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Congruences on Inverse Semigroups

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1. Introduction and Summary

The usual approach to congruences on inverse semigroups is to first notice that a congruence ρ on an inverse semigroup S is completely determined by the family of its classes containing idempotents. This family, called a kernel normal system, has been characterized abstractly.

We adopt a different approach: Scheiblich [19] proved that ρ is also uniquely determined by its restriction to the idempotents, called the trace of ρ, and the union of all its classes containing idempotents, called the kernel of ρ. The results proved here indicate that this way of looking at congruences has ceratin advantages. This leads, in a natural way, to the congruence θ on the lattice Λ of all congruences on S introduced by Reilly and Scheiblich [17]. In our terminology $\rho\theta\tau$ if and only if ρ and τ have the same trace. The congruence θ gives us a first decomposition of the lattice Λ that is useful in gaining some overview of the congruences on S. For example, the θ-class of the equality relation consists of all idempotent separating congruences, and the θ-class of the universal relation consists of all group congruences. Both of these θ-classes have attracted considerable attention. The last cited authors also give the form of the least and the greatest element of each θ-class. Hence for a given congruence ρ on S, one can identify the least and the greatest element of the θ-class containing ρ. As special cases, one obtains the greatest idempotent separating congruence and the least group congruence.

Besides some preliminaries on terminology, we summarize in Section 2 the results of Reilly and Scheiblich [17] that will be needed in the paper.

We begin in Section 3 by characterizing the congruence θ in several ways in terms of congruences and the $\mathscr{H}$-equivalence. We draw several interesting consequences of this result concerning θ-classes and their least and greatest elements.

In Section 4, we consider the trace and the kernel of a congruence ρ. These results are based on the fact that ρ is determined by its trace and its kernel. We establish necessary and sufficient conditions on a pair (ξ, K) for the existence of a

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congruence ρ on S such that ξ is the trace and K is the kernel of ρ. We call (ξ, K) a congruence pair and explore the relationship between congruences and congruence pairs.

We collect in Section 5 various results concerning kernels and congruences. Group congruences are taken up in Section 6 where it is proved that the mapping $\rho \to \rho \vee \sigma$, where σ is the least group congruence on S, is a homomorphism of Λ onto the lattice of all group congruences on S.

For a categorical ideal I, we characterize in Section 7 the θ-class of the greatest congruence on S having I as a class in terms of primitive homomorphic images. The θ-class of a Rees congruence of any ideal is also described.

We characterize in Section 8 all congruences on a P-semigroup. These are expressed directly in terms of parameters figuring in the definition of a P-semigroup. In Section 9 we construct all congruences on a polycyclic monoid. Some concluding remarks in Section 10 round up our study.

2. Preliminaries

We consider only congruences on inverse semigroups. As a background on inverse semigroups, we recommend the books by Clifford and Preston [2, Chap. 7]) and Howie [8, Chap. 5]), to whose notation and terminology we generally adhere.

We record only the most frequently used notation and nomenclature. Let S be any semigroup. If ρ is a congruence on S and $a \in S$, then $a\rho$ is the ρ-class containing a. If ρ and τ are congruences on S and $\rho \supseteq \tau$, then ρ/τ is the congruence on S/ρ defined by

$$a\tau \ \rho/\tau \ b\tau \Leftrightarrow a \ \rho \ b \qquad (a, b \in S).$$

If S/ρ is a group (semilattice of groups, etc.), then ρ is a *group* (*semilattice of groups*, etc.) *congruence* on S. A subset K of S is *saturated* for a congruence ρ on S if K is the union of some ρ-classes. A congruence ρ on S is *idempotent separating* if each ρ-class contains at most one idempotent. Following Vagner [19], we call an inverse semigroup S an *antigroup* if the equality relation is the only idempotent separating congruence on S; these semigroups are usually called fundamental inverse semigroups. For any sets A and B, we write $A \backslash B = \{a \in A \mid a \notin B\}$. The equality relation on any set is denoted by ι.

We now summarize some of the results of Reilly and Scheiblich [17] for inverse semigroups in somewhat different guise. On the lattice Λ of all congruences on an inverse semigroup S, the relation θ given by

$$\rho \, \theta \, \tau \Leftrightarrow \rho|_E = \tau|_E$$

is a congruence. Let $[\rho]$ stand for the θ-class containing ρ. Then the mapping $\rho \to [\rho]$ is a complete lattice homomorphism of Λ onto Λ/θ.

A congruence ξ on the semilattice E of idempotents of S is *normal* if

$$e \,\xi\, f \Rightarrow a^{-1}ea \,\xi\, a^{-1}fa \qquad (e, f \in E,\ a \in S).$$

For a normal congruence ξ on E, the relation $\xi^{\max}$ and $\xi^{\min}$ defined on S by

$$a\, \xi^{\max}\, b \Leftrightarrow a^{-1}ea \,\xi\, b^{-1}eb \qquad \text{for all } e \in E,$$
$$a\, \xi^{\min}\, b \Leftrightarrow ae = be \qquad \text{for some } e \in E, \quad e \,\xi\, a^{-1}a \,\xi\, b^{-1}b$$

are the greatest and the least elements of $[\rho]$, respectively, where $\rho \in \Lambda$ is such that $\rho\,|_E = \xi$.

It follows from this that for any $\rho \in \Lambda$, the relations $\rho_{\max}$ and $\rho_{\min}$ defined on S by

$$a\, \rho_{\max}\, b \Leftrightarrow a^{-1}ea \,\rho\, b^{-1}eb \qquad \text{for all } e \in E,$$
$$a\, \rho_{\min}\, b \Leftrightarrow ae = be \qquad \text{for some } e \in E, \quad e \,\rho\, a^{-1}a \,\rho\, b^{-1}b$$

are the greatest and the least elements of $[\rho]$, respectively. This can be easily verified directly.

For ρ the equality relation, we obtain $\rho_{\max} = \mu$, the greatest idempotent separating congruence on S, and for ρ the universal relation, we get $\rho_{\min} = \sigma$, the least group congruence on S. Normal congruences were used by Eberhart and Selden [3] to characterize the congruences on free one parameter semigroups and by Scheiblich [18] for congruences on symmetric inverse semigroups. The paper of Green [4] contains an extensive discussion of congruences on inverse semigroups.

We fix the notation S, E, Λ, θ, $[\rho]$, μ, σ as introduced above. However, we will denote by μ and σ the greatest idempotent separating and the least group congruences on any semigroup except when there is a need to emphasize the semigroup in question.

3. θ-Classes

We will first characterize the relation θ in several ways and then deduce some consequences of this result. This will be followed by a closer look at the structure of a θ-class.

THEOREM 3.1. *The following statements concerning congruences ρ and τ on S are equivalent.*

(i) $\rho\, \theta\, \tau$.

(ii) $\rho \subseteq \tau_{\max}$, $\tau_{\max}/\rho = \mu_{S/\rho}$.

(iii) $a\rho\, \mu\, b\rho \Leftrightarrow a\tau\, \mu\, b\tau \ (a, b \in S)$.

(iv) $a\rho\, \mathscr{H}\, b\rho \Leftrightarrow a\tau\, \mathscr{H}\, b\tau \ (a, b \in S)$.

(v) *$\rho \cap \tau\,|_{e\rho}$ and $\rho \cap \tau\,|_{e\tau}$ are group congruences $(e \in E)$.*

(vi) *$\rho/\rho \cap \tau$ and $\tau/\rho \cap \tau$ are idempotent separating congruences.*

Proof. (i) $\Rightarrow$ (ii). First note that $\rho_{\max} = \tau_{\max}$ so that $\rho \subseteq \tau_{\max}$. For any $a, b \in S$, we have

$$\begin{aligned} a\rho\ \rho_{\max}/\rho\ b\rho &\Leftrightarrow a\ \rho_{\max}\ b \\ &\Leftrightarrow a^{-1}ea\ \rho\ b^{-1}eb && \text{for all } e \in E \\ &\Leftrightarrow (a\rho)^{-1}(e\rho)(a\rho) = (b\rho)^{-1}(e\rho)(b\rho) && \text{for all } e\rho \in E_{S/\rho} \\ &\Leftrightarrow a\rho\ \mu_{S/\rho}\ b\rho. \end{aligned}$$

(ii) $\Rightarrow$ (i). Observe that $\rho\,|_E \subseteq \tau_{\max}\,|_E = \tau\,|_E$. Further, for any $e, f \in E$, we have

$$\begin{aligned} e\,\tau\, f &\Rightarrow e\ \tau_{\max} f \Rightarrow e\rho\ \tau_{\max}/\rho\, f\rho \Rightarrow e\rho\ \mu_{S/\rho}\, f\rho \\ &\Rightarrow e\rho = f\rho \Rightarrow e\,\rho\, f, \end{aligned}$$

and thus also $T\,|_E \subseteq P\,|_E$.

