x)\, y' & \text{for} \quad y \leqslant 0, \quad y' \geqslant 0 \end{cases} \tag{6.4}$$

and

$$G_2(x, y, y') = \begin{cases} k_2(x)\, y + \ell_2(x)\, y' & \text{for} \quad y \geqslant 0, \quad y' \geqslant 0, \\ k_2(x)\, y + \ell_1(x)\, y' & \text{for} \quad y \geqslant 0, \quad y' \leqslant 0, \\ k_1(x)\, y + \ell_1(x)\, y' & \text{for} \quad y \leqslant 0, \quad y' \leqslant 0, \\ k_1(x)\, y + \ell_2(x)\, y' & \text{for} \quad y \leqslant 0, \quad y' \geqslant 0. \end{cases} \tag{6.5}$$

From (6.3)-(6.5) it follows that $k_1(x) \leqslant k_2(x)$ and $\ell_1(x) \leqslant \ell_2(x)$ on I and that solutions of initial-value problems for $y'' = f(x, y, y')$ are unique and exist on all of I. The functions G_1 and G_2 can be written in the form

$$\begin{aligned} G_1(x, y, y') = \tfrac{1}{2}(k_1 + k_2)\, y + \tfrac{1}{2}(k_1 - k_2)\, |y| + \tfrac{1}{2}(\ell_1 + \ell_2)\, y' \\ + \tfrac{1}{2}(\ell_1 - \ell_2)\, |y'| \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} G_2(x, y, y') = \tfrac{1}{2}(k_1 + k_2)\, y + \tfrac{1}{2}(k_2 - k_1)\, |y| + \tfrac{1}{2}(\ell_1 + \ell_2)\, y' \\ + \tfrac{1}{2}(\ell_2 - \ell_1)\, |y'|\,. \end{aligned} \tag{6.7}$$

From these expressions for G_1 and G_2 it is clear that solutions of initial-value problems for $y'' = G_1(x, y, y')$ and $y'' = G_2(x, y, y')$ are unique and exist on all of I.

Theorem 6.2. *If solutions of boundary-value problems for $y'' = G_1(x, y, y')$ when they exist are unique, then solutions of boundary-value problems for $y'' = f(x, y, y')$ when they exist are unique.*

Proof. Assume that solutions of boundary-value problems for $y'' = G_1(x, y, y')$ are unique but that those of $y'' = f(x, y, y')$ are not. Then there is an interval $[x_1, x_2] \subset I$ and solutions $y_1(x)$ and $y_2(x)$ of $y'' = f(x, y, y')$ such that $y_1(x_i) = y_2(x_i)$ for $i = 1, 2$ and $y_1(x) > y_2(x)$ on (x_1, x_2). Then, setting $w(x) = y_1(x) - y_2(x)$ and using inequality (6.3), we have $w'' \geqslant G_1(x, w, w')$ on $[x_1, x_2]$, $w(x_1) = w(x_2) = 0$ and $w(x) > 0$ on (x_1, x_2). It follows from Theorem 3.12 that $w(x)$ is a subfunction on $[x_1, x_2]$ with respect to solutions of $y'' = G_1(x, y, y')$. Let $u(x)$ be the solution of the initial-value problem

$$y'' = G_1(x, y, y'), \qquad y(x_1) = w(x_1), \qquad y'(x_1) = w'(x_1).$$

Since solutions of initial-value problems for $y'' = f(x, y, y')$ are unique, $w'(x_1) > 0$. Consequently, since solutions of boundary-value problems for $y'' = G_1(x, y, y')$ are unique, $u(x_1) = 0$, $u'(x_1) > 0$, and $y(x) \equiv 0$ is a solution, it follows that $u(x) > 0$ on $(x_1, x_2]$. Thus $w(x_1) = u(x_1)$ and $w(x_2) < u(x_2)$ which, since $w(x)$ is a subfunction, implies $w(x) \leqslant u(x)$ on $[x_1, x_2]$. It then follows from Theorem 2.6 that $w(x) \equiv u(x)$ on $[x_1, x_2]$. From this contradiction we conclude that solutions of boundary-value problems for $y'' = f(x, y, y')$ when they exist are unique.

Corollary 6.3. *If solutions of boundary-value problems for $y'' = G_1(x, y, y')$ when they exist are unique, then any boundary-value problem on any proper subinterval $[x_1, x_2] \subset I$ for $y'' = f(x, y, y')$ has a solution.*

Proof. This is an immediate consequence of Theorems 6.1 and 6.2.

Let us consider now the case where $I = [a, b]$ is a compact interval. Assume that solutions of boundary-value problems for $y'' = G_1(x, y, y')$ on subintervals of I when they exist are unique. Then the conclusion of Corollary 6.3 follows but in this case the conclusion applies not only to proper subintervals of I but to the interval I itself. If one examines the proof of Theorem 6.1 and also takes Theorems 4.12 and 4.14 into account, it can be seen that it is sufficient to show that, for any boundary-value problem

$$y'' = f(x, y, y'), \qquad y(a) = A, \qquad y(b) = B, \tag{6.8}$$

there is an overfunction $\psi_0(x)$ and an underfunction $\varphi_0(x)$ with $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$. We will show that such functions can be constructed if solutions of boundary-value problems on subintervals of I for $y'' = G_1(x, y, y')$ when they exist are unique.

