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Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities

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Abstract

This paper introduces the notion of a quasi-hom-Lie algebra, or simply, a qhl-algebra, which is a natural generalization of hom-Lie algebras introduced in a previous paper [J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, math.QA/0408064]. Quasi-hom-Lie algebras include also as special cases (color) Lie algebras and superalgebras, and can be seen as deformations of these by maps, twisting the Jacobi identity and skew-symmetry. The natural realm for these quasi-hom-Lie algebras is generalizations-deformations of the Witt algebra $\mathfrak{d}$ of derivations on the Laurent polynomials $\mathbb{C}[t, t^{-1}]$. We also develop a theory of central extensions for qhl-algebras which can be used to deform and generalize the Virasoro algebra by centrally extending the deformed Witt type algebras constructed here. In addition, we give a number of other interesting examples of quasi-hom-Lie algebras, among them a deformation of the loop algebra.

Keywords: Deformations; Central extensions; Quasi-hom-Lie algebras; (Color) Lie algebras; Witt algebras; Virasoro algebras; Loop algebras

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1. Introduction

The classical Witt and Virasoro algebras are ubiquitous in mathematics and theoretical physics, the latter algebra being the unique one-dimensional central extension of the former [2,8,9,11–13,19,23]. Considering the origin of the Witt algebra this is not surprising: the Witt algebra $\mathfrak{d}$ is the infinite-dimensional Lie algebra of complexified polynomial vector fields on the unit circle S^1. It can also be defined as $\mathfrak{d} = \mathbb{C} \otimes \mathrm{Vect}(S^1) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} \cdot d_n$, where $d_n = -t^{n+1} d/dt$ is a linear basis for $\mathfrak{d}$, and the Lie product being defined on the generators d_n as $\langle d_n, d_m\rangle = (n-m)d_{n+m}$ and extended linearly to the whole $\mathfrak{d}$. This means in particular that any $\hat{f} \in \mathfrak{d}$ can be written as $\hat{f} = f \cdot d/dt$ with $f \in \mathbb{C}[t, t^{-1}]$, the algebra of Laurent polynomials, and hence $\mathfrak{d}$ can be viewed as the (complex) Lie algebra of derivations on $\mathbb{C}[t, t^{-1}]$. When the usual derivation operator is replaced by its difference discretization or deformation, the underlying algebra is also in general deformed, and the description and understanding of the properties of the new algebra becomes a problem of key importance.

To put the present article into the right perspective and to see where we are coming from we briefly recall the constructions from [14]. In that paper we considered deformations of $\mathfrak{d}$ using σ-derivations, that is, linear maps D satisfying a generalized Leibniz rule $D(ab) = Dab + \sigma(a)Db$. As we mentioned above the Witt algebra $\mathfrak{d}$ can be viewed as the Lie algebra of derivations on $\mathbb{C}[t, t^{-1}]$. This observation was in fact our starting point in [14] in constructing deformations of the Witt algebra. Instead of just considering ordinary derivations on $\mathbb{C}[t, t^{-1}]$ we considered σ-derivations. In fact, we did something even more general as we considered unital commutative associative $\mathbb{C}$-algebras $\mathcal{A}$ and a σ-derivation Δ on $\mathcal{A}$. Forming the cyclic left $\mathcal{A}$-module $\mathcal{A}\cdot\Delta$, a left submodule of the $\mathcal{A}$-module $\mathfrak{D}_\sigma(\mathcal{A})$ of all σ-derivations on $\mathcal{A}$, we equipped $\mathcal{A}\cdot\Delta$ with a bracket multiplication $\langle\cdot,\cdot\rangle_\sigma$ such that it satisfied skew-symmetry and a generalized Jacobi identity with six terms

$$\circlearrowleft_{x,y,z} \big(\langle\sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + \delta\cdot\langle x, \langle y, z\rangle_\sigma\rangle_\sigma\big) = 0, \tag{1}$$