(i) $\Rightarrow$ (iii). For any $a, b \in S$, we have

$$\begin{aligned} a\rho\ \mu\ b\rho &\Leftrightarrow (a\rho)^{-1}(e\rho)(a\rho) = (b\rho)^{-1}(e\rho)(b\rho) && \text{for all } e \in E \\ &\Leftrightarrow a^{-1}ea\ \rho\ b^{-1}eb && \text{for all } e \in E \\ &\Leftrightarrow a^{-1}ea\ \tau\ b^{-1}eb && \text{for all } e \in E \\ &\Leftrightarrow (aT^{-1})(e\tau)(a\tau) = (b\tau)^{-1}(e\tau)(b\tau) && \text{for all } e \in E \\ &\Leftrightarrow a\tau\ \mu\ b\tau. \end{aligned}$$

(iii) $\Rightarrow$ (i). For any $e, f \in E$, we obtain

$$\begin{aligned} e\,\rho\, f &\Leftrightarrow e\rho = f\rho \Leftrightarrow e\rho\ \mu\ f\rho \Leftrightarrow e\tau\ \mu\ f\tau \\ &\Leftrightarrow e\tau = f\tau \Leftrightarrow e\,\tau\, f. \end{aligned}$$

(i) $\Rightarrow$ (iv). Let $a, b \in S$ and assume that $a\rho\ \mathscr{H}\ b\rho$. Then $(a\rho)(a\rho)^{-1} = (b\rho)(b\rho)^{-1}$ and $(a\rho)^{-1}(a\rho) = (b\rho)^{-1}(b\rho)$, which implies that $aa^{-1}\,\rho\, bb^{-1}$ and $a^{-1}a\,\rho\, b^{-1}b$. The hypothesis implies that $aa^{-1}\,\tau\, bb^{-1}$ and $a^{-1}a\,\tau\, b^{-1}b$, which evidently implies that $a\tau\ \mathscr{H}\ b\tau$. By symmetry, $a\tau\ \mathscr{H}\ b\tau$ implies $a\rho\ \mathscr{H}\ b\rho$.

(iv) $\Rightarrow$ (i). Let $e, f \in E$ and assume that $e\,\rho\, f$. Then $e\rho\ \mathscr{H}\ f\rho$ so that, by hypothesis, $e\tau\ \mathscr{H}\ f\tau$, and hence $e\,\tau\, f$. Symmetrically, $e\,\tau\, f$ implies $e\,\rho\, f$.

(i) $\Rightarrow$ (v). It suffices to prove that $\rho\,\theta\,\tau$ and $\rho \subseteq \tau$ implies that $\rho\,|_{e\tau}$ is a group congruence for every $e \in E$. Hence suppose $\rho\,\theta\,\tau$, $\rho \subseteq \tau$, and let $e \in E$. If $f \in e\tau \cap E$, then $e\,\tau\,f$ and thus $e\,\rho\,f$. Hence all idempotents of $e\tau$ are ρ-related, which means that $\rho\,|_{e\tau}$ is a group congruence.

(v) $\Rightarrow$ (i). Let $e, f \in E$ and assume that $e\,\rho\,f$. Then $f \in e\rho$ and $\rho \cap \tau\,|_{e\rho}$ is a group congruence, so that $e\,\rho \cap \tau f$. Hence $\rho\,|_E \subseteq \tau\,|_E$, and symmetrically $\rho\,|_E \supseteq \tau\,|_E$.

(i) $\Rightarrow$ (vi). For $e, f \in E$, we obtain

$$\begin{aligned} e(\rho \cap \tau) \quad \rho/\rho \cap \tau \quad f(\rho \cap \tau) &\Rightarrow e\,\rho\,f \Rightarrow e\,\tau\,f \Rightarrow e\,\rho \cap \tau\,f \\ &\Rightarrow e(\rho \cap \tau) = f(\rho \cap \tau) \end{aligned}$$

which shows that $\rho/\rho \cap \tau$ is idempotent separating.

(vi) $\Rightarrow$ (i). For $e, f \in E$, we have

$$\begin{aligned} &e\,\rho\,f \Rightarrow e(\rho \cap \tau) \quad \rho/\rho \cap \tau \quad f(\rho \cap \tau) \Rightarrow e(\rho \cap \tau) = f(\rho \cap \tau) \\ &\quad \Rightarrow e\,\rho \cap \tau\,f \Rightarrow e\,\tau\,f, \end{aligned}$$

which shows that $\rho\,|_E \subseteq \tau\,|_E$. Symmetrically, we also have $\rho\,|_E \supseteq \tau\,|_E$.

Corollary 3.2. *A congruence ρ on S is the greatest element of its θ-class if and only if S/ρ is an antigroup.*

Proof. Using the equivalence of (i) and (ii) in Theorem 3.1, we obtain

$$\rho = \rho_{\max} \Leftrightarrow \rho_{\max}/\rho = \iota \Leftrightarrow \mu_{S/\rho} = \iota \Leftrightarrow S/\rho \text{ is an antigroup.}$$

Proposition 3.3. *Let $\rho, \tau \in \Lambda$. Then $\tau = \rho_{\min}$ if and only if $\rho \supseteq \tau$ and for every $e \in E$, $\tau\,|_{e\rho}$ is the least group congruence on $e\rho$.*

Proof. Necessity. Let $e \in E$ and $a, b \in e\rho$. Note that

$$a\,\tau\,b \Leftrightarrow af = bf \qquad \text{for some } f \in E, \quad f\,\rho\,a^{-1}a\,\rho\,b^{-1}b, \tag{1}$$

and denoting by σ_e the least group congruence on $e\rho$, we have

$$a\,\sigma_e\,b \Leftrightarrow af = bf \qquad \text{for some } f \in e\rho \cap E. \tag{2}$$

Let (1) hold. Then $e\,\rho\,a$ implies $e\,\rho\,a^{-1}a$ so that $e\,\rho\,f$. Thus $f \in e\rho \cap E$ and (2) holds. Conversely, let (2) hold. Then $f\,\rho\,a$ and $f\,\rho\,b$, which implies that $f\,\rho\,a^{-1}a\,\rho\,b^{-1}b$ and (1) holds. Consequently $\tau = \sigma_e$.

Sufficiency. By Theorem 3.1, we have $\rho\,\theta\,\tau$ and hence $\rho_{\min} \subseteq \tau$. Thus for any $e \in E$, $e\rho_{\min} \subseteq e\tau$. Conversely, let $a \in e\tau$. Then $af = ef$ for some $f \in e\tau \cap E$ since $\tau|_{e\rho}$ is the least group congruence on $e\rho$. In particular $e \mathrel{\rho} a$, which implies that $e \mathrel{\rho} a^{-1}a$. Since also $e \mathrel{\rho} f$, we obtain $a \mathrel{\rho_{\min}} e$, and hence $a \in e\rho_{\min}$. Consequently $e\tau = e\rho_{\min}$. Since $e \in E$ is arbitrary, it follows that $\tau = \rho_{\min}$.

4. Traces and Kernels

We establish here the main characterization theorem for congruences on an arbitrary inverse semigroup. In order to facilitate our discussion, we introduce the following concepts.

Definition 4.1. Let ρ be a congruence on an inverse semigroup S with the semilattice of idempotents E. The restriction $\rho|_E$ is the *trace* of ρ, to be denoted by $\operatorname{tr}\rho$, and the set

$$\ker \rho = \{a \in S \mid a \mathrel{\rho} e \text{ for some } e \in E\}$$

is the *kernel* of ρ.

We consider below necessary and sufficient conditions on a pair (ξ, K) in order that there exists a congruence ρ on S such that $\operatorname{tr}\rho = \xi$ and $\ker\rho = K$. Such conditions were first given by Scheiblich [19], and simpler ones by Green [4, Proposition 3.9]. Recall that a subsemigroup K of S is *full* if $E \subseteq K$, and *self-conjugate* if $a^{-1}Ka \subseteq K$ for all $a \in S$.

Definition 4.2. Let S be an inverse semigroup with the semilattice E of idempotents. A full, self-conjugate inverse subsemigroup K of S is a *normal subsemigroup*. If, in addition, ξ is a normal congruence on E satisfying

(i) $ae \in K$, $e \mathrel{\xi} a^{-1}a \Rightarrow a \in K$,

(ii) $a \in K \Rightarrow a^{-1}ea \mathrel{\xi} a^{-1}ae$,

for all $a \in S$, $e \in E$, then (ξ, K) is a *congruence pair* for S. In such a case, define a relation $\kappa_{(\xi,K)}$ on S by

$$a \mathrel{\kappa_{(\xi,K)}} b \Leftrightarrow a^{-1}a \mathrel{\xi} b^{-1}b,\ ab^{-1} \in K.$$

Note that if $E \subseteq K \subseteq S$, then K is a normal subsemigroup of S if and only if it is self-conjugate and $a, b \in K$ implies $ab^{-1} \in K$. Thus a normal subsemigroup of S is strongly reminiscent of a normal subgroup of a group, and $\kappa_{(\xi,K)}$ of the congruence induced by it. In order to show that $\kappa_{(\xi,K)}$ is a congruence, we first prove a technical lemma.

Lemma 4.3. *If (ξ, K) is a congruence pair for S, then*

(i) $aeb \in K,\ e \,\xi\, a^{-1}a \Rightarrow ab \in K,$

(ii) $ab \in K \Rightarrow aeb \in K,$

(iii) $ab^{-1} \in K,\ a^{-1}a \,\xi\, b^{-1}b \Rightarrow a^{-1}ea \,\xi\, b^{-1}eb,$

for all $a, b \in S,\ e \in E.$

Proof. (i) Let $aeb \in K$ and $e \,\xi\, a^{-1}a$ where $e \in E,\ a, b \in S$. Then

$$\begin{aligned} aeb &= ae(bb^{-1})\, b = ab(b^{-1}eb), \\ (ab)^{-1}(ab) &= b^{-1}(a^{-1}a)\, b \,\xi\, b^{-1}eb, \end{aligned} \tag{1}$$

which by Definition 4.2(i) yields $ab \in K$.

(ii) For $ab \in K$ and $e \in E$, we have $b^{-1}eb \in E$ so that $aeb \in K$ by (1) since K is full.