Let $y_1(x)$ be the solution of the initial-value problem $y'' = G_1(x, y, y')$, $y(a) = 1$, $y'(a) = 0$ and $y_2(x)$ the solution satisfying initial conditions $y_2(a) = 0$, $y_2'(a) = 1$. Then $y_2(x) > 0$ on $(a, b]$, consequently, for $h > 0$ sufficiently large $v(x) = y_1(x) + hy_2(x) > 0$ on $[a, b]$. Furthermore,

$$\begin{aligned} v'' = y_1'' + hy_2'' &= G_1(x, y_1, y_1') + hG_1(x, y_2, y_2') \\ &= G_1(x, y_1, y_1') + G_1(x, hy_2, hy_2') \\ &\leqslant G_1(x, v, v') \end{aligned}$$

on $[a, b]$. Let $z(x)$ be the solution of the initial-value problem

$$y'' = G_1(x, y, y') + f(x, 0, 0), \qquad y(a) = y'(a) = 0.$$

Then, given any A, B, there is an $r > 0$ such that, if $\psi_0(x) = rv(x) + z(x)$, then $\psi_0(a) \geqslant A$ and $\psi_0(b) \geqslant B$. Then

$$\begin{aligned} \psi_0'' = rv'' + z'' &= rG_1(x, v, v') + G_1(x, z, z') + f(x, 0, 0) \\ &\leqslant G_1(x, \psi_0, \psi_0') + f(x, 0, 0) \\ &\leqslant f(x, \psi_0, \psi_0') - f(x, 0, 0) + f(x, 0, 0) \\ &\leqslant f(x, \psi_0, \psi_0'). \end{aligned}$$

Thus $\psi_0(x)$ is an overfunction with respect to the boundary-value problem (6.8).

Now let $u(x)$ be the solution of the initial-value problem

$$y'' = G_2(x, y, y') + f(x, 0, 0), \qquad y(a) = y'(a) = 0.$$

Let $\varphi_0(x) = -qv(x) + u(x)$ where $v(x)$ is as above and $q > 0$ is chosen large enough that $\varphi_0(a) \leqslant A$, $\varphi_0(b) \leqslant B$, and $\varphi_0(x) \leqslant \psi_0(x)$ on $[a, b]$.

Then

$$\begin{aligned}\varphi_0'' = -qv'' + u'' &= -qG_1(x, v, v') + G_2(x, u, u') + f(x, 0, 0)\\ &= G_2(x, -qv, -qv') + G_2(x, u, u') + f(x, 0, 0)\\ &\geqslant G_2(x, \varphi_0, \varphi_0') + f(x, 0, 0) \geqslant f(x, \varphi_0, \varphi_0')\end{aligned}$$

on $[a, b]$ and it follows that $\varphi_0(x)$ is an underfunction with respect to the boundary-value problem (6.8). We have thus proven the following Theorem.

Theorem 6.4. *If $f(x, y, y')$ is continuous on $I \times R^2$, satisfies inequality (6.3) on $I \times R^2$, and solutions of boundary-value problems for $y'' = G_1(x, y, y')$ when they exist are unique, then all boundary-value problems for $y'' = f(x, y, y')$ on all subintervals of I have solutions and the solutions are unique.*

Now we consider the question of uniqueness of solutions of boundary-value problems for $y'' = G_1(x, y, y')$.

Theorem 6.5. *Solutions of boundary-value problems for $y'' = G_1(x, y, y')$ are unique if and only if for any $[x_1, x_2] \subset I$ the solution of the initial-value problem*

$$y'' = G_1(x, y, y'), \qquad y(x_1) = 0, \qquad y'(x_1) = 1 \tag{6.9}$$

satisfies $y(x) > 0$ on $(x_1, x_2]$.

Proof. Since $y(x) \equiv 0$ is a solution, the condition is obviously necessary.

Now assume that the condition is satisfied for each subinterval of I but that solutions of boundary-value problems are not unique. Then there is an interval $[x_1, x_2] \subset I$ and solutions $y_1(x)$ and $y_2(x)$ such that $y_1(x_i) = y_2(x_i)$ for $i = 1, 2$ and $y_1(x) > y_2(x)$ on (x_1, x_2). Let $w(x) = y_1(x) - y_2(x)$. Then $w(x_1) = w(x_1) = 0$ and $w'(x_1) > 0$ since solutions of initial value problems are unique. Also

$$w'' = G_1(x, y_1, y_1') - G_1(x, y_2, y_2') \geqslant G_1(x, w, w')$$

on $[x_1, x_2]$. If $y(x)$ is the solution of the initial-value problem (6.9), $u(x) = w'(x_1)\, y(x)$ is the solution satisfying initial conditions $u(x_1) = 0$, $u'(x_1) = w'(x_1)$; hence, $u(x) > 0$ on $(x_1, x_2]$. If $w(x) \leqslant u(x)$ on $[x_1, x_2]$, Theorem 2.6 can be applied to obtain the contradiction $w(x) \equiv u(x)$

on $[x_1, x_2]$. It follows that $w(x) > u(x)$ at some points in (x_1, x_2). In this case there is a $0 < \beta < 1$ such that $\beta w(x) \leqslant u(x)$ on $[x_1, x_2]$ and $\beta w(x_0) = u(x_0)$, $\beta w'(x_0) = u'(x_0)$ at some $x_1 < x_0 < x_2$. But then again Theorem 2.6 leads to the contradiction $\beta w(x) \equiv u(x)$ on $[x_1, x_2]$ since we still have $(\beta w)'' \geqslant G_1(x, \beta w, \beta w')$ on $[x_1, x_2]$. It follows that solutions of boundary-value problems for $y'' = G_1(x, y, y')$ are unique.