where $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to x, y, z and where $\delta \in \mathcal{A}$. In the case when $\mathcal{A}$ is a unique factorization domain (UFD) we showed that the whole $\mathcal{A}$-module $\mathfrak{D}_\sigma(\mathcal{A})$ is cyclic and can thus be generated by a single element Δ. Since $\mathbb{C}[t, t^{-1}]$ is a UFD this result applies in particular to the σ-derivations on the Laurent polynomials $\mathbb{C}[t, t^{-1}]$, and so we may regard $\mathfrak{D}_\sigma(\mathbb{C}[t, t^{-1}])$ as a deformation of $\mathfrak{d} = \mathfrak{D}_{\mathrm{id}}(\mathbb{C}[t, t^{-1}])$. As a result we have a Jacobi-like identity (1) on $\mathfrak{D}_\sigma(\mathbb{C}[t, t^{-1}])$.

Furthermore, in [14] we concentrated mainly on the case when $\delta \in \mathbb{C}\setminus\{0\}$ and so the Jacobi-like identity (1) simplified to the Jacobi-like identity with three terms

$$\circlearrowleft_{x,y,z} \langle(\varsigma + \mathrm{id})(x), \langle y, z\rangle_\varsigma\rangle_\varsigma = 0,$$

where $\varsigma = \bar{\sigma}/\delta$ is $1/\delta$-scaled version of $\bar{\sigma} : \mathcal{A}\cdot\Delta \to \mathcal{A}\cdot\Delta$, acting on this left module as $\bar{\sigma}(a\cdot\Delta) = \sigma(a)\cdot\Delta$. Motivated by this we called algebras with a three-term deformed Jacobi identity of this form *hom-Lie algebras*. Using that any non-zero algebra $\mathbb{C}$-endomorphism σ on $\mathbb{C}[t, t^{-1}]$ must be on the form $\sigma(t) = qt^s$ for $s \in \mathbb{Z}$ and

$q \in \mathbb{C} \setminus \{0\}$, we obtained a $\mathbb{Z}$-parametric family of deformations which, when $s = 1$, reduces to a q-deformation of the Witt algebra and becoming $\mathfrak{d}$ when $q = 1$. This deformation is closely related to the q-deformations of the Witt algebra introduced and studied in [1,3–7,20,21,26–28]. However, our defining commutation relations in this case look somewhat different, as we obtained them, not from some conditions aiming to resolve specifically the case of q-deformations, but rather by choosing $\mathbb{C}[t, t^{-1}]$ as an example of the underlying coefficient algebra and specifying σ to be the automorphism $\sigma_q : f(t) \mapsto f(qt)$ in our general construction for σ-derivations. By simply choosing a different coefficient algebra or basic σ-derivation one can construct many other analogues and deformations of the Witt algebra. The important feature of our approach is that, as in the non-deformed case, the deformations and analogues of the Witt algebra obtained by various choices of the underlying coefficient algebra, of the endomorphism σ and of the basic σ-derivation, are precisely the natural algebraic structures for the differential and integral type calculi and geometry based on the corresponding classes of generalized derivation and difference type operators.

We remarked in the beginning that the Witt algebra $\mathfrak{d}$ has a unique (up to isomorphism) one-dimensional central extension, namely the Virasoro algebra. In [14] we developed, for the class of hom-Lie algebras, a theory of central extensions, providing cohomological type conditions, useful for showing the existence of central extensions as well as for their construction. For natural reasons we required that the central extension of a hom-Lie algebra is also a hom-Lie algebra, i.e., that we extend within the category of hom-Lie algebras. In particular, the standard theory of central extensions of Lie algebras becomes a natural special case of the theory for hom-Lie algebras when no non-identity twisting is present. This implies that in the specific examples of deformation families of Witt and Virasoro type algebras constructed within the framework of [14], the corresponding non-deformed Witt and Virasoro type Lie algebras are included as the algebras corresponding to those specific values of deformation parameters which remove the non-trivial twisting. We rounded up [14], putting the central extension theory to the test applying it for the construction of a hom-Lie algebra central extension of the q-deformed Witt algebra producing a q-deformation of the Virasoro Lie algebra. For $q = 1$ one indeed recovers the usual Virasoro Lie algebra as is expected from our general approach.

