Induced Cyclic Cocycles and Higher Eta Invariants

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We construct some cyclic cocycles on the foliation algebra and show that the result of pairing them with a leafwise Dirac operator is a spectral invariant. This leads to a notation of higher eta invariants.

INTRODUCTION

Attaching invariants to operators on manifolds is the principal goal of index theory. Calculating these invariants and relating them to underlying topological and geometric properties of the manifold and operator has been an active area of research since the index theorem of Atiyah and Singer. In that case the manifold is compact, closed, and without boundary, the operator is elliptic, and the index is integer-valued. In more general situations, both the appropriate notion of index and the methods involved must be more sophisticated, but great strides have been made in this subject based on approaches involving concepts and machinery from operator algebras. This paper extends this development further in the context of Dirac operators on foliated compact manifolds.

The index of a leafwise elliptic operator lies in $K_*(C^*(\mathcal{F}))$, where $C^*(\mathcal{F})$ is the foliation $C^*$-algebra of Connes [4]. Invariants of the operator and the foliation can be obtained by pairing the index with cyclic cocycles on smooth subalgebras of $C^*(\mathcal{F})$. In [7], the case of a 0-cyclic cocycle—the trace, $\text{Tr}_\lambda$, associated with an invariant transverse measure—was studied.

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For a class of foliations transverse to the fibers of a flat principal bundle the
pairing with $\text{Tr}_A$ was interpreted as a spectral invariant of an elliptic
operator on the base of the bundle.

The present paper is intended to extend this to cocycles of higher degree.
We single out a class of cyclic cocycles such that the pairing with the
index of a leafwise elliptic operator has a spectral interpretation. They are
obtained from closed normal subgroups of the fiber of the principal bundle
by an inducing process. In case the subgroup is the entire group, the cyclic
cocycle obtained is the trace on the foliation algebra, while for the trivial
subgroup we recover the transverse fundamental class of Connes [5]. The
elliptic operator, whose spectrum is relevant, is obtained by twisting the
operator on the base as before by the Dirac operator on a homogeneous
space. In the final section we will explain how one can interpret the result
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1. FOLIATED FLAT PRINCIPAL BUNDLES

Let $G$ be a compact connected Lie group. Let $M$ be a closed, odd dimen-
sional Spin$^c$ manifold. Let $\pi: \tilde{M} \to M$ be the universal cover of $M$. Let
$\alpha: \pi_1(M) \to G$ be a representation of the fundamental group and assume
we are given a trivialization $\theta: \tilde{M} \to G$ of the associated flat principal $G$
bundle. Let $\text{Cl}(TM) \otimes \mathbb{C}$ be the complexified Clifford algebra bundle
associated to a metric on $TM$ (c.f. [15]). We recall the data necessary to
define a Dirac operator:

1. A metric $\rho$ on $TM$,
2. A Spin$^c$ structure on $M$ (i.e., an irreducible module, $S$, for
\text{Cl}(TM) \otimes \mathbb{C}$), and
3. A connection $\nabla_S$ on $S$ satisfying

$$
\nabla_S(f\sigma) = \nabla_{\text{Cl}}(f) \cdot \sigma + f \cdot \nabla_S(\sigma),
$$

where $\nabla_{\text{Cl}}$ is the connection on $\text{Cl}(TM) \otimes \mathbb{C}$ which extends the Levi-Civita
connection on $TM$ and $\cdot$ denotes Clifford multiplication.

We will refer to these data as $(\rho, S, \nabla_S)$. Let $\delta: C^\infty(S) \to C^\infty(S)$ be the
corresponding Dirac operator. The $K$-homology class of the Dirac
operator associated to the data does not depend on the choice of metric or connection, but it does depend on the particular Spin’ structure used.

We will now describe a suspension operation which associates to a Dirac operator as above a leafwise elliptic operator on an associated closed manifold with a foliation.

First we take the data used to construct the Dirac operator and lift them in a canonical way to the universal cover. Specifically, we use the data \((\pi^*(\rho), \pi^*(S), V_{\pi^*(S)})\) to construct the operator \(\hat{\delta} : C_c^*(\hat{M}, \pi^*(S)) \to C_c^*(\hat{M}, \pi^*(S))\) on \(\hat{M}\). Here \(\pi^*(\rho)\) is the metric on \(T\hat{M}\) making \(\delta^*\) an isometry. Then \(Cl(T\hat{M}) \otimes \mathbb{C} \cong \pi^*(Cl(TM) \otimes \mathbb{C})\) and \(V_{\pi^*(S)}\) is induced in a natural way. Let \(\hat{M} \times_s G\) denote the quotient of \(\hat{M} \times G\) by the action of \(\pi_1(M)\) via the representation \(\chi\) on the right and deck transformations on the left. The operator \(\hat{\delta} \times 1 : C_c^*(\hat{M} \times G, \text{pr}_1^* \pi^*(S)) \to C_c^*(\hat{M} \times G, \text{pr}_1^* \pi^*(S))\), where \(\text{pr}_1 : \hat{M} \times G \to \hat{M}\) is the projection on the first factor, is equivariant with respect to the action of \(\pi_1(M)\) and so descends to an operator on \(M \times_s G\). After applying the trivialization \(\theta\), we obtain the operator

\[
\hat{\delta} : C_c^*(M \times G, \text{pr}_1^* (\pi^*(S))) \to C_c^*(M \times G, \text{pr}_1^*(\pi^*(S))).
\] (1.1)

We will repeat this construction in various settings throughout the paper.

The manifold \(M \times G\) has two foliations, \(\mathcal{F}_u\) and \(\mathcal{F}_G\). The first, \(\mathcal{F}_u\), has leaves obtained by taking the image of \(M \times \{g\}\) under the projection to \(M \times G\) and applying the trivialization \(\theta\). The second, \(\mathcal{F}_G\), is simply the trivial foliation of the product \(M \times G\) by the fibers \(\{x\} \times G\). Let \(C_c^*(\mathcal{F}_u)\) and \(C_c^*(\mathcal{F}_G)\) denote the smooth convolution algebras of the holonomy groupoids of these foliations and \(C^*(\mathcal{F}_u)\) and \(C^*(\mathcal{F}_G)\) the associated \(C^*\)-algebras of the foliations, [4].

We now describe in terms of K-theory the suspension construction which associates \(\delta_u\) to \(\hat{\delta}\). It can be realized from the following sequence of maps, where \(II\) denote the image \(\pi_1(M) \subseteq G\),

\[
[s] \in K_1(M) = KK^1(C(M), \mathbb{C}) = KK_{1,0}(C(M), \mathbb{C})
\]

\[
\tau_{\text{C}} : KK_{1,0}(C(M), \mathbb{C}) \to KK_{1,0}(C(M, \mathbb{C}) \otimes C(G, \mathbb{C}))
\]

\[
\tau_{\text{II}} : KK_{1,0}(C(M, \mathbb{C}) \otimes C(G, \mathbb{C})) \to KK_{1,0}(C(M \times G, \mathbb{C}) \otimes C(G, \mathbb{C}))
\]

\[
\tau_{\text{II}} \circ \tau_{\text{C}} : KK_{1,0}(C(M), \mathbb{C}) \to KK_{1,0}(C(M \times G, \mathbb{C}) \otimes C(G, \mathbb{C}))
\]

Here \(\tau_{\text{C}}\) is obtained by tensoring by the algebra \(C(G)\) with its \(II\) action, and \(\tau_{\text{II}}\) is the map defined by Kasparov, [14]. The following proposition is a direct consequence of the definitions.

**Proposition 1.1.** The element \([\hat{\delta}] \in K_1(M)\) corresponds to \([\hat{\delta}_u] \in KK^1(C(M \times G, \mathbb{C}) \otimes C(G, \mathbb{C}))\) under the sequence of maps described in (1.2).
We will also need the operator on $M \times G$ obtained from the Laplacian $\Delta_G$ on (functions on) $G$,

$$A_G: C^\infty(M \times G) \to C^\infty(M \times G). \quad (1.3)$$

Note that $A_G$ is elliptic along the leaves of $\mathcal{F}_G$ and is invariant under the holonomy of the foliation $\mathcal{F}_G$. Similarly, $\mathcal{F}_e$ is elliptic along the leaves of $\mathcal{F}_G$ and invariant under the holonomy of $\mathcal{F}_G$. This phenomenon occurs for invariantly defined operators on commuting transverse foliations.

2. TRANSVERSE FOLIATIONS

Let $G$ be a compact, connected Lie group and let $H$ be a closed normal subgroup. One of our goals will be to construct a cyclic cocycle on $C^\infty_c(\mathcal{F}_G)$ which is determined by the subgroup $H$. We will assume that the dimension of $G/H$ is even.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of $H$. Let $\langle \cdot, \cdot \rangle$ be an Ad-invariant metric on $\mathfrak{g}$ and set $\mathfrak{f} = \mathfrak{h}^\perp$. Let $K$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{f}$. Let $p: G \to G/H$ be the projection on the space of cosets.