Let $[a, b] \subset I$ and on $[a, b]$ let $K_1 = \text{Min}\, k_1(x)$, $K_2 = \text{Max}\, k_2(x)$, $L_1 = \text{Min}\, \ell_1(x)$, and $L_2 = \text{Max}\, \ell_2(x)$. Let $G_1^*(x, y, y')$ be defined as in (6.4) using the constants L_1, L_2, K_1, and K_2. Then on $[a, b] \times R^2$

$$G_1^*(x, y_1 - y_2, y_1' - y_2') \leqslant G_1(x, y_1, y_1') - G_1(x, y_2, y_2'),$$

and it follows from Theorem 6.5 that solutions of boundary-value problems for $y'' = G_1(x, y, y')$ on subintervals of $[a, b]$ are unique in case for any $[x_1, x_2] \subset [a, b]$ the solution of the initial-value problem $y'' = G_1^*(x, y, y')$, $y(x_1) = 0$, $y'(x_1) = 1$ satisfies $y(x) > 0$ on $(x_1, x_2]$. This will be the case for any$[x_1, x_2] \subset [a, b]$ if

$$b - a < \alpha(L_1, K_1) + \beta(L_2, K_1),$$

where $\alpha(L_1, K_1)$ is the first positive zero of $u'(x)$ with $u(x)$ the solution of the initial-value problem

$$u'' = K_1 u + L_1 u', \qquad u(0) = 0, \qquad u'(0) = 1$$

and $-\beta(L_2, K_1)$ is the first negative zero of $v'(x)$ with $v(x)$ the solution of the initial-value problem

$$v'' = K_1 v + L_2 v', \qquad v(0) = 0, \qquad v'(0) = -1$$

(see [30], p. 312).

7. Further Existence Theorems for Solutions of Boundary-Value Problems

In this section we shall work with solutions of differential inequalities as in the previous sections but we will not impose conditions which imply the uniqueness of solutions of boundary value problems. In place of such conditions we will impose restrictions on the rate of growth of $f(x, y, y')$ with respect to y', in particular we will assume $f(x, y, y')$ satisfies a Nagumo condition. Nagumo [36] used such growth conditions to prove the existence of solutions of boundary-value problems but his methods also required that solutions of initial-value problems be unique. Our methods will not require this assumption. We will as before always assume that $f(x, y, y')$ is continuous.

Definition 7.1. $f(x, y, y')$ is said to satisfy a Nagumo condition on $[a, b]$ with respect to the pair $\alpha(x)$, $\beta(x) \in C[a, b]$ in case $\alpha(x) \leqslant \beta(x)$ on $[a, b]$ and there exists a positive continuous function $h(s)$ on $[0, \infty)$ such that $|f(x, y, y')| \leqslant h(|y'|)$ for all $a \leqslant x \leqslant b$, $a(x) \leqslant y \leqslant \beta(x)$, $|y'| < \infty$ and

$$\int_\lambda^\infty \frac{s\,ds}{h(s)} > \operatorname*{Max}_{a \leqslant x \leqslant b} \beta(x) - \operatorname*{Min}_{a \leqslant x \leqslant b} \alpha(x) \tag{7.1}$$

where

$$\lambda(b - a) = \operatorname{Max}[|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|]. \tag{7.2}$$

Lemma 7.2. *Assume that $f(x, y, y')$ satisfies a Nagumo condition on $[a, b]$ with respect to the pair $\alpha(x)$, $\beta(x) \in C[a, b]$. Then, for any solution $y(x) \in C^{(2)}[a, b]$ with $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, b]$, there is an $N > 0$ depending only on $\alpha(x)$, $\beta(x)$ and $h(s)$ such that $|y'(x)| \leqslant N$ on $[a, b]$.*

Proof. Choose $N > 0$ such that

$$\int_\lambda^N \frac{s\,ds}{h(s)} > \operatorname{Max} \beta(x) - \operatorname{Min} \alpha(x).$$

If $x_0 \in (a, b)$ is such that $(b - a)\, y'(x_0) = y(b) - y(a)$, then by (7.2), $|y'(x_0)| \leqslant \lambda$. Assume that $|y'(x)| \geqslant N$ at some points in $[a, b]$ and to deal with a specific case assume $y'(x) \geqslant N$ at some points. Then there is an interval $[c, d] \subset [a, b]$ such that $y'(c) = N$, $y'(d) = \lambda$, and $\lambda < y'(x) < N$ on (c, d) or $y'(c) = \lambda$, $y'(d) = N$, and $\lambda < y'(x) < N$ on (c, d). Let us consider the former case; then on $[c, d]$,

$$|y''(x)|\, y'(x) = |f(x, y(x), y'(x))|\, y'(x) \leqslant h(y'(x))\, y'(x)$$

and

$$\left| \int_c^d \frac{y''(x)\, y'(x)\, dx}{h(y'(x))} \right| \leqslant \int_c^d \frac{|y''(x)|\, y'(x)\, dx}{h(y'(x))} \leqslant \int_c^d y'(x)\, dx.$$

This leads to the contradiction

$$\int_\lambda^N \frac{s\,ds}{h(s)} \leqslant y(d) - y(c) \leqslant \operatorname{Max} \beta(x) - \operatorname{Min} \alpha(x).$$

Other possibilities can be dealt with in a similar way and we conclude that $|y'(x)| \leqslant N$ on $[a, b]$.

Theorem 7.3. *Assume that $f(x, y, y')$ satisfies a Nagumo condition with respect to the pair $\alpha(x)$, $\beta(x) \in C^{(1)}[a, b]$ which are, respectively, lower and upper solutions of the differential equation on $[a, b]$. Then, for any $\alpha(a) \leqslant c \leqslant \beta(a)$ and $\alpha(b) \leqslant d \leqslant \beta(b)$, the boundary-value problem*

$$y'' = f(x, y, y'), \qquad y(a) = c, \qquad y(b) = d \tag{7.3}$$

has a solution $y(x) \in C^{(2)}[a, b]$ with $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, b]$.