A number of examples of deformed algebras constructed in [14] do not satisfy the three-term Jacobi-like identity of hom-Lie algebras, but obey instead twisted six-term Jacobi-like identities of the form (1). These examples are recalled for the reader's convenience among other examples in Section 3. Moreover, there exists also many examples where skew-symmetry is twisted as well. Taking the Jacobi identity (1) as a stepping-stone we introduce in this paper a further generalization of hom-Lie algebras by twisting, not only the Jacobi identity, but also the skew-symmetry and the homomorphism σ itself (replaced by α in this paper). In addition, we let go the assumption that δ is an element of $\mathcal{A}$ and assume instead that it is a linear map β on $\mathcal{A}$. We call these algebras *quasi-hom-Lie algebras* or in short just *qhl-algebras* (see Definition 1). In this way we obtain a class of algebras which not only includes hom-Lie algebras but also color Lie algebras, Lie superalgebras and Lie algebras as well as other more exotic types of algebras, which then can be viewed as a kind of deformation of Lie algebras in some larger category.

The present paper is organized into two clearly distinguishable parts. The first, consisting of Sections 2 and 3 concerns the definition of qhl-algebras and some more or less elaborated examples of such. The second part, Section 4, is devoted to the (central) extension theory of qhl-algebras. Let us first comment some on the first part. In Section 3 we give, based on observations and results from [14], examples of qhl-algebras generalizations-deformations or analogues of the classical Witt algebra $\mathfrak{d}$, in addition to showing how the notion of a qhl-algebra also encompasses Lie algebras and superalgebras and, more generally, color Lie algebras by introducing gradings on the underlying linear space and by suitable choices of deformation maps. We also remark that we can define generalized color Lie algebras by admitting the twists α and β. As another, new, example of qhl-algebras we offer a deformed version of the loop algebra. Section 4 is devoted to the development of a central extension theory for qhl-algebras generalizing the theory for (color) Lie algebras and hom-Lie algebras. We give necessary and sufficient conditions for having a central extension and compare these results to the ones given in the existing literature, for example [14] for hom-Lie algebras and [29,30] for color Lie algebras. As a last example we consider central extensions of deformed loop qhl-algebras in Section 4.3.

2. Definitions and notations

Throughout this paper we let $\Bbbk$ be a field of characteristic zero and let $\mathcal{L}_{\Bbbk}(L)$ denote the linear space of $\Bbbk$-linear maps of the $\Bbbk$-linear space L.

Definition 1. A quasi-hom-Lie algebra (qhl-algebra) is a tuple $(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega)$ where

- L is a $\Bbbk$-linear space,
- $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map called a product or a bracket in L,
- $\alpha, \beta : L \to L$, are linear maps,
- $\omega : D_\omega \to \mathcal{L}_{\Bbbk}(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$,

such that the following conditions hold:

- (β-twisting) The map α is a β-twisted algebra homomorphism, that is,

$$\langle\alpha(x), \alpha(y)\rangle_L = \beta \circ \alpha\langle x, y\rangle_L, \quad \text{for all } x, y \in L;$$

- (ω-symmetry) The product satisfies a generalized skew-symmetry condition

$$\langle x, y\rangle_L = \omega(x, y)\langle y, x\rangle_L, \quad \text{for all } (x, y) \in D_\omega;$$

- (qhl-Jacobi identity) The bracket satisfies a generalized Jacobi identity

$$\circlearrowleft_{x,y,z} \big\{\omega(z, x)\big(\langle\alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\big)\big\} = 0,$$

  for all $(z, x), (x, y), (y, z) \in D_\omega$.