The restriction of $p$ to $K$, $p: K \to G/H$, is a covering homomorphism with covering group $\Gamma = H \cap K$. Since $K$ is connected and $\Gamma$ is discrete, it follows that $\Gamma$ is contained in the center of $K$, hence is abelian. Define the right action of $\Gamma$ on $K \times H$ by $(k, h)\gamma = (k\gamma, \gamma^{-1}h)$ and let $K \times \Gamma H$ denote the quotient.

**Proposition 2.1.** The principal bundle

$$H \to G \to G/H \quad (2.1)$$

is isomorphic to the flat bundle

$$H \to K \times \Gamma H \to G/H. \quad (2.2)$$

**Proof.** The multiplication map $m: K \times H \to G$ induces the commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{m} & H \\
\downarrow & & \downarrow \\
K \times \Gamma H & \xrightarrow{m} & G \\
\downarrow & & \downarrow \\
K/\Gamma & \xrightarrow{\bar{p}} & G/H
\end{array}
$$

(2.3)
One checks directly that $\tilde{m}$ is a homeomorphism and $\bar{p}$ is an isomorphism. This yields the result.

We will construct Dirac operators on $G, H, G/H$ and $K$ which satisfy certain compatibility conditions. We first obtain an invariant Spin$^c$ structure on $TG$ as follows. Take the Ad-invariant metric, $(\ , \ )$, used above on $T_eG = g$ and define a Riemannian metric on $TG$ by $\langle v, w \rangle = (dL_{e^{-1}}(v), dL_{e^{-1}}(w))$ if $v, w \in T_eG$. Here, $dL_g$ denotes the differential of left multiplication by $g$. Note that $TH$ and $TK$ are orthogonal with respect to this metric. Using this metric one constructs the Clifford algebra, $\text{Cl}(TG) \otimes \mathbb{C}$.

Note that the trivialization by left translation, $\iota : \text{Cl}(TG) \otimes \mathbb{C} \to G \times \text{Cl}(T_eG) \otimes \mathbb{C}$,

$$\iota : \text{Cl}(TG) \otimes \mathbb{C} \to G \times \text{Cl}(T_eG) \otimes \mathbb{C},$$

(2.4)
is an algebra isomorphism. Let $S_e \subseteq \text{Cl}(TG) \otimes \mathbb{C}$ be a minimal left ideal. Then $S^G = \iota^{-1}(G \times S_e)$ gives us a canonical invariant Spin$^c$ structure on $TG$. Since $TG \cong (G \times h) \otimes (G \times 1)$ one can choose a Spin$'$ structure, $S'H$, on $H$ and this will determine $S'^GH$ on $G/H$ and $S^K$ on $K$. Moreover, there are connections $\nabla_{S^G}, \nabla_{S'H}$, and $\nabla_{S^K}$ which satisfy

$$\nabla_{S^G} = \nabla_{S'H} \otimes 1 + 1 \otimes \nabla_{S^K}.$$

(2.5)

This enables one to construct the Dirac operators $\partial_H, \partial_K$, and $\partial_{G/H}$ and $\partial_G$. Note that $S_K$ is the lift of $S_{G/H}$ via the group covering map, and similarly for $\nabla_{S^K}$ and $\nabla_{S_{G/H}}$.

Now, the operator $\partial_K \otimes 1 + 1 \otimes \partial_H$ is the Dirac operator on $K \times H$ associated to the product Spin$'$ structure and it is equivariant under the action of $I$. Taking the quotient by $I$, one has that $\partial_K \otimes 1 + 1 \otimes \partial_H$ descends to an operator on $K \times \tilde{H} = G$ which we will refer to as $\partial_K \#_I \partial_H$. One then has the following result:

**Proposition 2.2.** We have the following:

1. $\partial_K$ is the lift of $\partial_{G/H}$, and
2. $\partial_G = \partial_K \#_I \partial_H$.

**Proof.** For (1), it suffices to note that the data on $K$ is the lift to the cover of the data on $G/H$. Statement (2) follows from the fact that the product Spin$'$ structure on $K \times H$ descends to that for $G$ under its identification with $K \times \tilde{H}$. Thus, $\partial_K \#_I \partial_H$ is equal to $\partial_G$.

Using $H$ we will construct a new pair of transverse foliations on $M \times G$.

The first, $\mathcal{F}_H$, is obtained from the integrable distribution $(T\mathcal{F}_H)(x, g) = dh \circ L_{\iota_e} \circ \iota_g(h)$. The leaves of $\mathcal{F}_H$ are copies of $H$. The second, $\mathcal{F}_K$, is obtained from the distribution $(T\mathcal{F}_K)(x, g) = (T\mathcal{F}_H)(x, g) \oplus dh \circ L_{\iota_e} \circ \iota_g(1)$. This
is also integrable and we obtain foliations that are related to each other in
the same way that \( \mathcal{F}_a \) and \( \mathcal{F}_G \) are—they are transverse and the leaves of
each are invariant under the holonomy of the other. Moreover, there are
immersions of foliations, \( \mathcal{F}_H \rightarrow \mathcal{F}_G \) and \( \mathcal{F}_a \rightarrow \mathcal{F}_a^K \) (cf. [10]).

We next construct operators related to the foliations \( \mathcal{F}_H \) and \( \mathcal{F}_a^K \). First
there is a technical point we must address. In what follows we need to work
with self-adjoint operators. Therefore, we will use the standard notion of
Dirac operator, even in the case of an even-dimensional manifold. In this
case the operator has the form \( (0 \ 0^*) \). The operator \( 0 \) is sometimes referred
to as “the Dirac operator.” Note that, since we need \( G/H \) to be even dimen-
sional, we will have \( \dim(G) = \dim(H) \mod 2 \). Thus, if \( \dim(G) \) is odd, then
the Dirac operator on \( K \) will be treated as above. Likewise, if \( \dim(G) \) is
even, then the Dirac operators on both \( H \) and \( K \) will also be as above.

We first define an operator \( (\hat{\vartheta} \# \hat{\vartheta}_K) \) on \( M \times G \) by the following
procedure. Consider \( \vartheta \otimes 1 \) on \( K \times H \). It is a \( \Gamma \)-invariant operator so its
descends to an operator \( \hat{\vartheta}_K \) on \( G = K \times H \). Let \( \hat{\vartheta} \) be the lift of \( \vartheta \) to \( \hat{M} \).
Consider the operator \( \hat{\vartheta} \otimes 1 + 1 \otimes \hat{\vartheta}_K \) on \( \hat{M} \times G \). It is invariant under the
action of \( \pi_1(M) \), so it descends to the quotient \( \hat{M} \times \times G \). Now, applying the
trivialization, \( \theta: \hat{M} \times \times G \rightarrow M \times G \), we obtain the desired operator \( (\hat{\vartheta} \# \hat{\vartheta}_K) \)
on \( M \times G \). Note that it is elliptic along the leaves of \( \mathcal{F}_a^K \) and elliptic
transverse to \( \mathcal{F}_H \).

We will need two more operators on \( M \times G \). In a manner similar to how
\( \hat{\vartheta}_K \) was constructed, we can obtain \( \hat{\vartheta}_H \) on \( G \). Then consider the operators
\( 1 \otimes \hat{\vartheta}_H \) and \( 1 \otimes \hat{\vartheta}_K \) on \( M \times G \). As above, they descend to \( \hat{M} \times \times G \) and
applying the trivialization we obtain the operators \( \hat{\vartheta}_H \) and \( \hat{\vartheta}_K \) on \( M \times G \).
The operator \( \hat{\vartheta}_H \) is elliptic along the leaves of \( \mathcal{F}_H \) and elliptic transverse to \( \mathcal{F}_a^K \).
Moreover, \( \hat{\vartheta}_K \) is a leafwise operator for \( \mathcal{F}_a^K \) whose is invertible on
the complement of \( T \mathcal{F} \) in \( T \mathcal{F}_a^K \). The following identity will be important
later and is verified by careful inspection of the constructions.

**Proposition 2.3.** One has the following:

\[
(\hat{\vartheta} \# \hat{\vartheta}_K) = \hat{\vartheta}_H \# \hat{\vartheta}_K.
\]

**3. Foliations Algebras**

Each of the four foliations we have considered has a smooth convolution
algebra and associated \( C^* \)-algebra,

\[
C^*(\mathcal{F}) \cong C^*_c(\mathcal{F}).
\]
We will always use the reduced $C^*$ algebra. In fact, it will be necessary to consider these algebras as built from sections of the endomorphism bundle of a bundle $E$ on $M \times G$. However, since $E$ never plays a role, we will suppress it from the notation and use $C^*(\mathcal{F})$ and $C^*_c(\mathcal{F})$ for $C^*(\mathcal{F}, E)$ and $C^*_c(\mathcal{F}, E)$.