Proof. By Lemma 7.2 there is an $N > 0$ depending only on $\alpha(x)$, $\beta(x)$, and the Nagumo function $h(s)$ such that $|y'(x)| \leqslant N$ for any such solution. Let $F(x, y, y')$ be the modification of $f(x, y, y')$ of Definition 2.3 associated with the triple $\alpha(x)$, $\beta(x)$, c_1 where $c_1 > 0$ is chosen so that $N < c_1$ and $|\alpha'(x)| < c_1$, $|\beta'(x)| < c_1$ on $[a, b]$. Then by Theorem 2.5 the boundary-value problem

$$y'' = F(x, y, y'), \qquad y(a) = c, \qquad y(b) = d$$

has a solution $y(x) \in C^{(2)}[a, b]$ with $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, b]$. By the mean-value theorem there is an $x_0 \in (a, b)$ such that

$$(b - a)\, y'(x_0) = y(b) - y(a)$$

and it follows that $|y'(x_0)| \leqslant \lambda < N < c_1$. It follows that there is an interval around x_0 in which $y(x)$ is a solution of $y'' = f(x, y, y')$. It follows from Lemma 7.2 that $|y'(x)| \leqslant N < c_1$ in this interval, but $y(x)$ is a solution of $y'' = f(x, y, x')$ as long as $|y'(x)| < c_1$. We conclude that $y(x)$ is a solution of (7.3) on $[a, b]$.

As an example to show that Theorem 7.3 yields results when solutions of boundary-value problems are not necessarily unique consider $y'' = |y'|^p$ where $0 < p < 1$. In this case, $h(s) = s^p + 1$ will serve as a Nagumo function and constants are upper and lower solutions.

Schrader [37] has shown that in certain cases combinations of "one-sided" Nagumo conditions can be used in Theorem 7.3.

Lemma 7.2 and Theorem 7.3 can be used to obtain solutions on infinite intervals.

Theorem 7.4. *Assume that for each $b > a$ $f(x, y, y')$ satisfies a Nagumo condition on $[a, b]$ with respect to the pair $\alpha(x)$, $\beta(x) \in C^{(1)}[a, \infty)$*

where $\alpha(x) \leqslant \beta(x)$ *on* $[a, \infty)$, *and* $\alpha(x)$ *and* $\beta(x)$ *are, respectively, lower and upper solutions on* $[a, \infty)$. *Then for any* $\alpha(a) \leqslant c \leqslant \beta(a)$ *the boundary-value problem*

$$y'' = f(x, y, y'), \qquad y(a) = c \tag{7.4}$$

has a solution $y(x) \in C^{(2)}[a, \infty)$ *with* $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ *on* $[a, \infty)$.

Proof. It follows from Lemma 7.2 and Theorem 7.3 that for each $n \geqslant 1$ there is a solution $y_n(x)$ on $[a, a + n]$ with $y_n(a) = c$, $y_n(a + n) = \beta(a + n)$, and $\alpha(x) \leqslant y_n(x) \leqslant \beta(x)$ on $[a, a + n]$ and there is an $N_n > 0$ such that $| y'(x) | \leqslant N_n$ on $[a, a + n]$ for any solution satisfying $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ on $[a, a + n]$. Thus, for any fixed $n \geqslant 1$, $y_m(x)$ is a solution on $[a, a + n]$ satisfying $\alpha(x) \leqslant y_m(x) \leqslant \beta(x)$ and $| y'_m(x) | \leqslant N_n$ on $[a, a + n]$ for all $m \geqslant n$. Hence, for $m \geqslant n$ the sequences $\{y_m(x)\}$ and $\{y'_m(x)\}$ are both uniformly bounded and equicontinuous on $[a, a + n]$. Then, employing standard diagonalization arguments, one obtains a subsequence which converges uniformly on all compact subintervals of $[a, \infty)$ to a solution $y(x)$. $y(x)$ is the desired solution of (7.4).

Theorem 7.5. *Assume that* $f(x, y, y')$ *satisfies a Nagumo condition on* $[- a, a]$ *for each* $a > 0$ *with respect to the pair* $\alpha(x), \beta(x) \in C^{(1)}(- \infty, + \infty)$, *where* $\alpha(x)$ *and* $\beta(x)$ *are lower and upper solutions on* $(- \infty, + \infty)$ *and* $\alpha(x) \leqslant \beta(x)$ *on* $(- \infty, + \infty)$. *Then there is a solution of* $y'' = f(x, y, y')$ *on* $(- \infty, + \infty)$ *with* $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ *on* $(- \infty, + \infty)$.

Proof. The proof is essentially the same as that of Theorem 7.4.

We consider now some applications of these results. The first application is in the establishment of comparison theorems for solutions of nonlinear equations. These results which constitute a different approach to results previously obtained by Knobloch [38] are contained in [21].

Definition 7.6. A solution $y(x)$ of $y'' = f(x, y, y')$ is said to have property (B) on $[a, b]$ in case there is a sequence of solutions $\{y_n(x)\}$ on $[a, b]$ such that

(i) $y_n \to y$ and $y'_n \to y'$ uniformly on $[a, b]$,

(ii) $\Delta_n = y - y_n \neq 0$ and has the same sign for all $n \geqslant 1$ and $a \leqslant x < b$ or for all $n \geqslant 1$ and $a < x \leqslant b$,

(iii) for each $0 < \delta < \frac{1}{2}(b - a)$ there is a constant $c > 0$ depending on δ but not on n and x such that $|\Delta_n'(x)| \leqslant c\,|\Delta_n(x)|$ for all $n \geqslant 1$ and all $a + \delta \leqslant x \leqslant b - \delta$.