(iii) Let $ab^{-1} \in K,\ a^{-1} \,\xi\, b^{-1}b,\ e \in E$. Then by Definition 4.2(ii), we have $(ab^{-1})^{-1} e(ab^{-1}) \,\xi\, (ab^{-1})^{-1}(ab^{-1})\, e$ which implies

$$b(a^{-1}ea)\, b^{-1} \,\xi\, b(a)\, b^{-1}e \,\xi\, b(b^{-1}b)\, b^{-1}e = bb^{-1}e,$$

whence

$$a^{-1}ea = (a^{-1}a)(a^{-1}ea)(a^{-1}a) \,\xi\, (b^{-1}b)(a^{-1}ea)(b^{-1}b) \,\xi\, (b^{-1})(bb^{-1}e)\, b = b^{-1}eb,$$

as required.

The main characterization theorem for congruences follows.

THEOREM 4.4. *If* (ξ, K) *is a congruence pair for an inverse semigroup* S, *then* $\kappa_{(\xi,K)}$ *is a congruence on* S *with trace* ξ *and kernel* K. *Conversely, if* ρ *is a congruence on* S, *then* $(\operatorname{te}\rho, \ker\rho)$ *is a congruence pair for* S *and* $\rho = \kappa_{(\operatorname{tr}\rho, \ker\rho)}$.

Proof. Let (ξ, K) be a congruence pair for S, and let $\kappa = \kappa_{(\xi,K)}$. It follows immediately that κ is reflexive and symmetric. Let $a \,\kappa\, b$ and $b \,\kappa\, c$. Then $a^{-1}a \,\xi\, b^{-1}b \,\xi\, c^{-1}c$ and $ab^{-1}, bc^{-1} \in K$. Hence $a(b^{-1}b)\, c^{-1} \in K$ which together with $b^{-1}b \,\xi\, a^{-1}a$ implies $ac^{-1} \in K$ by Lemma 4.3(i). Thus $a \,\kappa\, c$, and κ is transitive.

Next let $a \,\kappa\, b$ and $c \in S$. Then

$$(ac)^{-1}(ac) = c^{-1}(a^{-1}a)\, c \,\xi\, c^{-1}(b^{-1}b)\, c = (bc)^{-1}(bc),$$

since $a^{-1}a \,\xi\, b^{-1}b$ and ξ is a normal congruence, and

$$(ac)(bc)^{-1} = a(cc^{-1})\, b^{-1} \in K$$

by Lemma 4.3(ii) since $ab^{-1} \in K$. Consequently $ac \,\kappa\, bc$. Further,

$$(ca)^{-1}(ca) = a^{-1}(c^{-1}c)\, a \,\xi\, b^{-1}(c^{-1}c)\, b = (bc)^{-1}(bc)$$

by Lemma 4.3(iii) since $a^{-1}a \,\xi\, b^{-1}b$, and

$$(ca)\,(cb)^{-1} = c(ab^{-1})\,c^{-1} \in K$$

since $ab^{-1} \in K$ and K is self-conjugate. Therefore κ is a congruence on S.

It is obvious that $\operatorname{tr}\kappa = \xi$. If $a \,\kappa\, e$ for $e \in E$, then $a^{-1}a \,\xi\, e$, $ae \in K$ which by Definition 4.2(i) yields $a \in K$. Conversely, if $a \in K$, then $a \,\kappa\, a^{-1}a$. Consequently $\ker\kappa = K$.

Conversely, let ρ be a congruence on S. A simple verification shows that $(\operatorname{tr}\rho, \ker\rho)$ is a congruence pair for S. Let $a \,\kappa_{(\operatorname{tr}_\rho, \ker_\rho)}\, b$. Then $a^{-1}a \,\rho\, b^{-1}b$ and $ab^{-1} \,\rho\, e$ for some $e \in E$. Hence

$$\begin{aligned} a &= a(a^{-1}a)\,\rho\,(ab^{-1})\,b\,\rho\,eb, \\ b &= b(b^{-1}b)\,\rho\,(ba^{-1})\,a\,\rho\,ea, \end{aligned}$$

whence $a \,\rho\, eb \,\rho\, e(ea) = ea \,\rho\, b$. Thus $\kappa_{(\operatorname{tr}_\rho, \ker_\rho)} \subseteq \rho$, the opposite inclusion is trivial.

Other characterizations of congruences on inverse semigroups were given by Scheiblich [19] and Green [4].

COROLLARY 4.5. *Let $\mathscr{C}$ be the set of all congruence pairs for S ordered by*

$$(\xi, K) \leqslant (\xi', K') \Leftrightarrow \xi \subseteq \xi', K \subseteq K'.$$

Then the mappings

$$(\xi, K) \to \kappa_{(\xi, K)} \qquad \textit{and} \qquad \rho \to (\operatorname{tr}\rho, \ker\rho)$$

are mutually inverse lattice isomorphisms of $\mathscr{C}$ onto Λ and of Λ onto $\mathscr{C}$, respectively.

5. SUPPLEMENTS

Since the intersection of normal congruences is a normal congruence, the set Φ of all normal congruence on E is a complete lattice with meet equal to the intersection. We denote by $\vee$ the join both in Λ and in Φ. The first two parts of the next result are due to Reilly and Scheiblich [17, Theorem 5.1] and the third part to Green [4, Theorem 3.4].

PROPOSITION 5.1. *For any family $\mathscr{F}$ of congruences on S, we have*

$$\operatorname{tr}\bigcap_{\rho\in\mathscr{F}}\rho = \bigcap_{\rho\in\mathscr{F}}\operatorname{tr}\rho, \qquad \operatorname{tr}\bigvee_{\rho\in\mathscr{F}}\rho = \bigvee_{\rho\in\mathscr{F}}\operatorname{tr}\rho, \qquad \ker\bigcap_{\rho\in\mathscr{F}}\rho = \bigcap_{\rho\in\mathscr{F}}\ker\rho.$$

For a congruence ξ on E, the least normal congruence containing it does not seem to admit a simple expression. However, the greatest normal congruence contained in ξ is given by

$$e\,\hat{\xi}\,f \Leftrightarrow a^{-1}ea\ \xi\ a^{-1}fa \qquad \text{for all } a \in S,$$

which is easily verified.

LEMMA 5.2. *Let $\rho, \tau \in \Lambda$ be such that $\rho \subseteq \tau$ and $\rho\ \theta\ \tau$. Then*

$$a\ \rho\ b \Leftrightarrow a\ \tau\ b,\ ab^{-1} \in \ker \rho \qquad (a, b \in S).$$

Proof. The direct part is trivial. For the converse, assume that $a\ \tau\ b$ and $ab^{-1} \in \ker \rho$. Then $a^{-1}\ \tau\ b^{-1}$ and hence $a^{-1}a\ \tau\ b^{-1}b$. Since $\rho\ \theta\ \tau$, this implies that $a^{-1}a\ \rho\ b^{-1}b$. Further, $ab^{-1} \in \ker \rho$ implies $ab^{-1}\ \rho\ e$ for some $e \in E$. Then $a(b^{-1}b)\ \rho\ eb$ so that $a\ \rho\ eb$ since $b^{-1}b\ \rho\ a^{-1}a$. Also $ba^{-1}\ \rho\ e$ which implies $ba^{-1}a\ \rho\ ea$ and hence $b\ \rho\ ea$. Consequently

$$a\ \rho\ eb\ \rho\ e(ea) = ea\ \rho\ b,$$

as required.

This lemma and its proof immediately yield

COROLLARY 5.3. *For any $\rho \in \Lambda$, we have*

$$\begin{aligned} a\ \rho\ b &\Leftrightarrow a\ \rho_{\max}\ b,\ ab^{-1} \in \ker \rho \\ &\Leftrightarrow a^{-1}a\ \mathrm{tr}\ \rho\ b^{-1}b,\ ab^{-1} \in \ker \rho. \end{aligned}$$

PROPOSITION 5.4. *For $\rho \in \Lambda$, the following statements are true.*

(i) *S/ρ is a semilattice of groups if and only if $S/\rho_{\max}$ is a semilattice.*

(ii) *S/ρ is a bisimple inverse ω-semigroup if and only if $S/\rho_{\max}$ is a bicyclic semigroup.*

Proof. Note that Theorem 3.1 implies the statement

$$a\rho\ \mu\ b\rho \Leftrightarrow a\rho_{\max} = b\rho_{\max}. \tag{1}$$

If S/ρ is either a semilattice of groups or a bisimple inverse ω-semigroup, then $\mu = \mathscr{H}$ and hence (1) gives $(S/\rho)/\mathscr{H} \cong S/\rho_{\max}$, which proves the direct implications. Let $S/\rho_{\max}$ be a semilattice. For any $a \in S$, $e \in E$, we have by Corollary 5.3

$$ae\ \rho\ ea \Leftrightarrow ae\ \rho_{\max}\ ea, \qquad (ae)\,(ea)^{-1} \in \ker \rho.$$

The first part on the right is true since $S/\rho_{\max}$ is a semilattice. The second part is true since $(ae)(ea)^{-1} = (aea^{-1})\, e \in E$. Hence $ae \mathrel{\rho} ea$ and S/ρ is a semilattice of groups. Finally, let $S/\rho_{\max}$ be a bicyclic semigroup. Then Theorem 3.1 yields

$$a\rho \mathrel{\mathscr{H}} b\rho \Leftrightarrow a\rho_{\max} = b\rho_{\max}\,.$$

Hence $(S/\rho)/\mathscr{H}$ is a bicyclic semigroup. A simple argument shows that S/ρ is a bisimple inverse ω-semigroup.