Note that if $\alpha = \mathrm{id}_L$ then $\beta = \mathrm{id}|_{\langle L,L\rangle}$ on $\langle L, L\rangle \subseteq L$. To avoid writing all the maps $\langle\cdot,\cdot\rangle_L, \alpha_L, \beta_L$ and ω_L when presenting a quasi-hom-Lie algebra $(L, \langle\cdot,\cdot\rangle_L, \alpha_L, \beta_L, \omega_L)$ we simply write L, remembering that there are maps implicitly present.

Quasi-hom-Lie algebras form a category with morphisms (called strong morphisms) linear maps $\phi : L \to L'$ satisfying:

(M1) $\phi(\langle x, y\rangle_L) = \langle\phi(x), \phi(y)\rangle_{L'}$,
(M2) $\phi \circ \alpha = \alpha' \circ \phi$,
(M3) $\phi \circ \beta = \beta' \circ \phi$

in addition to:

(M4) $\phi \circ \omega_L(x, y) = \omega_{L'}(\phi(x), \phi(y)) \circ \phi$.

A weak quasi-hom-Lie algebra morphism is a linear map $L \to L'$ such that just condition (M1) holds. Note that (M4) is automatic on $\langle L, L\rangle_L$ if $(x, y) \in D_{\omega_L}$. In a similar fashion one can prove that $\beta_L \circ \alpha_L \circ \omega_L(x, y) = \omega_L(\alpha_L(x), \alpha_L(y)) \circ \beta_L \circ \alpha_L$, on $\langle L, L\rangle_L$ if $(x, y) \in D_{\omega_L}$, following from the β-twisting and the ω-symmetry. It is clear what we mean by weak and strong isomorphisms. By a short exact sequence of qhl-algebras $\mathfrak{a}$, E and L, we mean a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] \arrow[d, "\alpha_{\mathfrak{a}}"'] & E \arrow[r, "\mathrm{pr}"] \arrow[d, "\alpha_E"'] & L \arrow[r] \arrow[d, "\alpha_L"'] & 0 \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] & E \arrow[r, "\mathrm{pr}"] & L \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] \arrow[u, "\beta_{\mathfrak{a}}"] & E \arrow[r, "\mathrm{pr}"] \arrow[u, "\beta_E"] & L \arrow[r] \arrow[u, "\beta_L"] & 0
\end{tikzcd} \tag{2}$$

with exact rows and where ι and pr are strong morphisms. It is obviously a triviality to extend the above to arbitrary exact sequences of qhl-algebras.

Definition 2. A short exact sequence as (2) is a quasi-hom-Lie algebra extension of L by $\mathfrak{a}$, or by a slight abuse of language, we say that E is an extension of L by $\mathfrak{a}$.

3. Examples

Example 3. By taking β to be the identity id_L and $\omega = -\mathrm{id}_L$ we get the hom-Lie algebras discussed in a previous paper [14]. We recall the definition for the reader's convenience. A hom-Lie algebra is a non-associative algebra L equipped with an algebra endomorphism $\alpha : L \to L$ and with bracket multiplication $\langle\cdot,\cdot\rangle_\alpha$ such that

- $\langle x, y\rangle_\alpha = -\langle y, x\rangle_\alpha$ (skew-symmetry),

- $\circlearrowleft_{x,y,z} \langle (\alpha + \mathrm{id}_L)(x), \langle y, z\rangle_\alpha \rangle_\alpha = 0$ (α-deformed Jacobi identity)

for all $x, y, z \in L$. Specializing further, we get a Lie algebra by taking α equal to the identity id_L.

A Γ-graded algebra, with Γ an abelian group, is a Γ-graded $\Bbbk$-linear space $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ with bilinear multiplication $*$ respecting the grading in the sense that $V_{\gamma_1} * V_{\gamma_2} \subseteq V_{\gamma_1+\gamma_2}$. The elements $v_\gamma \in V_\gamma$ are called homogeneous of degree γ.