Theorem 7.7. *Assume that $f(x, y, y')$ satisfies a Lipschitz condition with respect to y and y' on each compact subset of $[a, b] \times R^2$ and satisfies a Nagumo condition on $[a, b]$ with respect to the pair $\alpha(x), \beta(x) \in C^{(1)}[a, b]$ which are lower and upper solutions on $[a, b]$. Assume further that $\alpha(a) < \beta(a)$ or $\alpha(b) < \beta(b)$. Then for any $\alpha(a) \leqslant c \leqslant \beta(a)$, $\alpha(b) \leqslant d \leqslant \beta(b)$ the boundary value problem (7.3) has a solution $y(x)$ having property* (B) *on $[a, b]$.*

Proof. Consider the case $\alpha(a) < c \leqslant \beta(a)$, $\alpha(b) \leqslant d \leqslant \beta(b)$. Choose $h > 0$ such that $\alpha(a) < c - h$ and consider the sequence of boundary-value problems

$$y'' = f(x, y, y'), \qquad y(a) = c - \frac{h}{n}, \qquad y(b) = d. \tag{7.5$_n$}$$

By Theorem 7.3 the problem $(7.5)_1$ has a solution $y_1(x)$ with $\alpha(x) \leqslant y_1(x) \leqslant \beta(x)$ on $[a, b]$. Using $y_1(x)$ as a lower solution and $\beta(x)$ as an upper solution, we can apply Theorem 7.3 again to obtain a solution $y_2(x)$ of the problem $(7.5)_2$ with $y_1(x) \leqslant y_2(x) \leqslant \beta(x)$ on $[a, b]$. Proceeding in this way we obtain a sequence $\{y_n(x)\} \subset C^{(2)}[a, b]$ such that $y_n(x)$ is a solution of $(7.5)_n$ for each $n \geqslant 1$ and

$$\alpha(x) \leqslant y_n(x) \leqslant y_{n+1}(x) \leqslant \beta(x)$$

on $[a, b]$ for each $n \geqslant 1$. Furthermore, since solutions of initial value problems are unique and $y_n(a) < y_{n+1}(a)$, it follows that $y_n(x) < y_{n+1}(x)$ on $[a, b)$ for each $n \geqslant 1$. By Lemma 7.2 there is an $N > 0$ such that $|y_n'(x)| \leqslant N$ on $[a, b]$ for all $n \geqslant 1$. Then there is a subsequence which we shall renumber as the original sequence such that $y_n \to y$ and $y_n' \to y'$ uniformly on $[a, b]$ where $y(x)$ is a solution of the boundary-value problem $y'' = f(x, y, y')$, $y(a) = c$, $y(b) = d$. Furthermore, this solution satisfies $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ and $|y'(x)| \leqslant N$ on $[a, b]$.

The solution $y(x)$ just obtained as the uniform limit of the subsequence $\{y_n(x)\}$ obviously satisfies parts (i) and (ii) of property (B). We claim that part (iii) of property (B) is also satisfied by $y(x)$ and the subsequence $\{y_n(x)\}$. To see this let $k > 0$ be a Lipschitz coefficient for $f(x, y, y')$ with respect to y and y' on the compact set

$$\{(x, y, y') : a \leqslant x \leqslant b,\ \alpha(x) \leqslant y \leqslant \beta(x),\ |y'| \leqslant N\}.$$

Then with $\Delta_n(x) = y(x) - y_n(x) \geqslant 0$ we have

$$| \Delta_n''(x) | \leqslant k(\Delta_n(x) + | \Delta_n'(x) |) \tag{7.6}$$

on $[a, b]$ for all $n \geqslant 1$. Let $0 < \delta < \frac{1}{2}(b - a)$ be given and let $x_0 \in [a + \delta, b - \delta]$. The hypotheses of Theorem 7.3 are satisfied by the equation $y'' = ky + k | y' |$ on each of the subintervals $[a, x_0]$ and $[x_0, b]$ with $\beta(x) = \Delta_n(x)$ as an upper solution and $\alpha(x) = 0$ as a lower solution. Consequently, the boundary-value problems

$$y'' = ky + k | y' |, \qquad y(a) = 0, \qquad y(x_0) = \Delta_n(x_0)$$

and

$$y'' = ky + k | y' |, \qquad y(b) = 0, \qquad y(x_0) = \Delta_n(x_0)$$

have solutions $y_1(x)$ and $y_2(x)$ with $0 \leqslant y_1(x) \leqslant \Delta_n(x)$ on $[a, x_0]$ and $0 \leqslant y_2(x) \leqslant \Delta_n(x)$ on $[x_0, b]$. Furthermore, it is not difficult to see that $y_1'(x) \geqslant 0$, $y_2'(x) \leqslant 0$, and that these solutions are unique. It follows from the above inequalities that

$$y_2'(x_0) \leqslant \Delta_n'(x_0) \leqslant y_1'(x_0).$$

Computing the solutions $y_1(x)$ and $y_2(x)$ we obtain

$$y_1'(x_0) = \tfrac{1}{2} k \, \Delta_n(x_0) \, [1 + \sqrt{5} \coth \tfrac{1}{2} \sqrt{5} \, k(x_0 - a)]$$

and

$$y_2'(x_0) = \tfrac{1}{2} k \, \Delta_n(x_0) \, [- 1 + \sqrt{5} \coth \tfrac{1}{2} \sqrt{5} \, k(x_0 - b)].$$

It follows that part (iii) of property (B) holds for the solution $y(x)$ and subsequence $\{y_n(x)\}$ with the constant

$$c = \tfrac{1}{2} k[1 + \sqrt{5} \coth \tfrac{1}{2} \sqrt{5} \, k\delta].$$

A different elementary proof that part (iii) of property (B) is satisfied is given in [21], Lemma 2.5.

Theorem 7.8. *Assume that $f(x, y, y')$ has continuous first partial derivatives f_y and $f_{y'}$ on $[a, b] \times R^2$. Let $y_0(x)$ be a solution of $y'' = f(x, y, y')$ having property* (B) *on $[a, b]$. Then the linear equation*

$$y'' = f_{y'}(x, y_0(x), y_0'(x)) \, y' + f_y(x, y_0(x), y_0'(x)) \, y \tag{7.7}$$

is disconjugate on (a, b).