Our first goal is to construct cyclic cocycles on $C^*_c(\mathcal{F})$.

Let us assume that our algebras are represented on $L^2(M \times G; E)$, where $E$ is an appropriate vector bundle. This can be a non-trivial assumption for $C^*(\mathcal{F})$. However, it will be sufficient to consider elements of the smooth convolution algebras, and these algebras are always representable in this way.

**Proposition 3.1.** If $T \in C^*_c(\mathcal{F})$ and $S \in C^*_c(\mathcal{F})$, then $ST$ and $TS$ are in $L^1(L^2(M \times G; E))$, the trace class operators. A similar statement holds for $C^*_c(\mathcal{F}_K)$ and $C^*(\mathcal{F}_H)$.

**Proof.** The result follows by representing the operators by kernels and noting that the resulting convolutions have the properties necessary for the conclusion.

Let $e^{-t\Delta_H}$ denote the heat kernel for the Laplacian along $H$. This is obtained, just as $\Delta_H$ was, by using the leafwise heat kernels to define a leafwise elliptic operator for $\mathcal{F}_H$.

**Proposition 3.2.** The heat kernel for the Laplacian along $H$ satisfies

\[ e^{-t\Delta_H} \in C_c^*(\mathcal{F}_H). \]

\[ (3.2) \]

**Proof.** We refer for the proof to [23] or [18].

We next recall the notion of asymptotic morphism introduced by Connes and Higson [6].

**Definition 3.3.** A family of functions $\varphi_t : A \to B$, where $A$ and $B$ are normed algebras and $1 \leq t < \infty$, is an asymptotic morphism if

1. $\|\varphi_t(a + \lambda b) - (\varphi_t(a) - \lambda \varphi_t(b))\| \to 0$ as $t \to \infty$,
2. $\|\varphi_t(a)b - \varphi_t(ab)\| \to 0$ as $t \to \infty$,
3. $\|\varphi_t(a^*) - \varphi_t(a)^*\| \to 0$ as $t \to \infty$, and
4. $t \to \|\varphi_t(a)\|$ is continuous for each fixed $a$.

Define two families of functions via multiplication on the right by $e^{-t\Delta_H}$.

\[ e_t : C^*_c(\mathcal{F}_K) \to L^1 \]

\[ (3.3) \]
and
\[ e'_t: C^\infty(M \times G) \to C^\infty(\mathcal{F}_H). \] (3.4)

By Proposition 3.1 the map in 3.3 is well defined. Since \( C^\infty(M \times G) \) acts as a set of multipliers on \( C^\infty(\mathcal{F}_H) \), the same holds for 3.4.

**Proposition 3.4.** Both \( e \) and \( e'_t \) are asymptotic morphisms with respect to the norms on the corresponding C*-algebras.

**Proof.** We sketch the proof for \( e_t \), since the steps are the same for \( e'_t \).

One needs the basic fact that for \( a \in C^\infty(\mathcal{F}_H) \) one has
\[ \|[a, e^{-tA_H}]\| \to 0 \] (3.5)
as \( t \to \infty \), since \( e^{-tA_H} \) is an approximate unit for \( C^\infty(\mathcal{F}_H) \) [7, 8]. Since \( e_t \) is actually linear and the continuity condition in 3.3(4) is straightforward, we consider 3.3(2) and 3.3(3). Note that
\[
\varphi_j(ab) - \varphi_j(a) \varphi_j(b) = ab e^{-tA_H} - ae^{-tA_H}be^{-tA_H} \\
= (ab)(e^{-tA_H} - e^{-2tA_H}) - a[e^{-tA_H}, b] e^{-tA_H}.
\]

For the first term on the right, we note that \( A_H \) is really a family of operators parametrized by \( x \in M \), and \( A_{H,x} \) is a self-adjoint operator. In fact, each of the operators \( A_{H,x} \) is the same. The function \( f_t(s) = e^{-ts} - e^{-2ts} \) converges uniformly to 0 on any interval \([x, \infty)\) for any \( x > 0 \). Choose \( x < \lambda_0 \), where \( \lambda_0 \) is the smallest positive eigenvalue of \( A_{H,x} \). Let
\[
\tilde{f}_t(s) = \begin{cases} 
0 & \text{if } 0 \leq s < x, \\
 f_t(s) & \text{if } s \geq x.
\end{cases} \] (3.6)

Then one easily shows that \( \|f_t(A_H)\| = \|\tilde{f}_t(A_H)\| \to 0 \) as \( t \to \infty \).

The second term goes to zero in norm by 3.5, yielding 3.3(2).

For 3.3(3), we have
\[
\varphi_j(a^*) = a^* e^{-tA_H} \\
= e^{-tA_H} a^* + [a^*, e^{-tA_H}] \\
= \varphi_j(a)^* + [a^*, e^{-tA_H}].
\]

Again, by 3.5, we obtain the result.
4. ASYMPTOTIC MORPHISMS AND CYCLIC COCYCLES

In this section we discuss how to use the asymptotic morphisms $e_t$ and $e'_t$ to pull back cyclic cocycles. The method may have more general applicability, but we will only need it for certain specific cocycles. It is based on a description due to John Roe [24] of the renormalization process in [7].

Consider first the asymptotic morphism $e_t$: $C^c_c(FK: L^1(L^2(M \times G)), Tr \circ C_0^*(L^1)$ and let $h \in C_c(\mathcal{F}_G^F)$. Then by (3.1) we have $e_t(h) \in L^1$. Thus, we may consider

$$\text{Tr}(e_t(h)) = \text{Tr}(h e_t),$$  

(4.1)

where $h e_t$ denotes the product operators.

We recall the following elementary fact.

**Proposition 4.1.** Let $M$ be a closed $n$-dimensional Riemannian manifold and let $e^{-t\Delta}$ be the heat kernel for the Laplacian, $\Delta$, on $M$. Then we have, for $f \in C^\infty(M)$,

1. $e^{-t\Delta} \in L^1$, and
2. $\text{Tr}(f e^{-t\Delta}) = \left( \int_M f(x) \, dx \right) \text{Tr}(e^{-t\Delta}) + O(t^{-1/2+1}).$

**Proof.** Statement (1) is clear. For (2), consider the family of pseudo-differential operators $f e^{-t\Delta}$. Then we have, according to [27, Chap. XII],

$$\text{Tr}(f e^{-t\Delta}) = \int_M K_t(x, x) \, dx,$$  

(4.2)

where

$$K_t(x, x) = \left( \int_{S^1(M)} \sigma(f e^{-t\Delta})^{-n/2} (x, \zeta) \, d\zeta \right) t^{-n/2} + O(t^{-n/2 + 1}),$$  

(4.3)

where $\sigma(f e^{-t\Delta})(x, \zeta) = f(x) \sigma(e^{-t\Delta})(x, \zeta)$ is the principal symbol. One may choose the metric so that $\sigma(e^{-t\Delta})(x, \zeta)$ is independent of $x$. Thus we obtain

$$\text{Tr}(f e^{-t\Delta})$$

$$= \int_M f(x) \, dx \left( \int_{S^1(M)} \left( \int_{S^1(M)} \sigma(f e^{-t\Delta})(x, \zeta) \right) t^{-n/2} + O(t^{-n/2 + 1}) \right)$$

(4.4)

Our next goal is to extend this to foliations.
Proposition 4.2. The element $e^{-tA}$ has an asymptotic expansion as $t \to 0$

$$e^{-tA} \sim t^{-p/2} a_0 + \text{higher order terms}, \quad (4.5)$$

where $a_0 \in C^\infty(\mathcal{F}_H)$ and $p = \dim H$.

Proof. The operator $e^{-tA}$ is represented by the kernel $k_t(x, u, v) \in C^\infty(\mathcal{F}_H)$ and $k_t(x, u, v)$ is actually independent of $x$. We have the standard asymptotic expansion of the heat kernel on $H$ given by

$$k_t(x, u, v) \sim t^{-p/2} a_0(x, u, v) + \text{higher order terms}. \quad (4.6)$$

Similarly, $a_0(x, u, v)$ is independent of $x$ and is an element of $C^\infty(\mathcal{F}_H)$.