Example 4. Lie algebras are covered by a more general notion, namely the color Lie algebras (or Γ-graded ε-Lie algebras). Here Γ is any abelian group and the color Lie algebra L with bracket $\langle\cdot,\cdot\rangle$ decomposes as $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ where $\langle L_{\gamma_1}, L_{\gamma_2}\rangle \subseteq L_{\gamma_1+\gamma_2}$, for $\gamma_1, \gamma_2 \in \Gamma$. In addition, the "color structure" includes a map $\varepsilon : \Gamma \times \Gamma \to \Bbbk$, called a commutation factor, satisfying

- $\varepsilon(\gamma_x, \gamma_y)\varepsilon(\gamma_y, \gamma_x) = 1$,
- $\varepsilon(\gamma_x + \gamma_y, \gamma_z) = \varepsilon(\gamma_x, \gamma_z)\varepsilon(\gamma_y, \gamma_z)$, and $\varepsilon(\gamma_x, \gamma_y + \gamma_z) = \varepsilon(\gamma_x, \gamma_y)\varepsilon(\gamma_x, \gamma_z)$,

for $\gamma_x, \gamma_y, \gamma_z \in \Gamma$. The color skew-symmetry and Jacobi condition are now stated, with the aid of ε, as

- $\langle x, y\rangle = -\varepsilon(\gamma_x, \gamma_y)\langle y, x\rangle$,
- $\varepsilon(\gamma_z, \gamma_x)\langle x, \langle y, z\rangle\rangle + \varepsilon(\gamma_x, \gamma_y)\langle y, \langle z, x\rangle\rangle + \varepsilon(\gamma_y, \gamma_z)\langle z, \langle x, y\rangle\rangle = 0$

for $x \in L_{\gamma_x}$, $y \in L_{\gamma_y}$ and $z \in L_{\gamma_z}$. Color Lie algebras are examples of qhl-algebras. This can be seen by grading L in the definition of qhl-algebras $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$, and putting $\alpha = \beta = \mathrm{id}_L$ and $\omega(x, y)v = -\varepsilon(\gamma_x, \gamma_y)v$ for $v \in L$, where $(x, y) \in D_\omega = (\bigcup_{\gamma\in\Gamma} L_\gamma) \times (\bigcup_{\gamma\in\Gamma} L_\gamma)$ and $\gamma_x, \gamma_y \in \Gamma$ are the graded degrees of x and y. The ω-symmetry and the qhl-Jacobi identity give the respective identities in the definition of a color Lie algebra. The Lie superalgebras are obtained when $\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\varepsilon(\gamma_x, \gamma_y) = (-1)^{\gamma_x\gamma_y}$, where $\gamma_x\gamma_y$ is the product in $\mathbb{Z}_2$.

Since $\alpha_L = \beta_L = \mathrm{id}_L$ for (color) Lie algebras, there is only one notion of morphism in this case, namely the usual (color) Lie algebra homomorphism. By not restricting α_L to be the identity in Example 4 we can define color hom-Lie algebras, and similarly, with $\beta_L \neq \mathrm{id}_L$, color qhl-algebras.

Example 5. The loop algebra $\check{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is defined to be the set of (Laurent) polynomial maps $f : S^1 \to \mathfrak{g}$, where S^1 is the unit circle, with multiplication defined by $\langle f, g\rangle(x) = \langle f(x), g(x)\rangle_{\mathfrak{g}}$, for $x \in X$. It is not difficult to see that $\check{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with bilinear multiplication given by

$$\langle f \otimes t^n, g \otimes t^m\rangle_{\check{\mathfrak{g}}} = \langle f, g\rangle_{\mathfrak{g}} \otimes t^{n+m}.$$

Loop algebras are important in physics, especially in conformal field theories and superstring theory [8,9,11,12].