As a consequence, we obtain a slight strengthening of a recent result of Mills [13] and Hardy and Tirasupa [6].

COROLLARY 5.5. *Let η denote the least semilattice congruence on S. Then $\eta_{\min}$ is the least semilattice of groups congruence on S.*

Proof. First, Proposition 5.4 implies that $S/\eta_{\min}$ is a semilattice of groups. Let ρ be any semilattice of groups congruence on S. Then $S/\rho_{\max}$ is a semilattice, again by Proposition 5.4, so that $\eta \subseteq \rho_{\max}$. Now let $a \mathrel{\eta_{\min}} b$. Then $ae = be$ for some $e \in E$ such that $e \mathrel{\eta} a \mathrel{\eta} b$. Hence $e \mathrel{\rho_{\max}} a$, which implies that $e \mathrel{\rho_{\max}} a^{-1}a$ and thus also $e \mathrel{\rho} a^{-1}a$. Analogously we have $e \mathrel{\rho} b^{-1}b$. Now $ae = be$ implies $ae \mathrel{\rho} be$, which together with $e \mathrel{\rho} a^{-1}a \mathrel{\rho} b^{-1}b$ yields $a \mathrel{\rho} b$. Thus $\eta_{\min} \subseteq \rho$, which establishes the minimality of $\eta_{\min}$ as a semilattice of groups congruence.

PROPOSITION 5.6. *For any $\rho \in \Lambda$, we have*

$$\ker \rho_{\max} = \{a \in S \mid ae \mathrel{\rho} ea \text{ for all } e \in E\},$$
$$\ker \rho_{\min} = \{a \in S \mid ae = e \text{ for some } e \in E,\ e \mathrel{\rho} a^{-1}a\}.$$

Proof. Let $a \in \ker \rho_{\max}$. Then $a \mathrel{\rho_{\max}} f$ for some $f \in E$ and thus $a^{-1}ea \mathrel{\rho} ef$ for all $e \in E$. For $e = aa^{-1}$, we get $a^{-1}a \mathrel{\rho} aa^{-1}f$ which implies $a \mathrel{\rho} a^2a^{-1}f$ and thus $a \mathrel{\rho} af$. For $e = f$, we have $a^{-1}fa \mathrel{\rho} f$ so that $fa \mathrel{\rho} af \mathrel{\rho} a$. But then $ea = a(a^{-1}ea) \mathrel{\rho} (af)\, e = ae$. Conversely, assume that $ae \mathrel{\rho} ea$ for all $e \in E$. Then $a^{-1}ea \mathrel{\rho} (a^{-1}a)\, e$ for all $e \in E$, which shows that $a \mathrel{\rho_{\max}} a^{-1}a$ and $a \in \ker \rho_{\max}$.

For the second equality, we have

$$\begin{aligned} a \in \ker \rho_{\min} &\Leftrightarrow a \mathrel{\rho_{\min}} f && \text{for some } f \in E \\ &\Leftrightarrow ae = fe && \text{for some } f \in E \quad \text{and} \quad e \in E, \quad e \mathrel{\rho} a^{-1}a \mathrel{\rho} f \\ &\Leftrightarrow ae = e && \text{for all } e \in E, \quad e \mathrel{\rho} a^{-1}a. \end{aligned}$$

COROLLARY 5.7. *For any normal congruence ξ on E, we have*

$$\ker \xi^{\max} = \{a \in S \mid a^{-1}ea \mathrel{\xi} a^{-1}ae \text{ for all } e \in E\}.$$

For ρ the equality relation on S, we obtain $\mu = \rho_{\max}$, the greatest idempotent separating congruence on S, and for ρ the universal relation on S, we get $\sigma = \rho_{\min}$, the least group congruence on S. They have the familiar form

$$\begin{aligned} a\,\mu\,b &\Leftrightarrow a^{-1}ea = b^{-1}eb && \text{for all } e \in E,\\ a\,\sigma\,b &\Leftrightarrow ae = be && \text{for some } e \in E, \end{aligned}$$

and their kernels are, respectively,

$$\begin{aligned} E\zeta &= \{a \in S \mid ae = ea \text{ for all } e \in E\},\\ E\omega &= \{a \in S \mid ae = e \text{ for some } e \in E\}. \end{aligned}$$

Here $E\zeta$ is the centralizer of E in S, and $E\omega$ is the closure of E in S relative to the partial order which we will discuss below. For an extensive discussion of these congruences, consult Howie [7], in particular cf. Theorem 3.2 with the next proposition.

Proposition 5.8. *A semigroup S is a subdirect product of a group and an antigroup if and only if $E\omega \cap E\zeta = E$.*

Proof. Let S be a subdirect product of a group G and an antigroup A, and let $(g, a) \in E\omega \cap E\zeta$. Then $(g, a) \in E\omega$, which implies $(g, a)(1, e) = (1, e)$ for some $e \in E_A$ so that $g = 1$. Further, $(g, a) \in E\zeta$ implies that $(g, a)(1, e) = (1, e)(g, a)$ for all idempotents e in A, and hence $ae = ea$ for all $e \in E_A$. Since A is an antigroup, we deduce that $a \in E_A$. We have shown that $(g, a) \in E$.

Conversely, if $E\omega \cap E\zeta = E$, then by the last assertion of Proposition 5.1 we have $\ker(\sigma \cap \mu) = E$ which evidently implies that $\sigma \cap \mu$ is the equality relation. But then S is a subdirect product of the group S/σ and the antigroup S/μ.

For the next result, we need a lemma.

Lemma 5.9. *Let ϕ be an idempotent separating homomorphism of S onto T. Then $a\,\mu\,b \Leftrightarrow a\phi\,\mu\,b\phi$ for all $a, b \in S$.*

Proof. The direct implication is obvious. Let $a\phi\,\mu\,b\phi$. Then $(a\phi)^{-1}(e\phi)(a\phi) = (b\phi)^{-1}(e\phi)(b\phi)$ for all $e \in E_S$, so that $(a^{-1}ea)\phi = (b^{-1}eb)\phi$ for all $e \in E_S$. Since ϕ is idempotent separating, we must have $a^{-1}ea = b^{-1}eb$ for all $e \in E_S$, and $a\,\mu\,b$.

As a consequence of [5, Theorem 13], this lemma remains valid if we substitute μ by Green's relations $\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$ or $\mathscr{J}$. It is easy to see that it is also valid for $\mathscr{D}$. In particular, $|S/\mathscr{K}| = |T/\mathscr{K}|$ for $\mathscr{K} = \mathscr{H}, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}$.

Proposition 5.10. *For every inverse semigroup S there exists a group G, a subdirect product T of S/μ and G, and an idempotent separating homomorphism ϕ of T onto S.*

Proof. Let S be an inverse semigroup. According to [11, Corollary 2.5], S is an idempotent separating homomorphic image of an E-unitary inverse semigroup T. That T is E-unitary means that $\ker \sigma = E_T$. Hence

$$\ker(\mu \cap \sigma) \subseteq \ker \sigma = E_T\,,$$

$$\operatorname{tr}(\mu \cap \sigma) \subseteq \operatorname{tr} \mu = \iota,$$

so that $\mu \cap \sigma$ is the equality relation, and T is a subdirect product of the antigroup T/μ and the group T/σ. It follows from Lemma 5.9 that $T/\mu \cong S/\mu$.

COROLLARY 5.11. *Every inverse semigroup is an idempotent separating homomorphic image of a subdirect product of a group and an antigroup.*

PROPOSITION 5.12. *For any $\rho \in \Lambda$ and $e \in E$, we have*

$$e\rho = e(\operatorname{tr} \rho)^{\max} \cap \ker \rho.$$

Proof. This formula is equivalent to

$$e\rho = e\rho_{\max} \cap \ker \rho.$$

Let $a \in e\rho_{\max} \cap \ker \rho$. Then $a\ \rho_{\max}\ e$ and $a\ \rho\ f$ for some $f \in E$. Then $a^{-1}a\ \rho\ e$ and $a^{-1}a\ \rho\ f$ so that $e\ \rho\ f$, whence $a\ \rho\ e$. Hence $e\ \rho_{\max} \cap \ker \rho \subseteq e\rho$, the opposite inclusion is obvious.

This result provides an explicit demonstration of the fact that the trace and the kernel determine the classes of the congruence containing the idempotents, and thus the entire congruence.

Inverse semigroups S have a *natural partial order* defined by

$$a \geqslant b \Leftrightarrow ab^{-1} = bb^{-1}.$$

For any subset K of S, the set

$$K\omega = \{a \in S \mid a \geqslant b \text{ for some } b \in K\}$$

is the *closure* of K in S. Subsets K of S for which $K\omega = K$ are *closed in S*.

The next result provides an alternative characterization of congruence pairs (cf. [19, Theorem 2.1]).

PROPOSITION 5.13. *Let ξ be a normal congruence on E and K be a normal subsemigroup of S.*

(i) *Condition* (i) *in Definition* 4.2 *is equivalent to: for all* $a \in S$, $a\xi^{\max} \cap K$ *is closed in* $a\xi^{\max}$.

(ii) *Condition* (ii) *in Definition* 4.2 *is equivalent to:* $K \subseteq \ker \xi^{\max}$.

Proof. (i) *Direct part.* Let $a \in S$, $b \in a\xi^{\max} \cap K$, $c \in a\xi^{\max}$, $c \geqslant b$. Then $c(b^{-1}b) = (cb^{-1})\, b = bb^{-1}b = b \in K$ and $b\, \xi^{\max}\, a\, \xi^{\max}\, c$, which implies $b^{-1}b\, \xi\, c^{-1}c$. Hence condition (i) in Definition 4.2 yields $c \in K$ so that $c \in a\xi^{\max} \cap K$.