We may now substitute (4.5) into (4.1). Note that, by (3.1), $ha_0 \in L^1$ so that the following formula makes sense.

$$\text{Tr}(e^{-tA}) \sim t^{-p/2} \text{Tr}(ha_0) + \text{higher order terms}. \quad (4.7)$$

We want to identify the leading coefficient as a functional of $h$. To this end recall the trace, $\text{Tr}_{\varphi} : C^\infty(\mathcal{F}_K) \to \mathbb{C}$, which is determined by Haar measure, $\mu$ on $H$ [4]. It is obtained as follows. Let $f \in C^\infty(\mathcal{F}_K)$ be supported in a coordinate chart $(\varphi, U)$ on the holonomy groupoid $\mathcal{G}$,

$$\varphi : U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k. \quad (4.8)$$

Then

$$\text{Tr}_{\varphi}(f) = \int \left( \int_{\mathcal{G}} f(x, x, h) \, dx \right) \, d\mu(h). \quad (4.9)$$

This extends to all of $C^\infty_c(\mathcal{F}_K)$ using a partition of unity and can be shown to yield a trace.

Proposition 4.3. The coefficient of the leading order term in the asymptotic expansion (4.7) yields a trace on $C^\infty_c(\mathcal{F}_K)$ which coincides with the trace obtained from Haar measure on $H$.

Proof. Let $f \in C^\infty(\mathcal{F}_K)$ be supported in a coordinate chart. Let $e^{-tA} \in C^\infty(\mathcal{F}_H)$ be the operator obtained from the heat kernel on $H$. Then $fe^{-tA} \in L^1(L^2(M \times G))$ and has a smooth kernel, $L$. We will compute the trace by integrating its kernel over the diagonal

$$\text{Tr}_{L^1(L^2(M \times G))}(fe^{-tA}) = \int_{M \times G} L(x, g, x, g) \, dx \, dg = \sum_i \int_{M \times G} \varphi_i L(x, g, x, g) \, dx \, dg,$$
where we have chosen a partition of unity \( \{ \varphi_i \} \) so each of the integrals is over a flow box. Let \( L_i = \varphi_i L \). Locally, there are coordinates \((x, k, h)\) and we will evaluate \( \int L_i \), explicitly.

\[
\int L_i(x, k, h, x, k, h) \, dx \, dk \, dh
\]

\[
= \int \varphi_i(x, k, h) \, f(x, k, x, k, h) \, e^{-i\lambda h}(x, k, h, h) \, dx \, dk \, dh
\]

\[
= \int \left[ \int \varphi_i(x, k, h) \, f(x, k, x, k, h) \, dx \, dk \right] e^{-i\lambda h}(h, h) \, dh
\]

\[
= \text{Tr}_{L_i H}(\hat{f}e^{-i\lambda h}),
\]

(4.10)

where \( \hat{f}(h) = \int \varphi_i(x, h, k) \, f(x, k, x, k, h) \, dx \, dk \). The result now follows from (4.1) and (4.9), for we have

\[
\text{Tr}_{L_i H}(\hat{f}e^{-i\lambda h}) = \left( \int_H \hat{f} \right) \left( \text{Tr}_{L_i H}(e^{-i\lambda h}) \right) + O(t^{-\eta^2 + 1})
\]

(4.11)

by (4.1) and

\[
\int_H \hat{f} = \text{Tr}_{\delta}(f)
\]

(4.12)

by (4.9). }

The result of Proposition 4.3 can be viewed as describing a way to pull back cyclic cocycles via asymptotic morphisms. Thus, one has

\[
e_t^*: C^\infty_0(M \times G) \to \mathcal{L}^1
\]

(4.13)

and

\[
e_t^*: C^\infty_0(\mathcal{L}^1) \to C^\infty_0(C^\infty_0(M \times G))
\]

(4.14)

is defined by taking the leading coefficient in the asymptotic expansion of \( e_t^*(c) \) for \( c \in C^\infty_0(\mathcal{L}^1) \). In the case above we have that

\[
e_t^*(\text{Tr}) = \text{Tr}_{\delta}.
\]

(4.15)

It is easy to check in this case that \( e_t^* \) defines a homomorphism which commutes with \( h \), the boundary map in the cyclic complex. We will next develop the analogous result for the asymptotic morphism

\[
e_t^*: C^\infty(M \times G) \to C^\infty(\mathcal{L}_H).
\]

(4.16)
and certain higher dimensional cocycles. To describe them we recall the construction of the transverse cocycle. The operator $\tilde{\mathcal{D}}_K$ is elliptic transverse to the foliation $\mathcal{F}_H$. As such it has a Chern character which is a cyclic cohomology class $ch[\tilde{\mathcal{D}}_K] \in H^*_\text{c}(C^\infty(\mathcal{F}_H))$. In [7] it was necessary to choose a specific cyclic cocycle representing this class. It was called the transverse cocycle and was cohomologous to, but not the same as, the analogous cocycle defined by Connes in [5]. In the case when $\tilde{\mathcal{D}}_K$ is self-adjoint, the cocycle can be written in the form

$$c^\infty(k_0, \ldots, k_p) = c_p \text{Tr}(P[P, k_0] \cdots [P, k_p]),$$  (4.17)

where $P$ is a carefully chosen phase for $\tilde{\mathcal{D}}_K$, (see [7, Sect. 5] for details), $c_p$ is a constant, and $k_i \in C^\infty(\mathcal{F}_H)$.

We will study the way that $e^t$ induces a mapping

$$e^t : C_p^{p+1}(C^\infty(\mathcal{F}_H)) \to C_p^{p+1}(C^\infty(M \times G)).$$  (4.18)

This can be viewed as an interpretation of the renormalization process of [7].

To this end consider

$$c^\infty(e^t(f_0), \ldots, e^t(f_p)) = \text{Tr}([P, f_0 e^{-tH}] \cdots [P, f_p e^{-tH}]).$$  (4.19)

We will again substitute the asymptotic expansion of $e^{-tH}$ into (4.19) and consider the leading coefficient

$$c^\infty(e^t(f_0), \ldots, e^t(f_p)) \sim t^{-p/2}c_0^\infty(f_0, \ldots, f_p) + \text{higher order terms}. $$  (4.20)

Our intention is to define

$$e^t : c^\infty = c_0^\infty.$$  (4.21)

To do this requires us to show that the right hand side is a cyclic cocycle. To prove this we need the following fact.

**Proposition 4.4.** One has, for $f \in C^\infty(M \times G)$ and $P$ as above,

1. $\text{Tr}([P, e^{-tH}]) = o(\text{Tr}(e^{-tH}))$
2. $\text{Tr}([f, e^{-tH}]) = o(\text{Tr}(e^{-tH}))$

as $t \to \infty$.

With this in hand we can now show that the leading coefficient defines a cyclic cocycle.

**Proposition 4.5.** $c_0^\infty$ is a cyclic cocycle on the algebra $C^\infty(M \times G)$. 


Proof. By repeated application of Proposition 4.4, one shows that
\[ \text{Tr}(P[p, f_a e^{-i\lambda u}] \cdots [P, f_p e^{-i\lambda u}]) \] (4.22)
can be expressed as
\[ \text{Tr}(P[p, f_a] \cdots [P, f_p] e^{-i(p+1)\lambda u}) + o(\text{Tr}(e^{-i(p+1)\lambda u})). \] (4.23)
Now, the operator $P$ was chosen precisely so that
\[ \text{Tr}(P[p, f_a] \cdots [P, f_p] e^{-i(p+1)\lambda u}) \in C^\infty(F^K) \subseteq L^2(M \times G). \] (4.24)
Therefore, it follows from Proposition 4.3 that (4.23) is equal to
\[ \text{Tr}_d(P[p, f_a] \cdots [P, f_p]) \text{Tr}(e^{-i(p+1)\lambda u}) + o(\text{Tr}(e^{-i(p+1)\lambda u})). \] (4.25)
Dividing by $\text{Tr}(e^{-i(p+1)\lambda u})$ one sees that the constant term is
\[ \text{Tr}_d(P[p, f_a] \cdots [P, f_p]). \] (4.26)
This is clearly a cyclic cochain. Using the properties of the specific choice of $P$, [7, Sect. 6], one sees that it is a cyclic cocycle. \[\square\]

**Corollary 4.6.** $e'_*(e^\xi)$ is the same cyclic cocycle as that obtained via the renormalization process [7].

5. CHERN CHARACTERS

We will next study the Chern characters of several of the operators we have introduced. It will be necessary to use the bivariant Chern character as developed by [20, 28].

Consider the foliations, $F_1$ and $F^K_2$. They are related via an inclusion $i: F_1 \to F^K_2$. It will be necessary to apply the following general facts. The first is a restatement of A.7.5 in [10].

**Proposition 5.1.** Let $F_1 \subseteq F_2$ be foliations of $M$ with $i: F_1 \to F_2$ the inclusion. Let $D$ be a 1st order differential operator satisfying

1. $D$ is a leafwise operator for $F_2$, and
2. $D$ is elliptic transverse to the foliation induced by $F_1$ on $F_2$.