Proof. Assume that Eq. (7.7) is not disconjugate on (a, b). Then (7.7) has a nontrivial solution with zeros at x_1, x_2 with $a < x_1 < x_2 < b$.

Now to be specific assume that the sequence of solutions referred to in property (B) satisfies $\Delta_n(x) = y_0(x) - y_n(x) > 0$ on $[a, b)$ for all $n \geqslant 1$. Then applying the mean-value theorem we obtain

$$\Delta_n'' = f_{y'}(x, y_0(x), y_0'(x))\, \Delta_n' + f_y(x, y_0(x), y_0'(x))\, \Delta_n + q_n \mid \Delta_n \mid + p_n \mid \Delta_n' \mid \qquad (7.8)$$

where p_n, $q_n \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given and pick $\delta > 0$ such that x_1, $x_2 \in (a + \delta, b - \delta)$. Then it follows from part (iii) of property (B) that for n sufficiently large

$$\Delta_n'' \leqslant f_{y'}(x, y_0(x), y_0'(x))\, \Delta_n' + [f_y(x, y_0(x), y_0'(x)) + \epsilon]\, \Delta_n \qquad (7.9)$$

on $[a + \delta, b - \delta]$ with $\Delta_n > 0$ on $[a + \delta, b - \delta]$. We conclude from Theorems 7.3 and 2.6 that there is a solution $z(x)$ of the equality form of the differential inequality (7.9) with $z(x) > 0$ on $[a + \delta, b - \delta]$. It follows that the equality form of (7.9) is disconjugate on $[a + \delta, b - \delta]$ for each $\epsilon > 0$. However, since (7.7) has a nontrivial solution with zeros at $a + \delta < x_1 < x_2 < b - \delta$, it follows that for $\epsilon > 0$ sufficiently small the equality form of (7.9) must have a nontrivial solution with two distinct zeros in $[a + \delta, b - \delta]$. From this contradiction we conclude that (7.7) is disconjugate on (a, b).

With slightly different hypotheses Knobloch [38] proves the existence of at least one solution $y_0(x)$ of $y'' = f(x, y, y')$ such that the corresponding equation (7.7) is disconjugate on the closed interval $[a, b]$. One might conjecture that our results could be strengthened to conclude disconjugacy on $[a, b]$ with the same hypotheses. The following illustration to show that this is not the case is also due to Keith Schrader.

Consider the equation $y'' = -y + y^3$ on the interval $[0, \pi]$. On this interval $\beta(x) \equiv 1$ and $\alpha(x) \equiv 0$ are upper and lower solutions and the hypotheses of Theorem 7.7 are satisfied. Consequently, the boundary-value problem

$$y'' = -y + y^3, \qquad y(0) = y(\pi) = 0$$

has a solution $y_0(x)$ which has property (B) and is such that $0 \leqslant y_0(x) \leqslant 1$ on $[0, \pi]$. It follows that the first variational equation (7.7)

$$y'' = [-1 + 3y_0^2(x)]\, y$$

is disconjugate on $(0, \pi)$. However, this equation is not disconjugate on

$[0, \pi]$ since $y_0(x) \equiv 0$ on $[0, \pi]$. To see this assume $y_0(x) \not\equiv 0$ on $[0, \pi]$. Then, since $y_0(x) \geqslant 0$, $y_0(x) > 0$ on $(0, \pi)$ and $y_0''(x) \geqslant -y_0(x)$ on $[0, \pi]$. Let $y_1(x)$ be the solution of the initial value problem $y'' = -y$, $y(\frac{1}{2}\pi) = y_0(\frac{1}{2}\pi)$, $y'(\frac{1}{2}\pi) = y_0(\frac{1}{2}\pi)$. Then, since $y_0''(\frac{1}{2}\pi) > -y_0(\frac{1}{2}\pi)$, $y_0(x) > y_1(x)$ on an interval to the left and on an interval to the right of $x = \frac{1}{2}\pi$. However on $[0, \frac{1}{2}\pi]$ and on $[\frac{1}{2}\pi, \pi]$ a lower solution of $y'' = -y$ is a subfunction. Consequently, $y_1(0) < y_0(0) = 0$ and $y_1(\pi) < y_0(\pi) = 0$. This implies $y_1(x)$ has two distinct zeros on $(0, \pi)$ which is impossible. We conclude that $y_0(x) \equiv 0$ on $[0, \pi]$.

The above results may be useful in dealing with questions of oscillation of solutions of nonlinear equations. For example, suppose that the equation $y'' + a(x)\,y^{2n-1} = 0$, n a positive integer, has a solution which is positive on $[x_0, \infty)$. Then it follows that it also has a positive solution $y_0(x)$ on $[x_0, \infty)$ such that $y'' + (2n-1)\,a(x)\,y_0^{2n-2}(x)\,y = 0$ is disconjugate on (x_0, ∞).

Next we consider an application of the results of this Section that was suggested by a recent paper of J. D. Schuur [40]. We consider the third-order linear equation

$$y''' + p_0(x)\,y'' + p_1(x)\,y' + p_2(x)\,y = 0, \tag{7.10}$$

where $p_j(x) \in C[x_0, \infty)$ for $j = 0, 1, 2$. The substitution $z = y'/y$ transforms (7.10) into the nonlinear second-order equation

$$\begin{aligned} z'' &= -3zz' - p_0(x)\,z' - (z^3 + p_0(x)\,z^2 + p_1(x)\,z + p_2(x)) \\ &= f(x, z, z'). \end{aligned} \tag{7.11}$$

Clearly, if $z(x)$ is a solution of (7.11) on $[x_0, \infty)$ then

$$y(x) = y(x_0) \exp\left[\int_{x_0}^{x} z(s)\,ds\right]$$

is a solution of (7.10) on $[x_0, \infty)$.