Converse. Let $ae \in K$, $e\, \xi\, a^{-1}a$ where $e \in E$, $a \in S$. Then $e\, \xi^{\max}\, a^{-1}a$ and hence $ae\, \xi^{\max}\, a$, so that $ae \in a\xi^{\max} \cap K$. Further, $a(ae)^{-1} = (ae)\,(ae)^{-1}$, which says that $a \geqslant ae$. The hypothesis implies that $a \in K$.

(ii) This follows directly from Corollary 5.7.

Note that in terms of the natural partial order, we have for any $a, b \in S$, $\rho \in \Lambda$ that $a\, \rho_{\min}\, b$ is equivalent to the existence of $c \in S$ such that $a \geqslant c$, $b \geqslant c$, $a\, \rho\, b\, \rho\, c$.

Let Δ denote the lattice of all idempotent separating congruences on S. Green [4, Theorem 3.4] proved a result related to the following

PROPOSITION 5.14. *For any normal subsemigroup* K *of* S, *contained in* $E\zeta$ *define a relation* κ_K *on* S *by*

$$a\, \kappa_K\, b \Leftrightarrow a\, \mathscr{H}\, b,\ ab^{-1} \in K.$$

Then $\kappa_K \in \Delta$ *and* $\ker \kappa_K = K$. *Conversely, if* $\rho \in \Delta$, *then* $\ker \rho$ *is a normal subsemigroup of* S *contained in* $E\zeta$ *and* $\rho = \kappa_{\ker_\rho}$. *Define a function* ker *on* Δ *by*

$$\ker\colon \rho \rightarrow \ker \rho.$$

Then ker *is a complete lattice isomorphism of* Δ *onto the lattice of all normal subsemigroups of* S *contained in* $E\zeta$.

Proof. First note that idempotent separating congruences are precisely those whose traces coincide with the equality relation. For any normal subsemigroup K of S, and ξ the equality relation, condition (i) of Definition 4.2 is automatically satisfied. For $K \subseteq E\zeta = \ker \mu$, by Proposition 5.13, condition (ii) of Definition 4.2 is also satisfied. It remains to prove that $\kappa_K = \kappa_{(\iota, K)}$, and for this it suffices to show that $a^{-1}a = b^{-1}b$, $ab^{-1} \in K \subseteq E\zeta$ implies $aa^{-1} = bb^{-1}$. Indeed

$$\begin{aligned} aa^{-1} &= a(a^{-1}a)\, a^{-1} = a(b^{-1}b)\, a^{-1} = (ab^{-1})\,(bb^{-1})\,(ba^{-1}) \\ &= (bb^{-1})\, ab^{-1}ba^{-1} = (bb^{-1})\, aa^{-1} \end{aligned}$$

and analogously $bb^{-1} = (aa^{-1})\,(bb^{-1})$ so that $aa^{-1} = bb^{-1}$.

Since for any $\rho \in \Delta$, $\ker \rho$ is a normal subsemigroup contained in $\ker \mu = E\zeta$, in order to establish the second statement, it suffices to prove that the meet and the join in Δ coincide with those in the poset Σ of all subsemigroups of S. Let $\{K_\alpha\}_{\alpha\in A}$ be a family of normal subsemigroups of S contained in $E\zeta$. By Proposition 5.1, $\cap K_\alpha$ is a normal subsemigroup contained in $E\zeta$. The join $\vee K_\alpha$ in Σ consists of all $x_1x_2 \cdots x_n \in S$ such that $x_1, x_2, \ldots, x_n \in \cup K_\alpha$. It is immediate that $\vee K_\alpha$ is a full inverse subsemigroup of S contained in $E\zeta$. For any $k = x_1x_2 \cdots x_n$ with $x_i \in \cup K_\alpha$ and $a \in S$, we have

$$\begin{aligned} a^{-1}ka &= a^{-1}x_1x_2 \cdots x_na = a^{-1}[(aa^{-1})\,x_1]\,x_2 \cdots x_na \\ &= a^{-1}[x_1(aa^{-1})]\,x_2 \cdots x_na = (a^{-1}x_1a)\,a^{-1}x_2 \cdots x_na \\ &= \cdots = (a^{-1}x_1a)\,(a^{-1}x_2a) \cdots (a^{-1}x_na) \in \vee K_\alpha \end{aligned}$$

since all x_i are in the centralizer $E\zeta$ of E. Hence $\vee K_\alpha$ is self-conjugate and thus a normal subsemigroup of S.

6. Group Congruences

These form the θ-class of congruences whose trace is the universal relation on E. We denote by Γ the lattice of all group congruences on S. We will see below that there exists a homomorphism of Λ onto Γ which leaves Γ elementwise fixed. We start with characterizations of $\rho \vee \sigma$ where $\rho \in \Lambda$ and σ is the least element of Γ.

Proposition 6.1. *For ρ a congruence on S and a, b elements of S, each of the following statements is equivalent to $a\ \rho \vee \sigma\ b$.*

(i) *$ae\ \rho\ be$ for some $e \in E$.*

(ii) *$a(\ker \rho) \cap b(\ker \rho) \neq \varnothing$.*

(iii) *$aEb^{-1} \cap \ker \rho \neq \varnothing$.*

Proof. According to ([7], Theorem 3.9), $\rho \vee \sigma = \sigma\rho\sigma$. Assume first that $a\ \sigma\rho\sigma\ b$. Then $a\ \sigma\ x$, $x\ \rho\ y$, $y\ \sigma\ b$ for some $x, y \in S$. Then $af = xf$ and $yg = bg$ for some $f, g \in E$. Letting $e = fg$, we obtain $ae = xe\ \rho\ ye = be$ which proves (i).

Next assume that (i) holds and let $K = \ker \rho$. Then $(ae)\,(be)^{-1} \in K$ and thus $aeb^{-1} \in K$ which implies that $b^{-1}(aeb^{-1})\,b \in K$. But then $b(b^{-1}aeb^{-1}b) \in bK$, that is $(bb^{-1})\,(aa^{-1})\,(ae)\,(b) \in bK$. It follows that $ac \in bK$ where $c = [a^{-1}(bb^{-1})\,a]\,e(b^{-1}b) \in E$. Consequently $ac \in aK \cap bK$ and (ii) holds.

Now suppose that (ii) is valid, say $ak = bl$ where $k, l \in K$. Then $akk^{-1}b^{-1} = b(lk^{-1})\,b^{-1} \in K$ since $lk^{-1} \in K$. Hence $aEb^{-1} \cap K \neq \varnothing$ and (iii) holds.

Finally, assume that (iii) holds, say $aeb^{-1} \in K$ where $e \in E$. Then $aeb^{-1}\ \rho\ f$ for some $f \in E$. It follows that $ae(b^{-1}b)\ \rho\ fb = b(b^{-1}fb)$. Letting $g = e(b^{-1}fb)$, we obtain $ag \, \rho \, bg$ with $g \in E$, whence $a \, \sigma \, ag$, $ag \, \rho \, bg$, $bg \, \sigma \, b$. Consequently $a \, \sigma\rho\sigma \, b$, as required.

PROPOSITION 6.2. *For any $\rho \in \Lambda$, we have*

$$\begin{aligned}\ker(\rho \vee \sigma) &= \{a \in S \mid ae \, \rho \, e \text{ for some } e \in E\}\\ &= \{a \in S \mid a(\ker \rho) \cap \ker \rho \neq \varnothing\}\\ &= \{a \in S \mid aE \cap \ker \rho \neq \varnothing\}\\ &= (\ker \rho)\, \omega.\end{aligned}$$

Proof. The first three equalities follow directly from Proposition 6.1. According to [7, Theorem 3.9], we have $\rho \vee \sigma = \sigma\rho\sigma$. Let $a \in \ker(\sigma\rho\sigma)$. Then $a \, \sigma \, x$, $x \, \rho \, y$, $y \, \sigma \, e$ for some $x, y \in S$ and $e \in E$. It follows that $af = xf$ and $yg = eg$ for some $f, g \in E$. Letting $t = efg$, we obtain $at = xt$, $xt \, \rho \, yt$, $yt = t$ with $t \in E$. Hence $at \, \rho \, t$ and $at \in \ker \rho$. Since $a \geqslant at$, it follows that $a \in (\ker \rho)\, \omega$.

Conversely, let $a \in (\ker \rho)\, \omega$. Then $a \geqslant b$ for some $b \in \ker \rho$. Hence $ab^{-1} = bb^{-1}$ and $b \, \rho \, e$ for some $e \in E$. Thus $a(b^{-1}b) = b(b^{-1}b)$ so that $a \, \sigma \, b$, $b \, \rho \, e$, $e \, \sigma \, e$ which shows that $a \, \sigma\rho\sigma \, e$, that is $a \in \ker(\sigma\rho\sigma)$.

THEOREM 6.3. *Let Γ denote the lattice of all group congruences on S. Then the mapping ϕ defined by*

$$\phi\colon \rho \to \rho \vee \sigma \qquad (\rho \in \Lambda)$$

is a homomorphism of Λ onto Γ.