Then $D$ defines a $p$-summable Kasparov bimodule. This determines an element $[D] \in KK^*(C^*(F_1), C^*(F_2))$. 

5. Higher eta invariants
Proof. Except for the summability, this is proved in [10]. For the summability, one uses standard arguments.

If the symbol of the operator $D$ is given as the symbol of the Dirac operator on the complement of $T\mathcal{F}$ in $T\mathcal{F}$, then the element, $[D]$, above agrees with $i$, as defined in [10]. We apply this to the foliations $\mathcal{F} \subseteq T\mathcal{F}_\mathcal{K}$.

The operator $\hat{\mathcal{K}}_\mathcal{K}$ satisfies the hypothesis of Proposition 5.1. Thus we obtain

**Proposition 5.2.** The operator $\hat{\mathcal{K}}_\mathcal{K}$ defines an element $[\hat{\mathcal{K}}_\mathcal{K}] \in KK(C^*(\mathcal{F}), C^*(\mathcal{F}_{\mathcal{K}}))$. This element is represented by a $(p+1)$-summable Kasparov bimodule, where $p = \dim(G/H)$.

We next study the bivariant Chern character of this element. We recall some of the formalism associated to that theory. Let $A^\infty$ and $B^\infty$ be unital locally convex algebras. There is a bivariant cyclic group, $HC^*(A^\infty, B^\infty)$ defined by Jones and Kassel in [12]. An important property, proved there, is the existence of a pairing, analogous to the Kasparov product,

$$HC^*(A^\infty, B^\infty) \times HC^*(B^\infty, C^\infty) \to HC^*(A^\infty, C^\infty).$$

(5.1)

Combining this with the fact that $HC^*(A^\infty, C) = HC^*(A^\infty)$, one gets, for any element $u \in HC^*(A^\infty, B^\infty)$, a homomorphism

$$u: HC^*(B^\infty) \to HC^*(A^\infty).$$

(5.2)

Assume now that $A^\infty$ and $B^\infty$ are dense and holomorphically closed subalgebras of $C^*$-algebras. Let $KK_{\text{fin}}^*(A, B) \subseteq KK^*(A, B)$ denote the elements of $KK^*(A, B)$ represented by finitely summable Kasparov bimodules. There is a bivariant Chern character defined by Nistor [19, 20] and Wang [28, 29]. If $x \in KK_{\text{fin}}^*(A, B)$ then one has

$$Ch(x): HC^*(B^\infty) \to HC^*(A^\infty).$$

(5.3)

It will be necessary to apply this at the level of cochains.

Thus, we may apply the bivariant character to get

$$Ch[\hat{\mathcal{K}}_\mathcal{K}]: HC^0(C^\infty_\mathcal{K}(\mathcal{F}_{\mathcal{K}})) \to HC^0(C^\infty(\mathcal{F}_\mathcal{K})).$$

(5.4)

One must next relate the action of this homomorphism to taking the Kasparov product with $[\hat{\mathcal{K}}_\mathcal{K}]$. We recall the following result from [19].

**Proposition 5.3.** Let $x \in KK^*(A, B)$ be $p$-smooth with respect to $A^\infty \subseteq A$, and $B^\infty \subseteq B$. Then we have the following commutative diagram:
We want to extend this to the case in which $C$ is replaced by a commutative $C^*$-algebra $D$ with $D^\otimes \subseteq D$. This will yield

$$
\begin{array}{c}
KK^*(D, A) @>\otimes x>> KK^*(D, B) \\
\downarrow \text{ch} @. \downarrow \text{ch} \\
HC^*_a(D^\otimes, A^\otimes) @>\text{Ch}^{(c)}>> HC^*_a(D^\otimes, B^\otimes)
\end{array}
$$

(5.6)

where now the vertical arrows are bivariant Chern characters as well. This will be obtained by applying the Künneth formula [25]. Recall that this implies that if $K^*(D)$ is finitely generated and $D$ is commutative, then the natural map induced by Kasparov product

$$
\beta \otimes 1 : K^*(D) \otimes K_0(A) \otimes \mathbb{C} \rightarrow KK^*(D, A) \otimes \mathbb{C}
$$

(5.7)

is an isomorphism. There is an analogous product map for cyclic theory, [12].

$$
\tilde{\beta} : HC^*(A^\otimes) \otimes HC_0(B^\otimes) \rightarrow HC^*(A^\otimes, B^\otimes).
$$

(5.8)

**Proposition 5.4.** The diagram

$$
\begin{array}{c}
K^*(D) \otimes K_0(A) \otimes \mathbb{C} @>\beta \otimes 1>> KK(D, A) \otimes \mathbb{C} \\
\downarrow \text{ch}^* \otimes \text{ch}_a @. \downarrow \text{ch} \\
HC^*(D^\otimes) \otimes HC_0(A^\otimes) @>\tilde{\beta}>> HC^*(D^\otimes, A^\otimes)
\end{array}
$$

(5.9)

commutes.

We apply this to obtain the main result of this section.

**Proposition 5.5.** Let $x \in KK^*(A, B)$ be $p$-smooth. Let $D$ be a commutative algebra with $K_0(D)$ and $K^*(D)$ finitely generated. Then

$$
\begin{array}{c}
KK^*(D, A) \otimes \mathbb{C} @>\otimes x>> KK(D, B) \otimes \mathbb{C} \\
\downarrow \text{ch} @. \downarrow \text{ch} \\
HC^*(D^\otimes, A^\otimes) @>\text{Ch}^{(c)}>> HC^*(D^\otimes, B^\otimes)
\end{array}
$$

(5.10)

commutes.
Proof. We consider the diagram, where we have commutativity already in all but the square corresponding to the conclusion of the theorem.

\[
\begin{array}{c}
\text{K}(D) \otimes \text{K}(A) \otimes \text{C} \\
\text{K}(D) \otimes \text{K}(B) \otimes \text{C}
\end{array}
\]

A direct analysis yields the result. \(\blacksquare\)

We will apply Proposition 5.5 to the case when \(x = [\partial_+^*]\). To this end consider \([\partial_+^*] \in KK^*(C(M \times G), C^*(\mathcal{F}_x))\) and \([\partial_+^*] \in KK^*(C^*(\mathcal{F}_x), C^*(\mathcal{F}_x^K))\). Then we have the following

**Proposition 5.6.**

\[ [\partial_+^*] \otimes [\partial_+^*] = [\partial_+^* \#_x \partial_+^*] \] (5.12)

**Proof.** This follows from Proposition 2.3. \(\blacksquare\)

**Corollary 5.7.** We have

\[ \text{Ch}([\partial_+^*]) \cdot (\text{Ch}([\partial_+^*])) = \text{Ch}([\partial_+^* \#_x \partial_+^*]). \] (5.13)

**Proof.** Consider the commutative diagram

\[
\begin{array}{c}
\text{KK}(C(M \times G), C^*(\mathcal{F}_x)) \otimes \text{Ch}[\mathcal{F}_x^*] \\
\text{KK}(C(M \times G), C^*(\mathcal{F}_x^K))
\end{array}
\]

\[
\begin{array}{c}
\text{HC}^*(C^*(M \times G), C^*_x(\mathcal{F}_x)) \text{Ch}[\mathcal{F}_x^*] \\
\text{HC}^*(C^*(M \times G), C^*_x(\mathcal{F}_x^K))
\end{array}
\]

(5.14)

316 DOUGLAS AND KAMINKER
Take the element \([\hat{s}_n] \in \text{KK}(C(M \times G), C^*(\mathcal{F}_\gamma))\) and follow it around the diagram both ways. One gets
\[
\text{Ch}(\hat{s}_n \otimes \hat{s}_K) = \text{Ch}([\hat{s}_K] \cdot (\text{Ch}([\hat{s}_n])).
\] (5.15)

By 5.6 the statement follows.

Let \(\text{Tr}^K\) denote the trace on \(C^*(\mathcal{F}_\gamma)\) obtained from Haar measure on \(H\). Thus, \([\text{Tr}^K]\) is in \(HC^*(C^*(\mathcal{F}_\gamma))\). Applying the above formulas, one obtains
\[
\langle \text{Ch}([\hat{s}_n] \otimes \hat{s}_K), [\text{Tr}^K]\rangle = \langle \text{Ch}([\hat{s}_K])(\text{Ch}([\hat{s}_n])), [\text{Tr}^K]\rangle = \langle \text{Ch}([\hat{s}_n]) \cdot \text{Ch}([\hat{s}_K]), [\text{Tr}^K]\rangle.
\]

This has the following important consequence.