Theorem 7.9. *Assume that there exist lower and upper solutions $\alpha(x)$, $\beta(x)$ of (7.11) on $[x_0, \infty)$ such that $\alpha(x) < \beta(x)$ on $[x_0, \infty)$. Then (7.11) has two positive linearly independent solutions on $[x_0, \infty)$ and (7.11) is disconjugate on $[x_0, \infty)$.*

Proof. From the form of equation (7.11) it is clear that $f(x, z, z')$ satisfies a Nagumo condition on $[x_0, x_0 + n]$ with respect to the pair $\alpha(x)$, $\beta(x)$ for each positive integer n. By Theorem 7.4 there is a solution

$z_1(x)$ of (7.11) on $[x_0, \infty)$ such that $z_1(x_0) = \beta(x_0)$ and $\alpha(x) \leqslant z_1(x) \leqslant \beta(x)$ on $[x_0, \infty)$. Furthermore, by Theorem 2.6 $\alpha(x) < z_1(x)$ on $[x_0, \infty)$. Consequently, applying Theorem 7.4 again we conclude that there is a solution $z_2(x)$ on $[x_0, \infty)$ such that $z_2(x_0) = \alpha(x_0)$ and $\alpha(x) \leqslant z_2(x) < z_1(x)$ on $[x_0, \infty)$. Now let $y_1(x) = \exp[\int_{x_0}^{x} z_1(s)\, ds]$, $y_2(x) = \exp[\int_{x_0}^{x} z_2(s)\, ds]$. Then $y_1(x_0) = y_2(x_0) = 1$ and $y_1'(x_0) = \beta(x_0) \neq \alpha(x_0) = y_2'(x_0)$ so that $y_1(x)$ and $y_2(x)$ are positive linearly independent solutions of (7.10) on $[x_0, \infty)$.

Using the procedure for reducing the order of a linear equation when a solution is known and the fact that $y_1(x)$ and $y_2(x)$ are solutions of (7.10), we find that $y_3(x) = y_2(x)\, u(x)$ is a solution of (7.10) on $[x_0, \infty)$ where

$$u(x) = \int_{x_1}^{x} v(s)\, w(s)\, ds, \qquad x_0 \leqslant x_1 < \infty,$$

$$v(x) = \left(\frac{y_1(x)}{y_2(x)}\right)' = \frac{(z_1(x) - z_2(x))\, y_1(x)}{y_2(x)},$$

$$w(x) = \int_{x_1}^{x} \frac{A(s)}{v^2(s)}\, ds,$$

and

$$A(x) = \exp\left(- \int_{x_1}^{x} [3z_2(s) + p_0(s)]\, ds\right).$$

From this it follows that

$$y_3(x_1) = y_3'(x_1) = 0 \qquad \text{and} \qquad y_3''(x_1) = \frac{y_2(x_1)}{v(x_1)} > 0.$$

Hence, $y_3(x)$ is a positive multiple of the Cauchy function for (7.10) with zero at $x = x_1$. Since $w(x) > 0$ for $x > x_1$, $w(x) < 0$ for $x < x_1$, and $v(x) > 0$ for $x \geqslant x_0$, it follows that $y_3(x) > 0$ for $x \geqslant x_0$, $x \neq x_1$. From this we conclude that (7.10) is disconjugate on $[x_0, \infty)$ ([28], p. 630).

The results of Theorem 7.9 apply equally well on any finite interval in place of $[x_0, \infty)$.

Theorem 7.10. *If there exist constants $\alpha < \beta$ such that*

$$\beta^3 + p_0(x)\, \beta^2 + p_1(x)\, \beta + p_2(x) \leqslant 0$$

and

$$\alpha^3 + p_0(x)\,\alpha^2 + p_1(x)\,\alpha + p_2(x) \geqslant 0$$

on $[x_0, \infty)$, *then (7.10) has two positive linearly independent solutions on* $[x_0, \infty)$ *and is disconjugate on* $[x_0, \infty)$ [39], [40].

Proof. In this case $\beta(x) \equiv \beta$ and $\alpha(x) \equiv \alpha$ are respectively upper and lower solutions of (7.11) and the result follows from Theorem 7.9.

Theorem 7.11. *If there is a* $\delta > 0$ *such that*

$$p_1(x) + (x + \delta)\,p_2(x) \leqslant 0$$

and

$$(x + \delta)\,p_0(x) + (x + \delta)^3\,p_2(x) \geqslant 3$$

on $[0, \infty)$, *then (7.10) has two positive linearly independent solutions on* $[0, \infty)$ *and is disconjugate on* $[0, \infty)$.

Proof. In this case a computation shows that $\beta(x) = 1/(x + \delta)$ and $\alpha(x) = -1/(x + \delta)$ are upper and lower solutions of (7.11) on $[0, \infty)$.