Proof. Let $\rho, \tau \in \Lambda$ and $a \, (\rho \vee \sigma) \cap (\tau \vee \sigma) \, b$. Then by Proposition 6.1, we have $ae \, \rho \, be$ and $af \, \tau \, vf$ for some $e, f \in E$. Hence $aef \, \rho \cap \tau \, bef$ with $ef \in E$ which by Proposition 6.1 implies $a \, (\rho \cap \tau) \vee \sigma \, b$. Consequently $(\rho \vee \sigma) \cap (\tau \vee \sigma) \subseteq (\rho \cap \tau) \vee \sigma$; the opposite inclusion is trivial. Hence ϕ is a $\cap$-homomorphism; it is obvious that ϕ is a $\vee$-homomorphism. Finally, ϕ maps Λ onto Γ since ϕ leaves Γ elementwise fixed.

For a characterization of σ itself, we first need a lemma.

LEMMA 6.4. *For any nonempty subset K of S, we have $K\sigma = (KE)\, \omega$.*

Proof. Let $a \in K\sigma$. Then $a \, \sigma \, k$ for some $k \in K$, so that $ae = ke$ for some $e \in E$. Hence

$$a(ke)^{-1} = aek^{-1} = kek^{-1} = (ke)\,(ke)^{-1};$$

that is, $a \geqslant ke \in KE$ and thus $a \in (KE)\, \omega$. Conversely, let $a \in (KE)\, \omega$. Then

$a \geqslant ke$ for some $k \in K$, $e \in E$. It follows that $aek^{-1} = kek^{-1}$ which implies $a(ek^{-1}k) = k(ek^{-1}k)$ where $ek^{-1}k \in E$. Thus $a \, \sigma \, k$ and $a \in K\omega$.

COROLLARY 6.5. *The classes of σ are the sets $(aE)\,\omega$. In particular,*

$$a \, \sigma \, b \Leftrightarrow (aE)\,\omega = (bE)\,\omega.$$

7. θ-CLASSES OF AN IDEAL

To each ideal I of S we can associate the θ-class of the greatest (resp. least) congruence on S having I as a class.

An inverse semigroup S with zero all of whose nonzero idempotents are primitive is a *primitive inverse semigroup* (briefly S is *primitive*). An ideal I of any semigroup S is *categorical* if for any $a, b, c \in S$, $ab, bc \notin I$ implies $abc \notin I$. If ϕ is a homomorphism of an inverse semigroup S onto a primitive inverse semigroup T, then the complete inverse image of 0 of T under ϕ is a categorical ideal of S. Conversely, to each categorical ideal I of S once can in a natural way associate a homomorphism onto a primitive inverse semigroup T with the property enunciated above.

An ideal I of S is *prime* if for any $a, b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$. Categorical prime ideals are in the same correspondence to homomorphic images of S which are 0 or Brandt semigroups as the categorical ideals vs. primitive homomorphic images. For a complete discussion, consult [2, Sect. 7.7].

THEOREM 7.1. *Let I be a proper categorical ideal of S. The set Σ of all congruences ρ on S having I as a class and for which S/ρ is primitive forms a θ-class with the greatest element γ:*

$$a \, \gamma \, b \Leftrightarrow (xay \in I \Leftrightarrow xby \in I \textit{ for all } x, y \in S^1)$$

and the least element λ:

$$a \, \lambda \, b \Leftrightarrow ae = be \notin I \quad \textit{for some } e \in E \quad \textit{or} \quad a, b \in I.$$

Moreover,

$$\ker \gamma = \{a \in S \mid eae \notin I \textit{ for some } e \in E\} \cup I,$$
$$\ker \lambda = \{a \in S \mid ae = e \notin I \textit{ for some } e \in E\} \cup I.$$

The common trace of all congruences in Σ is given by

$$e \, \xi \, f \Leftrightarrow eg = fg \notin I \quad \textit{for some } g \in E \quad \textit{or} \quad e, f \in I.$$

The kernels K of congruences in Σ are normal subsemigroups of S satisfying

$$I \subset K \subseteq \ker \gamma, \qquad ae \in K \backslash I,\ e \in E \Rightarrow a \in K.$$

Proof. It is easy to verify that γ is the greatest congruence on S having I as one of its classes. Further [2, Lemma 7.64] implies that λ is the least congruence on S for which I is a class and whose quotient semigroup is primitive. Hence λ is the least element of Σ. It follows from the minimality of λ that S/γ is a homomorphic image of S/λ, which implies that S/γ is also primitive. Hence γ is the greatest element of Σ. Further, the congruence γ/λ is a 0-restricted congruence on S/λ. Since S/λ is a primitive inverse semigroup, it follows easily that γ/λ is idempotent separating. Now Theorem 3.1 implies that $\gamma\ \theta\ \lambda$.

Obviously tr λ coincides with the relation ξ in the statement of the theorem, and hence ξ is the trace of any congruence in Σ. Next let $\rho \in \Lambda$ be such that $\rho\ \theta\ \gamma$. By Theorem 3.1, we have for any $a, b \in S$,

$$\begin{aligned} a\rho\ \mathscr{H}\ b\rho &\Leftrightarrow a\gamma\ \mathscr{H}\ b\gamma \Leftrightarrow a\gamma = b\gamma \\ &\Leftrightarrow a\ \gamma\ b \Leftrightarrow a\rho\ \gamma/\rho\ b\rho \end{aligned}$$

since $\mathscr{H}$ coincides with the equality relation in the primitive inverse antigroup S/γ. Consequently $\mathscr{H} = \gamma/\rho$ in S/ρ, which evidently implies that S/ρ is primitive and that I is a class of ρ. Hence $\rho \in \Sigma$ which proves that $[\gamma] \subseteq \Sigma$. But then Σ coincides with the θ-class $[\gamma]$.

$$\begin{gathered} \text{For any } a \in S, \text{ we have } a \in \ker \gamma \text{ if and only if} \\ xay \in I \Leftrightarrow xey \in I \qquad \text{for some } e \in E \quad \text{and all} \quad x, y \in S^1. \end{gathered} \tag{1}$$

If $e \in I$, then $e(a^{-1}a) \in I$ and thus $a = a(a^{-1}a) \in I$. If $e \notin I$, then $eee \notin I$ implies $eae \notin I$. Conversely, assume that $eae \notin I$; we will show that (1) holds. Let $xay \notin I$. Then $xa, ea \notin I$ which implies that $(xa)\,a^{-1}, a^{-1}e \notin I$. Since I is categorical, it follows that $xaa^{-1}e \notin I$. But then $xe(aa^{-1}) \in I$ and thus $xe \notin I$. Analogously, we have $ey \notin I$ which yields $xey \notin I$. A similar argument can be used to show that $xey \notin I$ implies $xay \notin I$. This proves (1) and hence ker γ has the form stated in the theorem. A simple argument shows that ker λ has the form in the statement of the theorem.

In order to prove the last assertion of the theorem, in view of Proposition 5.13(ii), it suffices to show that for any $a \in S$, $e \in E$, the statements

$$ae \in K,\ e\ \xi\ a^{-1}a \Rightarrow a \in K, \tag{2}$$

$$ae \in K \backslash I \Rightarrow a \in K \tag{3}$$

are equivalent. Assume that (2) holds and let $ae \in K\backslash I$. Then $a^{-1}ae \notin I$ and hence

$(a^{-1}e)(a^{-1}ae) = e(a^{-1}ae) \notin I$. It follows that $e \mathrel{\xi} a^{-1}a$ and (2) implies that $a \in K$. Conversely, assume that (3) holds and let $ae \in K$, $e \mathrel{\xi} a^{-1}a$. If $a \in I$, then $a \in K$. Suppose that $a \notin I$. Then $e \mathrel{\xi} a^{-1}a$ implies that $e \notin I$ and also that $e \mathrel{\xi} a^{-1}ae$. But then $a^{-1}ae \notin I$, so $ae \in K \backslash I$. Now (3) yields $a \in K$, as required.

The special case of the above theorem when I is also a prime ideal yields Σ in which all congruences ρ have the property that S/ρ is a Brandt semigroup.

The theorem characterize the θ-class of the greatest congruence on S for which the categorical ideal I is saturated. The next proposition describes the greatest and the least elements of the θ-class of the Rees congruence ρ_I for any ideal I of S. Recall that

$$a \, \rho_I \, b \Leftrightarrow a, b \in I \qquad \text{or} \qquad a = b.$$

PROPOSITION 7.2. *Let I be an ideal of S.*

(i) *I is saturated for every congruence in the θ-class of ρ_I.*

(ii) *$(\rho_I)_{\max}$ is the greatest congruence on S contained in $\mathscr{H} \cup (I \times I)$.*

(iii) *$\rho = (\rho_I)_{\min}$ has the property: $\rho|_I$ is the least group congruence on I, $\rho|_{S \backslash I}$ is the equality relation.*

Proof. (i) Let $\rho \, \theta \, \rho_I$, $a \, \rho \, b$, $a \in I$. Then $a^{-1}a \, \rho \, b^{-1}b$ so that $a^{-1}a \, \rho_I \, b^{-1}b$. Since $a^{-1}a \in I$, it follows that $b^{-1}b \in I$ and hence $b \in I$.

(ii) According to Theorem 3.1, $(\rho_I)_{\max}/\rho_I$ is the greatest idempotent separating congruence on S/ρ_I, hence the greatest congruence on S/ρ_I contained in $\mathscr{H}$. The assertion in (ii) now follows from the correspondence of congruences on S and on S/ρ_I.