**Proposition 5.8.** \(\langle \text{Ch}([\hat{s}_n] \otimes \hat{s}_K), [\text{Tr}^K]\rangle \in HC^*(C^*(M \times G))\) is represented by
\[
c(f_0, \ldots, f_p) = c_p \text{Tr}^K(P[P, f_0] \cdots [P, f_p]).
\] (5.16)

**Proof.** One adapts the proof that for \([\hat{s}] \in \text{KK}(C(M), \mathcal{H})\) with \(\text{Ch}([\hat{s}] \in HC^*(C^*(M), \mathcal{L}_1)\) and \([\text{Tr}] \in HC^*(\mathcal{L}_1)\) we have \(\langle \text{Ch}([\hat{s}], [\text{Tr}] \rangle = \text{ch}([\hat{s}]), [\text{Tr}]\rangle\), where the right side is Connes' Chern character,
\[
\text{ch}([\hat{s}] = c_p \text{Tr}(P[P, f_0] \cdots [P, f_p]).
\] (5.17)

Then one extends to the case where the compact operators \(\mathcal{H}\) is replaced by a general \(C^*\)-algebra \(A\).

Now we must consider a new situation. We will bring into play the fact that \(\hat{s}_K\) is a leafwise operator along \(\mathcal{F}_G\) and is elliptic transverse to the foliation induced by \(\mathcal{F}_H\) on the leaves of \(\mathcal{F}_G\). Thus \(\hat{s}_K\) satisfies the hypothesis of Proposition 5.1 and so defines an element
\[
[\hat{s}_K] \in \text{KK}(C^*(\mathcal{F}_H), C^*(\mathcal{F}_G)).
\] (5.18)

Similar reasoning shows that \(\hat{s}_n\), being elliptic transverse to \(\mathcal{F}_G\), determines an element
\[
[\hat{s}_n] \in \text{KK}(C^*(\mathcal{F}_H), C).\] (5.19)

Finally, \(\hat{s}_n \cdot \hat{s}_K\) is elliptic transverse to \(\mathcal{F}_H\), hence it defines
\[
[\hat{s}_n \cdot \hat{s}_K] \in \text{KK}(C^*(\mathcal{F}_H), C).\] (5.20)

By an argument similar to that above we obtain the following two results.
Proposition 5.9.

\[ [\mathcal{E}]^n \oplus [\mathcal{E}]^n = [\mathcal{E} \#_n \mathcal{E}]^n \]  

(5.21)

Proposition 5.10. \( \text{Ch}([\mathcal{E}]^n) = \text{Ch}(\mathcal{E} \#_n \mathcal{E}) \in HC^n(C^\omega(M \times G)). \)

In the transverse case the following result is a direct consequence of the definitions.

Proposition 5.11. \( \text{Ch}(\mathcal{E} \#_n \mathcal{E}) \) is represented by \( c_n \text{Tr}(P[k_0], \cdots [P, k_n]) \) where \( P \) is the phase of \( \mathcal{E} \#_n \mathcal{E} \) chosen in Section 4.

6. HOMOGENEOUS COCYCLES

In this section we will construct the cyclic cocycles on \( C^\omega_+(\mathcal{F}_x) \) we have been aiming for. Let \( H \subseteq G \) be a closed normal subgroup. Let \( p = \dim(G/H) \). We will construct a cyclic cocycle

\[ c_H \in Z^p_1(C^\omega_+(\mathcal{F}_x)) \]  

(6.1)

with the property that the result of the pairing

\[ \langle [c_H], \text{Index} \rangle \in \mathbb{C} \]  

(6.2)

has an interpretation as a spectral invariant. Here \( \text{Index} \in KK^s(\mathcal{F}_x, C^\omega_+(\mathcal{F}_x)) \) is defined to be \([1] \otimes [\mathcal{E}] \) in the odd case and is \([\varphi] \otimes [\mathcal{E}] \), for some suitably chosen \([\varphi] \), in the odd case.

Consider the composition

\[ C^0(L^1) \overset{\epsilon^t}{\longrightarrow} C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\text{Ch}^t}{\longrightarrow} C^0(\mathcal{C}^\omega_+(\mathcal{F}_x)), \]  

(6.3)

where \( \text{Ch}^t(\mathcal{E}_x) \) is a representative of the bivariant Chern character of \( \mathcal{E}_x \).

We apply this to \( \text{Tr} \in C^0(L^1) \) and set

\[ c_H = \text{Ch}^t(\mathcal{E}_x) \cdot \epsilon^t(\text{Tr}) = \text{Ch}^t(\mathcal{E}_x)(\text{Tr}^x). \]  

(6.4)

Thus \( c_H \) is a cyclic cocycle and we will study its properties. The most important is how it fits into the following basic diagram.

\[ C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\epsilon^t}{\longrightarrow} C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\text{Ch}^t}{\longrightarrow} C^0(\mathcal{C}^\omega_+(\mathcal{F}_x)) \]

\[ \text{Ch}^t(\mathcal{E}_x) \]

\[ C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\epsilon^t}{\longrightarrow} C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\text{Ch}^t}{\longrightarrow} C^0(\mathcal{C}^\omega_+(\mathcal{F}_x)) \]

\[ \text{Ch}^t(\mathcal{E}_x) \]

\[ C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\epsilon^t}{\longrightarrow} C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\text{Ch}^t}{\longrightarrow} C^0(\mathcal{C}^\omega_+(\mathcal{F}_x)) \]

\[ \epsilon^t \]

\[ C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\epsilon^t}{\longrightarrow} C^0_+(C^\omega_+(\mathcal{F}_x)) \overset{\text{Ch}^t}{\longrightarrow} C^0(\mathcal{C}^\omega_+(\mathcal{F}_x)) \]

\[ \epsilon^t \]
**Theorem 6.1.** Let $\text{Tr} \in C^0_0(L^1)$ be the trace. Then diagram (6.5) commutes when applied to $\text{Tr}$:

$$
\text{Ch}(\tilde{\varphi}) \cdot \text{Ch}(\tilde{\varphi}_K) \cdot e^K(\text{Tr}) = e^K(\text{Ch}(\tilde{\varphi}_K)) \cdot \text{Ch}(\tilde{\varphi}_K)(\text{Tr}). \quad (6.6)
$$

**Proof.** We first consider the two outside triangles. They commute for all elements, not only the trace. To see that the right hand triangle commutes, we simply apply Proposition 5.6 and its corollary Proposition 5.13 which say that

$$
[\tilde{\varphi}_K] \otimes [\tilde{\varphi}_K'] = [\tilde{\varphi} \#_K \tilde{\varphi}_K']
$$

so that we have

$$
\text{Ch}(\tilde{\varphi}_K) \cdot \text{Ch}(\tilde{\varphi}) = \text{Ch}(\tilde{\varphi} \#_K \tilde{\varphi}_K). \quad (6.8)
$$

This shows that the right hand triangle commutes. For the left hand triangle we recall that we may view the same operators as above as being transversally elliptic to the appropriate foliations. Then by Propositions 5.9 and 5.10 the analogous results hold.

For the inner rhombus we must interpret the results of [7] in the present context. A principal result of that paper implies that

$$
\text{Ch}(\tilde{\varphi} \#_K \tilde{\varphi}_K) = e^K(\text{Ch}(\tilde{\varphi} \#_K \tilde{\varphi}_K)) (\text{Tr}). \quad (6.9)
$$

To clarify this, note that by Proposition 5.11 the expression on the right is the renormalized transverse cocycle. Moreover, the one on the left is the longitudinal cocycle by Proposition 5.8. The required result in [7, Theorem 6.11] states that these two cocycles are equal. This shows that the inner rhombus commutes.

Following the diagram to the right and then down takes $\text{Tr} \in C^0_0(L^1)$ to a cyclic cocycle on $C^\infty(M \times G)$. Recall that $[\tilde{\varphi}_K]$ is a self-adjoint operator, so $[\tilde{\varphi}_K] \in KK^1(C(M \times G), C^*(\mathcal{F}_K))$. Let $[\varphi] \in K^1(M \times G)$. Consider $[\varphi], [\tilde{\varphi}_K], [c_H] \in KK^1(C(M \times G), C)$. One has

$$
\langle [\varphi] \otimes [\tilde{\varphi}_K], [c_H] \rangle = \langle [\varphi], [\tilde{\varphi}_K], [c_H] \rangle. \quad (6.10)
$$

Moreover, $\text{ch}(\langle [\varphi], [c_H] \rangle) = \text{Ch}(\tilde{\varphi}_K)(c_H)$. Thus,

**Proposition 6.2.** We have

$$
\langle [\varphi] \otimes [\tilde{\varphi}_K], [c_H] \rangle = \langle \text{ch}(\varphi), \text{Ch}(\tilde{\varphi}_K)(c_H) \rangle.
$$
As we indicate in the introduction, the construction of the cyclic cocycle \( c_H \) for a closed, normal subgroup \( H \) can be viewed as generalizing that of the foliation trace of Connes since one obtains the latter in case \( H = G \). There is another sense in which our construction can be viewed as generalizing the one that yields the trace. If the Dirac operator on \( G/H \) is lifted to an invariant operator on \( G \), then its Chern character yields an invariant cyclic cocycle on the transversal \( G \) to the foliation \( F \). If one had a procedure for using such data to define a cyclic cocycle on the smooth foliation algebra, then \( c_H \) is the appropriate candidate. However, at present there is no such construction except for invariant zero cyclic cocycles which are simply invariant measures.