References

1. O. Perron, Ein Neuer Existenzbeweis fur die Integrale der Differentialgleichung $y' = f(x, y)$, *Math. Ann.* **76** (1915), 471–484.
2. O. Perron, Eine neue Behandlung der ersten Randwert-aufgabe fur $\Delta u = 0$, *Math. Z.* **18** (1923), 42–54.
3. F. Riesz, Uber Subharmonische Funktionen und ihre Rolle in der Funktionentheorie und in der Potential-theorie. *Acta. Litt. Sci. Szeged.* **2** (1925), 87–100.
4. F. Riesz, Sur les fonctions subharmoniques et leur rapport à la théorie du Potential, I, *Acta Math.* **48** (1926), 329–343.
5. F. Riesz, Sur les fonctions subharmoniques et leur rapport à la théorie du Potential, II, *Acta Math.* **54** (1930), 321–360.
6. G. Tautz, Zur Theorie der elliptischen Differentialgleichungen II, *Math. Ann.* **118** (1943), 733–770.
7. E. F. Beckenbach and L. K. Jackson, Subfunctions of several variables, *Pacific J. Math.* **3** (1953), 291–313.
8. M. Inoue, Dirichlet problem relative to a family of functions, *J. Inst. Polytech., Osaka City Univ.* **7** (1956), 1–16.
9. L. K. Jackson, Subfunctions and the Dirichlet problem, *Pacific J. Math.* **8** (1958), 243–255.
10. J. B. Serrin, Jr., On the Harnack inequality for linear elliptic equations, *J. Analyse Math.* **4** (1955–56), 292–308.
11. S. E. Bohn, Equicontinuity of solutions of a quasi-linear equation, *Pacific J. Math.* **12** (1962), 1193–1202.

12. S. E. Bohn and L. K. Jackson, The Liouville theorem for a quasi-linear elliptic partial differential equation, *Trans. Am. Math. Soc.* **104** (1962), 392–397.
13. J. O. Herzog, Phragman-Lindelof theorems for second order quasi-linear elliptic partial differential equations, *Proc. Am. Math. Soc.* **15** (1964), 721–729.
14. E. F. Beckenbach, Generalized Convex functions, *Bull. Am. Math. Soc.* **43** (1937), 363–371.
15. F. F. Bonsall, The characterization of generalized convex functions, *Quart. J. Math.* [*Oxford Ser.* (2)] **1** (1950), 100–111.
16. J. W. Green, Generalized convex functions, *Proc. Am. Math. Soc.* **4** (1953), 391–396.
17. M. M. Peixoto, Generalized convex functions and second order differential inequalities, *Bull. Am. Math. Soc.* **55** (1949), 563–572.
18. W. T. Reid, Variational aspects of generalized convex functions, *Pacific J. Math.* **9** (1959), 571–581.
19. B. N. Babkin, Solution of a boundary value problem for an ordinary differential equation of second order by Caplygin's method, *Prikl. Math. Meh. Akad. Nauk SSSR* **18** (1954), 239–242.
20. L. Fountain and L. Jackson, A generalized solution of the boundary value problem for $y'' = f(x, y, y')$, *Pacific J. Math.* **12** (1962), 1251–1272.
21. L. K. Jackson and K. W. Schrader, Comparison theorems for nonlinear differential equations, *J. Diff. Eqs.* **3** (1967), 248–255.
22. Ralph P. Boas, A primer of real functions, *in* "Carus Mathematical Monographs." The Mathematical Association of America, New York, 1961.
23. J. W. Bebernes, A subfunction approach to boundary value problems for ordinary differential equations, *Pacific J. Math.* **13** (1963), 1053–1066.
24. K. W. Schrader, On second order differential inequalities, *Proc. Am. Math. Soc.* (to be published).
25. M. Lees, A boundary value problem for nonlinear ordinary differential equations, *J. Math. Mech.* **10** (1961), 423–430.
26. Z. Nehari, On an inequality of Lyapunov, *in* "Studies in Mathematical Analysis and Related Topics," pp. 256–261. Stanford University Press, Stanford, California, 1962.
27. A. Lasota, Sur la distance entre les zéros de l'équation différentielle linéaire du troisième ordre, *Ann. Polon. Math.* **13** (1963), 129–132.
28. R. M. Mathsen, A disconjugacy condition for $y''' + a_2 y'' + a_1 y' + a_0 y = 0$, *Proc. Am. Math. Soc.* **17** (1966), 627–632.
29. J. W. Bebernes and L. K. Jackson, Infinite interval boundary value problems for $y'' = f(x, y)$, *Duke Math. J.* **34** (1967), 39–48.
30. J. D. Schuur, The existence of proper solutions of a second order ordinary differential equation, *Proc. Am. Math. Soc.* **17** (1966), 595–597.
31. W. J. Coles and T. L. Sherman, Two-point Problems for Non-linear Second Order Ordinary Differential Equations. Report 513, Mathematics Research Center, Univ. of Wisconsin, 1964.
32. P. Bailey and P. Waltman, On the distance between consecutive zeros for second order differential equations, *J. Math. Anal. Appl.* **14** (1966), 23–30.
33. P. Bailey and P. Waltman, Existence and uniqueness of solutions to the first boundary value problem for non-linear second order differential equations, *Arch. Ratl. Mech. Anal.* **12** (1966), 310–320.
34. P. Bailey, L. Shampine, and P. Waltman, The first and second boundary value

problems for nonlinear second order differential equations, *J. Diff. Eqs.* **2** (1966), 399–411.
35. P. Bailey, L. Shampine, and P. Waltman, Existence and uniqueness of solutions of the second order boundary value problem, *Bull. Am. Math. Soc.*, **72** (1966), 96–98.
36. M. Nagumo, Uber die differentialgleichung $y'' = f(x, y, y')$, *Proc. Phys.-Math. Soc. Japan*, [*Ser.* 3] **19** (1937), 861–866.
37. K. W. Schrader, Solutions of second order ordinary differential equations, *J. Diff. Eqs.* (to be published).
38. H. Knobloch, Comparison theorems for nonlinear second order differential equations, *J. Diff. Eqs.* **1** (1965), 1–26.
39. L. Jackson, Disconjugacy conditions for linear third order differential equations, *J. Diff. Eqs.* (to be published).
40. J. D. Schuur, Asymptotic behavior of a solution of the third order linear differential equation, *Proc. Am. Math. Soc.* **18** (1967), 391–393.

Printed in Belgium