(iii) We know that

$$a \, (\rho_I)_{\min} \, b \Leftrightarrow ae = be \qquad \text{for some } e \in E, \quad e \, \rho_I \, a^{-1}a \, \rho_I \, b^{-1}b.$$

If $a \notin I$, then we must have $e = a^{-1}a = b^{-1}b$, so $a = b$. If $a \in I$, then so are both b and e. Conversely, if $ae = be$ with $a, b, e \in I$, then $e \, \rho_I \, a^{-1}a \, \rho_I \, b^{-1}b$ trivially. Hence the restriction of $(\rho_I)_{\min}$ to I coincides with the least group congruence on I.

8. CONGRUENCES ON P-SEMIGROUPS

We apply here the results of Section 4 to P-semigroups constructed by McAlister [10]; the full importance of this class of semigroups is exhibited in [11]. A description of these semigroups goes as follows.

Let $\mathscr{X}$ be a partially ordered set. The meet of elements A and B of $\mathscr{X}$ is denoted by $A \wedge B$ if it exists. Let $\mathscr{Y}$ be an ideal and a subsemilattice of $\mathscr{X}$, that is, if $A \in \mathscr{Y}$, $B \in \mathscr{X}$, $B \leqslant A$, then $B \in \mathscr{Y}$ and if $A, B \in \mathscr{Y}$, then $A \wedge B$ exists and belongs to $\mathscr{Y}$. Let G be a group acting on $\mathscr{X}$ by order automorphisms on the left. Let

$$P(G, \mathscr{X}, \mathscr{Y}) = \{(A, g) \in \mathscr{Y} \times G \mid A \wedge gB \text{ and } g^{-1}(A \wedge B) \text{ exist and belong to } \mathscr{Y}\},$$

with multiplication

$$(A, g)(B, h) = (A \wedge gB, gh).$$

Then $S = P(G, \mathscr{X}, \mathscr{Y})$ is an E-unitary inverse semigroup (that is, $\ker \sigma = E$). Conversely every E-unitary inverse semigroup is isomorphic to one so constructed.

For the rest of this section, we fix a semigroup $S = P(G, \mathscr{X}, \mathscr{Y})$ and denote by E the semilattice of its idempotents. Then $E \cong \mathscr{Y}$ by the isomorphism $A \to (A, 1)$, where 1 is the identity of G. We first characterize normal congruences on E.

Lemma 8.1. *Let ξ be an equivalence relation on $\mathscr{Y}$ satisfying*

$$(\alpha)\ (A, g) \in S, B \,\xi\, C \Rightarrow A \wedge gB \,\xi\, A \wedge gC.$$

Define on E a relation $\bar{\xi}$ by

$$(A, 1)\, \bar{\xi}\, (B, 1) \Leftrightarrow A \,\xi\, B.$$

Then $\bar{\xi}$ is a normal congruence on E. Conversely, every normal congruence on E can be so obtained.

Proof. First let ξ satisfy (α). For $g = 1$, we get that ξ is a congruence on $\mathscr{Y}$, and hence $\bar{\xi}$ is a congruence on E. Let $(A, g) \in S$ and $(B, 1)\, \bar{\xi}\, (C, 1)$. Then

$$(A, g)(B, 1)(A, g)^{-1} = (A \wedge gB, g)(g^{-1}A, g^{-1}) = (A \wedge gB, 1)$$

and analogously $(A, g)(C, 1)(A, g)^{-1} = (A \wedge gC, 1)$, which by (α) implies that

$$(A, g)(B, 1)(A, g)^{-1}\, \bar{\xi}\, (A, g)(C, 1)(A, g)^{-1}$$

so that $\bar{\xi}$ is a normal congruence on E. The converse follows by reversing the steps.

Next we characterize all normal subsemigroups of S. By $\mathscr{I}(\mathscr{Y})$ denote the set of all ideals of $\mathscr{Y}$.

LEMMA 8.2. *Let N be a normal subgroup of G and $\pi\colon N \to \mathscr{I}(\mathscr{Y})$ be a function satisfying*

(β) $1\pi = \mathscr{Y}$,

(γ) $A \in g\pi,\ B \in h\pi \Rightarrow h^{-1}(A \wedge B) \in (h^{-1}g)\,\pi$,

(δ) $(A, g) \in S,\ B \in h\pi \Rightarrow A \wedge gB \wedge ghg^{-1}A \in (ghg^{-1})\,\pi$.

Then

$$N_\pi = \{(A, g) \in S \mid A \in g\pi\}$$

is a normal subsemigroup of S. Conversely, every normal subsemigroup of S can be so obtained for unique N and π.

Proof. Let $(A, g) \in N_\pi$. Then $A \in g\pi$ and $A \in 1\pi$ so that $g^{-1}A \in g^{-1}\pi$ by (γ), and hence $(g^{-1}A, g^{-1}) \in N_\pi$. Thus N_π is closed under the taking of inverses.

Next let $(A, g), (B, h) \in N_\pi$. Then $g^{-1}A \in g^{-1}\pi$ and $B \in h\pi$ which implies that $g(g^{-1}A \wedge B) \in (gh)\,\pi$ again by (γ). Hence $A \wedge gB \in (gh)\,\pi$, that is, $(A \wedge gB, gh) \in S$. Thus N_π is closed under multiplication.

Let $(A, g) \in S$ and $(B, h) \in N_\pi$. Then $B \in h\pi$ and (δ) implies

$$(A, g)\,(B, h)\,(A, g)^{-1} = (A \wedge gB \wedge ghg^{-1}A, ghg^{-1}) \in N_\pi\,.$$

Hence N_π is self-conjugate. Condition (α) ensures that N_π will be full. Therefore N_π is a normal subsemigroup of S.

Conversely, let K be a normal subsemigroup of S, and let

$$N = \{g \in G \mid (A, g) \in K \text{ for some } A \in \mathscr{Y}\}$$

and define π on N by

$$g\pi = \{A \in \mathscr{Y} \mid (A, g) \in K\}.$$

It is straightforward to show that N and π satisfy all the conditions spelled out above and that $N_\pi = K$. For example, (γ) is verified by computing $(B, h)^{-1}\,(A, g)$; also (γ) implies that each $g\pi$ is an ideal of $\mathscr{Y}$.

LEMMA 8.3. *With the notation of Lemmas* 8.1 *and* 8.2, (ξ, N_π) *is a congruence pair for S if and only if*

(ϵ) $(A, g) \in S,\ A\,\xi\,B,\ A \wedge B \in g\pi \Rightarrow A \in g\pi$.

(η) $A \in g\pi,\ B \in \mathscr{Y} \Rightarrow A \wedge gB\ \xi\ A \wedge B$.

Proof. It is a notational convenience to take the left–right duals of conditions (i) and (ii) in Definition 4.2, viz.,

$$ea \in N_\pi, \; e \in E, \; e \; \xi \; aa^{-1} \Rightarrow a \in N_\pi,$$
$$a \in N_\pi, \; e \in E \Rightarrow aea^{-1} \; \xi \; aa^{-1}e.$$

It is easily checked that these two definitions are equivalent. Now

$$(B, 1)(A, g) \in N_\pi, \qquad (B, 1) \; \bar{\xi} \; (A, g)(A, g)^{-1} \Rightarrow (A, g) \in N_\pi$$

is clearly equivalent to (ϵ); analogously for (η).

When all the conditions (α)–(η) are fulfilled, we have a congruence pair $(\bar{\xi}, N_\pi)$, and conversely. Hence the above lemmas, in conjunction with Theorem 4.4, yield the following characterization of congruences on S.

THEOREM 8.4. *Let $S = P(G, \mathscr{X}, \mathscr{Y})$ be a P-semigroup. Let ξ be an equivalence relation on $\mathscr{Y}$ satisfying condition (α), π be a function mapping a normal subgroup N of G into the ideals of $\mathscr{Y}$ satisfying (β), (γ), (δ), and assume that (ϵ), (η) hold. Define a relation $\rho = \rho_{(\xi, \pi)}$ on S by*

$$(A, g) \; \rho \; (B, h) \Leftrightarrow A \; \xi \; B, \qquad A \wedge gh^{-1}B \in (gh^{-1}) \pi.$$

Then $\rho_{(\xi,\pi)}$ is a congruence on S. Conversely, every congruence on S can be uniquely written in this form.

For ξ as above, we have

$$(A, g) \; \bar{\xi}^{\max} \; (B, h) \Leftrightarrow A \wedge gC \; \xi \; B \wedge hC \qquad \text{for all } C \in \mathscr{Y},$$
$$(A, g) \; \bar{\xi}^{\min} \; (B, h) \Leftrightarrow A \; \xi \; B, \qquad g = h.$$

Another characterization of congruences on P-semigroups was given by McAlister [12].

9. Congruences on Polycyclic Monoids

Munn and Reilly [14] proved that a congruence on a bisimple ω-semigroup S is either idempotent separating or is a group congruence. They characterized the former as well as the resulting quotient semigroups. Group congruences on these semigroups were constructed by Ault [1].

Among the numerous generalizations of a bisimple inverse ω-semigroup, a very natural one is that of a polycyclic monoid, which we now proceed to describe.

Let X be any nonempty set. Denote by X^* the free monoid on X, that is the set of all finite sequences (words) over X, including the empty sequence, under concatanation as multiplication. Let G be a group and ϕ be a homomorphism of X^* into the monoid of endomorphisms on C (that is, the empty word $\varnothing$ maps onto the identity endomorphism on G). For any $g \in G$ and $w \in X^*$, we write g^w instead of $g\phi(w)$. On the set $S = (X^* \times G \times X^*) \cup \{0\}$ where 0 is an extra symbol, define a multiplication by

$$\begin{aligned}(u, g, v)(w, h, z) &= (u, gh^t, tz) \qquad \text{if} \qquad v = tw \\ &= (tu, g^t h, z) \qquad \text{if} \qquad w = tv\end{aligned}$$

and all other products are equal to 0. Then $S = S(X, G, \phi)$ is a polycyclic monoid.