In the next section we show that the result of going down and then to the right in the diagram (6.5), when applied to \( \text{Tr} \), can be expressed in terms of the spectrum of certain self-adjoint elliptic operators on closed manifolds. Explicitly,

\[
\psi^* \cdot \text{Ch}(\tilde{\sigma} \#_K \tilde{\sigma})^\partial(\text{Tr}) = \text{spectral invariant. (6.11)}
\]

7. SPECTRAL INVARIANTS

In this section we will show that pairing with \( c_H \) can yield a spectral invariant of an elliptic operator. The construction will use a variant of the one in [7]. Recall that the data needed there was a self-adjoint elliptic operator on a closed manifold along with a trivialized flat bundle. Given that, one showed that a spectral invariant obtained from the operator and the trivialized flat bundle was equal to the result of pairing the index of a leafwise elliptic operator with a 0-cyclic cocycle. To apply these ideas here we must construct an operator and flat bundle.

We first construct the “twisted Dirac operator”, \( \tilde{\sigma}_M \#_x \tilde{\sigma}_{G/H} \), on \( M \times G/H \) as follows. Consider the operators \( \tilde{\sigma}_{G/H} \) on \( G/H \) and \( \tilde{\sigma} \) on \( \tilde{M} \) and form the operator \( \tilde{\sigma}_M \#_x \tilde{\sigma}_{G/H} \). As before, this operator is invariant under the action of \( \pi_1(M) \) on \( M \times G/H \). Descending to the quotient and applying the trivialization, we arrive at the operator \( \tilde{\sigma}_M \#_x \tilde{\sigma}_{G/H} \) on \( M \times G/H \). Note that it is elliptic and self-adjoint, since \( M \) is odd-dimensional and \( G/H \) is even-dimensional.

Next we must construct a trivialized flat bundle over \( M \times G/H \). To this end recall that the principal bundle \( H \to G \to G/H \) is isomorphic to \( H \to K \times H \to G/H \) and hence is flat. Let \( \rho : H \to \mathfrak{g}_K \) be a representation and consider \( E(\rho) = (M \times G) \times_{\rho} \mathbb{C}_N \), the total space of the flat bundle

\[
\mathbb{C}^N \to E(\rho) \to M \times G/H.
\]
This bundle is determined by the representation $\rho \beta$, where $\beta$ is the composition

$$\beta: \pi_1(M \times G/H) \to \pi_1(G/H) \to \Gamma = K \cap H \subseteq H,$$

(7.2)

and the map from $\pi_1(G/H)$ to $\Gamma$ is given by the holonomy. Since $E(\rho)$ is a flat vector bundle, there is an $m \geq 1$ and a trivialization $\kappa$ so that

$$\kappa: E(mp) = mE(\rho) \cong (M \times G/H) \times \mathbb{C}^m.$$

(7.3)

We will study the relative eta invariant $\eta(\mathcal{D}_M \#_s \mathcal{D}_{G/H}, mp, \kappa)$. This is a real number which is equal modulo $\mathbb{Z}$ to the difference of eta invariants

$$\eta(\mathcal{D}_M \#_s \mathcal{D}_{G/H}, mp, \kappa) = \eta(\mathcal{D}_M \#_s \mathcal{D}_{G/H} \otimes I) - \eta(\mathcal{D}_M \#_s \mathcal{D}_{G/H} \otimes E(mp)) \mod \mathbb{Z}.$$

(7.4)

The $\eta$ invariant is determined by the spectrum of the elliptic operator. By applying the results of [7] we will be able to express this in terms of pairing with the cocycle $c_H$. It is in this sense that $c_H$ yields a spectral invariant.

The last piece of data we need is an element of $K^1(M \times G)$. Let $\varphi: M \times G \to \mathbb{U}_{mN}$ denote the composition

$$M \times G \to (M \times K) \times_{mg} H \to (M \times G/H) \times \mathbb{U}_{mN} \to \mathbb{U}_{mN}.$$

(7.5)

Then $[\varphi] \in K^1(M \times G)$ is the element we require. Note that $\varphi$ depends on the integer $m$. With all this in hand we may state the main result relating our cocycle $c_H$ to spectral invariants.

**Theorem 7.1.** For $[\varphi] \in K^1(M \times G)$ we have

$$\langle [\varphi] \otimes [\mathcal{D}_s], [c_H] \rangle = \eta(\mathcal{D}_M \#_s \mathcal{D}_{G/H}, mp, \kappa).$$

(7.6)

**Proof.** Note first that

$$\langle [\varphi] \otimes [\mathcal{D}_s], [c_H] \rangle = \langle [\varphi], \langle [\mathcal{D}_s], [c_H] \rangle \rangle \equiv \langle \text{ch}([\varphi]), \text{Ch}(\langle [\mathcal{D}_s], [c_H] \rangle) \rangle.$$

Now $\text{Ch}(\langle [\mathcal{D}_s], [c_H] \rangle) = \text{Ch}([\mathcal{D}_s]) \cdot c_H$ and by Proposition 6.5 this is equal to

$$(e')^* \text{Ch}((\mathcal{D}_s \#_s \mathcal{D}_K)^h).$$

(7.7)

But by [7] we have

$$\langle [\varphi], (e')^* (\text{Ch}(\mathcal{D}_s \#_s \mathcal{D}_K)^h) \rangle.$$  

(7.8)
is the relative eta invariant of the operator whose suspension to $M \times G$ is $\tilde{\theta} \# s \tilde{\theta}_G$. It is easy to see that this is the twisted Dirac operator $\tilde{\theta}_M \# s \tilde{\theta}_{G,H}$.

This completes the proof.

Remark 7.2. The cocycle $c_H$ has a dual in the following sense: to $[c_H] \in HC^0(C^*(\mathcal{F}_G))$ is associated a dual class $D[c_H] \in HC^0(C^*(M \times G), C^*(\mathcal{F}_G))$. The class $D[c_H]$ is represented by $c_H^* \cdot \text{Ch}(\tilde{\theta}_K)$. Note that these classes appear in the basic diagram (6.5) as the compositions along the top and bottom rows, respectively.

The relation between $D[c_H]$ and $[c_H]$ is the cyclic theory version of a natural correspondence between certain elements of $KK$-groups that is obtained when one has transverse foliations. In the present case this sends elements of $KK^0(C^*(M \times G), C^*(\mathcal{F}_G))$ to $KK^0(C^*(\mathcal{F}), C)$. One sends the class of a leafwise elliptic operator to itself, viewed as a transversally elliptic operator for the transverse foliation. It would be interesting to know to what extent the existence of such a dual is related to the ability to obtain a spectral interpretation for pairing with the cocycle.

8. HIGHER SPECTRAL FLOW

Despite our use of the terminology "higher eta invariant", it is actually a version of spectral flow that we are dealing with. In this section we will clarify this point of view and describe some generalizations.

Spectral flow is an invariant defined for a loop of self-adjoint Fredholm operators,

$$ F_t \in \mathcal{F}^{sa}, \quad 0 \leq t \leq 1 $$

by means of the isomorphism

$$ \pi_1(\mathcal{F}^{sa}) \cong \mathbb{Z}. $$

A development of these ideas can be found in [1–3, 17]. This can be generalized to the case of self-adjoint Fredholm operators over a Hilbert module. If $A$ is a $C^*$-algebra, then one considers the standard Hilbert module over $A$, $\mathcal{H}_A$, and the adjointable operators $\mathcal{D}(\mathcal{H}_A)$. One can define the self-adjoint Fredholm operators,

$$ \mathcal{F}^{sa}(\mathcal{H}_A) \subseteq \mathcal{D}(\mathcal{H}_A), $$

which have been analyzed by V. Perera in [22]. He showed that

$$ \pi_1(\mathcal{F}^{sa}(\mathcal{H}_A)) \cong K_0(A). $$
This provides a way to define spectral flow for a loop $F_t \in \mathcal{F}^{sa}(\mathcal{H}_A)$ as an element $[F_t] \in K_0(A)$. Applying a homomorphism to from $K_0(A)$ to $\mathbb{R}$ provides a numerical invariant. The results of [7] show that this is a useful notion. To see this, consider the suspended operator $\partial_s$. It is a leafwise elliptic, self-adjoint, differential operator for the foliation $\mathcal{F}_A$. Let $\varphi: M \times G \to U_N$ be a unitary multiplier. Then using functional calculus and the Kasparov stabilization theorem we get a path in $\mathcal{F}^{sa}(\mathcal{H}_{C^*(\mathcal{F}_A)})$ corresponding to the path $(1-t)\partial_s + t\varphi^{-1}\partial_s\varphi$. Now, the unitary operator $\varphi \in \mathcal{L}(\mathcal{H}_{C^*(\mathcal{F}_A)})$ can be connected to $I$ in the contractible space $\mathcal{H}(\mathcal{H}_{C^*(\mathcal{F}_A)})$. Thus, the path can be completed to a loop $F_t \in \mathcal{F}^{sa}(C^*(\mathcal{F}_A))$. The main result of [7] can be rephrased in the following form.