This monoid was introduced by Nivat and Perrot [15] who established an abstract characteristic of it; cf. [9, Theorem 3.3]. When card $X = 1$, this essentially reduces to the bisimple ω-semigroup constructed by Reilly [16] with a zero adjoined. Since the congruences for this case have been characterized, as mentioned above, we consider only the case card $X > 1$. For any subgroup H of G and $x \in X$, we write

$$H^x = \{g^x \mid g \in H\}.$$

If $H^x \subseteq H$ for a normal subgroup H of G and all $x \in X$, we can define a homomorphism σ/H from G/H to the endomorphism monoid of G by letting

$$(gH)^w = g^w H \qquad \text{for all} \quad w \in X^*, \quad g \in G.$$

THEOREM 9.1. *Let $S = S(X, G, \phi)$ and* card $X > 1$. *Let H be a normal subgroup of G such that $H^x \subseteq H$ for all $x \in X$. On S define a relation ρ_H by*

$$(u, g, v)\, \rho_H\, (w, h, z) \Leftrightarrow u = w, \qquad gh^{-1} \in H, \qquad v = z,$$

and $0\, \rho_H\, 0$. Then ρ_H is a congruence on S and $S/\rho_H \cong S(X, G/H, \phi/H)$. Conversely, every nonuniversal congruence on S can be uniquely written as ρ_H for some H.

Proof. Under the hypotheses of the direct part, let

$$K = \{(u, h, u) \mid u \in X^*, h \in H\} \cup \{0\}.$$

The hypothesis on H clearly implies that K is closed under multiplication and the taking of inverses. Further,

$$(u, g, v)^{-1}(z, h, z)(u, g, v) = (v, g^{-1}, u)\begin{cases}(z, hg^t, tv) & \text{if} \quad z = tu\\ (tz, h^t g, v) & \text{if} \quad u = tz\\ 0 & \text{otherwise}\end{cases}$$

$$= \begin{cases}(tv, (g^t)^{-1} hg^t, tv) & \text{if} \quad z = tu,\\ (v, g^{-1}h^t g, v) & \text{if} \quad u = tz,\\ 0 & \text{otherwise.}\end{cases}$$

The hypotheses on H now show that if $(z, b, z) \in K$, then also $(u, g, v)^{-1} \cdot (z, h, z)(u, g, v) \in K$. Hence K is a normal subsemigroup of S. Noting that

$$(u, g, v) \mathscr{H} (w, h, z) \Leftrightarrow u = w, \qquad v = z,$$

we see that Proposition 5.14 yields

$$(u, g, v)\, \kappa_K\, (w, h, z) \Leftrightarrow u = w, \qquad (u, g, v)(z, h^{-1}, w) \in K, \qquad v = z,$$

which evidently implies that $\kappa_K = \rho_H$. Proposition 5.14 now gives that ρ_H is an idempotent separating congruence.

Define a mapping χ on S by

$$\chi\colon \begin{cases}(u, g, v) \to (u, gH, v),\\ 0 \to 0.\end{cases}$$

A simple verification shows that χ is a homomorphism of S onto $S(X, G/H, \phi/H)$ which induces the congruence ρ_H. Hence $S/\rho_H \cong S(X, G/H, \phi/H)$.

Conversely, let ρ be a congruence on S, and assume that $(u, 1, u)\, \rho\, (v, 1, v)$ for some $u, v \in X^*$, $u \neq v$. We distinguish several cases.

Case 1. u is a right factor of v. Letting x be the first variable occurring in v, we can write $v = xwu$ for some $w \in X^*$. We now calculate

$$(wu, 1, \varnothing)^{-1}(u, 1, u)(wu, 1, \varnothing) = (\varnothing, 1, wu)(wu, 1, \varnothing) = (\varnothing, 1, \varnothing),$$
$$(wu, 1, \varnothing)^{-1}(v, 1, v)(wu, 1, \varnothing) = (\varnothing, 1, wu)(v, 1, x) = (x, 1, x),$$

which implies $(\varnothing, 1, \varnothing)\, \rho\, (x, 1, x)$. By hypothesis there is $y \in X$ such that $y \neq x$. We obtain

$$(\varnothing, i, \varnothing)(y, 1, y)\, \rho\, (x, 1, x)(y, 1, y)$$

so that $(y, 1, y)\, \rho\, 0$. Since S has no proper nonzero ideals, it follows that ρ is the universal relation.

Case 2. v is a right factor of u; this is symmetric to Case 1.

Case 3. Neither Case 1 nor Case 2 occurs. Then

$$(u, 1, u)(u, 1, u)\,\rho\,(v, 1, v)(u, 1, u)$$

implies $(u, 1, u)\,\rho\,0$, and again ρ is the universal congruence.

Hence let ρ be idempotent separating, and let $K = \ker \rho$. Then

$$K \subseteq E\zeta = \{(u, g, u) \mid u \in X^*, g \in G\} \cup \{0\};$$

let

$$H = \{h \in G \mid (\varnothing, h, \varnothing) \in K\}.$$

It is clear that H is a normal subgroup of G. Further,

$$(x, 1, \varnothing)^{-1}(\varnothing, h, \varnothing)(x, 1, \varnothing) = (\varnothing, h^x, \varnothing)$$

shows that $H^x \subset H$ for all $x \in X$. Finally,

$$(\varnothing, 1, u)^{-1}(\varnothing, h, \varnothing)(\varnothing, 1, u) = (u, h, u)$$
$$(u, 1, \varnothing)^{-1}(u, h, u)(u, 1, \varnothing) = (\varnothing, h, \varnothing)$$

shows that

$$K = \{(u, h, u) \mid u \in X^*, h \in H\} \cup \{0\}.$$

The direct part of the proof now easily shows that $\rho = \rho_H$.

COROLLARY 9.2 (Perrot). *Every nonuniversal congruence on* $S = S(X, G, \phi)$, *where* card $X > 1$, *is idempotent separating.*

10. CONCLUDING REMARKS

Among the special congruences on a general inverse semigroup we have encountered idempotent separating congruences, group congruences, and congruences associated with an ideal. A class of congruences we did not discuss are the *idempotent determined* congruences, that is those congruences ρ with the property: $a\,\rho\,e$, $e \in E$ implies $a \in E$ (a better name would be idempotent pure). In our present notation these are precisely those congruences ρ for which $\ker \rho = E$.

In the diagram below, we give a review of the congruences with extremal properties of the trace and the kernel. We use the following notation:

$\omega =$ the universal relation (on S and E),

$\sigma =$ the least group congruence,

$\eta =$ the least semilattice congruence,

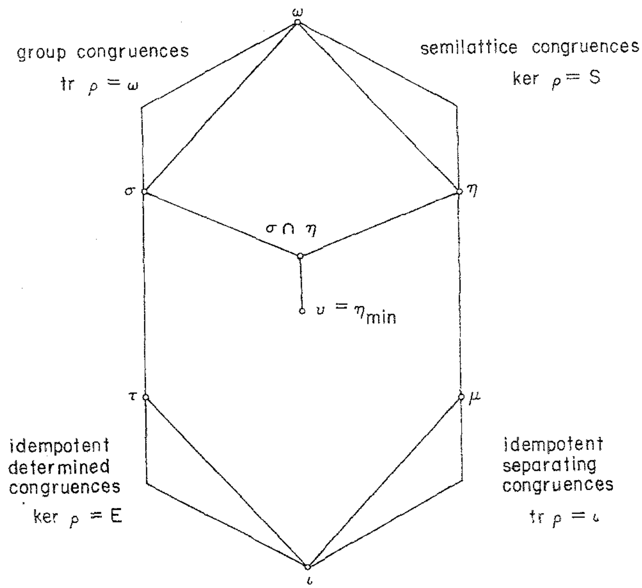
$\nu =$ the least semilattice of groups congruence,

$\tau =$ the greatest idempotent determined congruence,

$\mu =$ the greatest idempotent separating congruence,

$\iota =$ the equality relation (on S and E),

for an inverse semigroup S with the semilattice of idempotents E.



Each of the upper two classes of congruences forms a filter of Λ, and each of the lower classes forms an ideal of Λ. For each semilattice congruence ρ, we have $\ker \rho = S$ and hence $\rho = \rho_{\max}$; dually for each idempotent determined congruence ρ, we have $\ker \rho = E$ and hence $\rho = \rho_{\min}$. In contradistinction, the group congruences and the idempotent separating congruences each constitute a θ-class.

It is easy to see that the following statements are equivalent:

(i) S is E-unitary,

(ii) $\sigma = \tau$,

(iii) for any $\rho \in \Lambda$, $\ker \rho = E \Leftrightarrow \rho = \rho_{\min}$;

and that also the following statements are equivalent:

(i) S is a semilattice of groups,

(ii) $\eta = \mu$,

(iii) for any $\rho \in \Lambda$, $\ker \rho = S \Leftrightarrow \rho = \rho_{\max}$.

A portion of the above discussion can be found in Ref. [4]. Using the methods developed here, we have constructed all congruences on any simple inverse ω-semigroup. In view of the considerable length of the considerations involved, this will appear in a separate communication in *Glasgow Math. J.*

Note added in proof. Congruences on P-semigroups were also characterized by R. P. Jones. The lattice of inverse subsemigroups of a reduced inverse semigroup, *Glasgow Math. J.* **17** (1976), 161–172.

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