**Theorem 8.1.** Let $\text{Tr}_\text{r}: K_0(C^*(\mathcal{F}_A)) \to \mathbb{R}$ be the trade corresponding to Haar measure on $G$. Let $G = U_N$ and let $\varphi: M_0 \to U_N$ be projection onto the second factor. Then, if $[F_t] \in K_0(C^*(\mathcal{F}_A))$ is the generalized spectral flow described above, we have

$$\text{Tr}_\text{r}([F_t]) = \eta(\partial_s, z, \theta),$$

where the right hand side is the relative eta invariant.

Since the von Neumann algebra associated to $C^*(\mathcal{F}_A)$ and $\text{Tr}_\text{r}$ is often a Type II$_1$ factor we factor the left hand side as Type II spectral flow. That is, the relative eta invariant should be viewed as real valued spectral flow. In this context we see that pairing the $K$-theoretic spectral flow $[F_t]$ with other cyclic cocycles can reasonably be called higher spectral flow.

This leads one to study a generalization of this notion. Since it is so strongly motivated by the treatment in the present work we describe here some of the definitions. In a later paper, we will develop these ideas more completely and describe applications to the cohomology of the gauge group and anomalies. To this end let $\mathcal{A}$ be the space of connections on the principal bundle $p: M \times G \to M$, and let $\mathcal{G} = \text{Map}(M, G)$ be the gauge group. Let $k$ be odd and choose $\varphi_1, \ldots, \varphi_k \in \mathcal{G}$. We will need to fix a unitary representation, $\varphi: G \to U_N$, and consider the compositions $\rho\varphi_i$, which we will still denote by $\varphi_i$. Consider the operators

$$\partial_s, \varphi_1^{-1}\partial_s\varphi_1, \ldots, \varphi_k^{-1}\partial_s\varphi_k.$$

Note that $\partial_s$ is obtained from the flat connection on $M \times G$ corresponding to $\mathcal{F}_A$, and we have that

$$\varphi_i^{-1}\partial_s\varphi_i = \partial_s \varphi_i^{-1}\mathcal{V}\varphi_i.$$

Now, let $A_k$ denote the standard $k$-simplex and define a map $\tilde{\mathcal{V}}: A_k \to \mathcal{A}$ by linearly extending the map taking the $i$th vertex to $\varphi_i^{-1}\mathcal{V}\varphi_i$. We
associate to this family of connections the family of operators \( \tilde{F}(t_0, \ldots, t_k) \), for \((t_0, \ldots, t_k) \in A_k\). This family defines a map \( F: A_k \to \mathcal{F}^{\text{sa}}(C^*(\mathcal{F})) \) as before. 

Next, one chooses paths from each \( \varphi \), to the identity operator, \( I \), in \( \mathcal{U}(\mathcal{H}_H, C^*(\mathcal{F})) \). We then obtain another map \( F': A_k \to \mathcal{F}^{\text{sa}}(C^*(\mathcal{F})) \) which agrees with \( F \) on \( \partial A_k \). Patching these maps together we obtain \( \tilde{F} = F \cup F' \in \pi_{k+1}(\mathcal{F}^{\text{sa}}(C^*(\mathcal{F}))) = K_0(C^*(\mathcal{F})) \). We will show in later paper that this defines a cocycle for the group homology of the gauge group with coefficients in \( K_0(C^*(\mathcal{F})) \).

\[ [\tilde{F}] \in H^1(\mathcal{F}, K_0(C^*(\mathcal{F})). \quad (8.8) \]

Pairing with cyclic cocycles will yield elements in \( H^*(\mathcal{F}, \mathbb{R}) \) which can be viewed as generalized higher spectral flow. They can be related to more familiar descriptions of the cohomology of the gauge group, (cf. [9, 26]). Moreover, this construction allows us to lift our invariants to algebraic K-theory, \( K_{\text{alg}}^*((C^*(\mathcal{F}))) \).

If we take \( k = 1 \) then the above construction provides an element \( [\tilde{F}] \in H^1(\mathcal{F}, K_0(C^*(\mathcal{F}))) \) and \( \text{Tr}_\mathcal{F}([\tilde{F}]) \in H^1(\mathcal{F}, \mathbb{R}) \) is analogous to the generator described by Singer in [26].

9. CONCLUDING REMARKS

1. The notion of “higher eta invariant” appears in the title of the paper. Our reasons for using it are based on the theory, introduced in [7], which yields a representation of the relative eta invariant associated to the pair consisting of a self-adjoint elliptic operator and a trivialized flat vector bundle. The ingredients are a foliated compact manifold obtained from the flat vector bundle and a leafwise elliptic operator obtained from the original operator. The index of this operator is an element of \( K_0(C^*(\mathcal{F})) \) and the relative eta invariant is obtained by applying the trace, \( \text{Tr} \in HC^{\text{sa}}(C^*(\mathcal{F})) \). We have shown in the present paper that pairing with certain other cyclic cocycles yields invariants with spectral properties similar to the relative eta invariant.

2. There have been other proposals for notions of higher eta invariant. We mention here one discussed by J. Lott in [16] (see also F. Wu [31]). In a broad sense that invariant and the one described in the present paper are obtained by the same process. The setting is the following. Let \( M \) be a closed smooth, odd-dimensional Spin\(^c\) manifold. Let \( F: \pi_1(M) \) denote the Dirac operator on \( M \). Assume that we have a homomorphism \( \pi: \Gamma \to \text{Diff}^+ (F) \), where \( F \) is a compact oriented manifold, and an equivariant map \( \theta: \tilde{M} \to F \). We have the suspension operation yielding \( S_{\pi, \theta}(\tilde{\varphi}) \in KK^1(C(M \times F), C(F), \pi_\text{sa} \Gamma) \).
Consider now the following two cases. One can take $F = \#_s$, as is done in the present paper, and obtain an element of $K_*(C^*(\mathcal{F}_s))$ to which one applies a cyclic cocycle to obtain a numerical invariant. Alternatively, one can take $F = pt$ as in Lott's work [16] and obtain an element of $K_*(C^*(\pi_1(M)))$ and pair with cyclic cocycles on $\mathcal{F} \subseteq C^*(\pi_1(M))$. At this point both approaches must analyze the respective K-theory elements using a Chern character. In the present paper, as in [7], $S_{\alpha, \nu}(\mathcal{F})$ is represented by a finitely summable Fredholm module. Thus, we are able to use Connes' Chern character formulas which involve commutators. In Lott's case, the Fredholm modules may only be theta-summable so it is necessary to use a Chern character based on the JLO cocycle. We use an averaging process to relate this to the eta invariant while in Lott's case it is obtained directly from the formulas. There should be a direct approach relating the two cyclic cocycles.

3. One can obtain a topological formula for the pairing $\langle [\mathcal{F}] \otimes [\mathcal{F}_s], [c_H] \rangle$ by applying the Atiyah–Patodi–Singer theorem to the twisted Dirac operator and unitary multiplier of (7.1). The result is

$$
\langle [\mathcal{F}] \otimes [\mathcal{F}_s], [c_H] \rangle = \int_{g \in G/H} \phi^{-1}(\text{ch}(\mathcal{F}_M \#_s \mathcal{F}_G/H)) \wedge T \text{ch}(mp, \kappa) \wedge T d(M \times G/H) dA.
$$

Topological formulas for pairings with a large class of cocycles have been obtained by Jiang [11]. Connes has shown how to associate a cocycle on the foliation algebra to a cocycle in the double complex $C^*(\mathcal{F}, \mathcal{O}_\mathcal{F}(G))$, [4]. It is possible to apply Jiang’s work to identify the elements in this double complex which correspond to the cocycles $c_M$. His results can then be used to yield another topological formula for the pairing. It is not yet known if the pairings by all of the cocycles considered by Jiang have a spectral interpretation.

In [21], Nistor studies the pairing $\langle [\mathcal{F}_s], [c] \rangle$ for general cyclic cohomology classes, $[c] \in HC^*(C_c^*(\mathcal{F}_s))$. He also obtains explicit formulas.

REFERENCES