Let $\mathfrak{g}$ be a qhl-algebra. Then the vector space $\check{\mathfrak{g}} := \mathfrak{g} \otimes \Bbbk[t, t^{-1}]$ can be considered as the algebra of Laurent polynomials with coefficients in the qhl-algebra $\mathfrak{g}$. Put $\alpha_{\check{\mathfrak{g}}} := \alpha_{\mathfrak{g}} \otimes \mathrm{id}$, $\beta_{\check{\mathfrak{g}}} := \beta_{\mathfrak{g}} \otimes \mathrm{id}$ and $\omega_{\check{\mathfrak{g}}} := \omega_{\mathfrak{g}} \otimes \mathrm{id}$ and define a product on $\check{\mathfrak{g}}$ by $\langle x \otimes t^n, y \otimes t^m \rangle_{\check{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}$. With these definitions $\check{\mathfrak{g}}$ is a qhl-algebra. The verification of this consists of checking the axioms from Definition 1 of qhl-algebras. The $\omega_{\check{\mathfrak{g}}}$-skew symmetry is checked as follows:

$$\begin{aligned}\langle x \otimes t^n, y \otimes t^m \rangle_{\check{\mathfrak{g}}} &= \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m} = (\omega_{\mathfrak{g}}(x, y) \langle y, x \rangle_{\mathfrak{g}}) \otimes t^{n+m} \\ &= (\omega_{\mathfrak{g}}(x, y) \otimes \mathrm{id})(\langle y, x \rangle_{\mathfrak{g}} \otimes t^{n+m}) = \omega_{\check{\mathfrak{g}}}(x, y) \langle y \otimes t^m, x \otimes t^n \rangle_{\check{\mathfrak{g}}}.\end{aligned}$$

Next we prove the $\beta_{\check{\mathfrak{g}}}$-twisting of $\alpha_{\check{\mathfrak{g}}}$. First $\alpha_{\check{\mathfrak{g}}}(x \otimes t^n) = \alpha_{\mathfrak{g}}(x) \otimes t^n$, and so

$$\begin{aligned}\langle \alpha_{\check{\mathfrak{g}}}(x \otimes t^n), \alpha_{\check{\mathfrak{g}}}(y \otimes t^m) \rangle_{\check{\mathfrak{g}}} &= \langle \alpha_{\mathfrak{g}}(x) \otimes t^n, \alpha_{\mathfrak{g}}(y) \otimes t^m \rangle_{\check{\mathfrak{g}}} \\ &= \langle \alpha_{\mathfrak{g}}(x), \alpha_{\mathfrak{g}}(y) \rangle_{\mathfrak{g}} \otimes t^{n+m} = (\beta_{\mathfrak{g}} \circ \alpha_{\mathfrak{g}} \langle x, y \rangle_{\mathfrak{g}}) \otimes t^{n+m} \\ &= \beta_{\check{\mathfrak{g}}} \circ (\alpha_{\mathfrak{g}} \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}) = \beta_{\check{\mathfrak{g}}} \circ \alpha_{\check{\mathfrak{g}}}(\langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}).\end{aligned}$$

The qhl-Jacobi identity lastly, is as follows. The left-hand side is

$$\begin{aligned}&\circlearrowleft \omega_{\check{\mathfrak{g}}}(z \otimes t^l, x \otimes t^n)\big(\langle \alpha_{\check{\mathfrak{g}}}(x \otimes t^n), \langle y \otimes t^m, z \otimes t^l \rangle_{\check{\mathfrak{g}}} \rangle_{\check{\mathfrak{g}}} \\ &\quad + \beta_{\check{\mathfrak{g}}} \langle x \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_{\check{\mathfrak{g}}} \rangle_{\check{\mathfrak{g}}}\big),\end{aligned}$$

where the notation $\circlearrowleft$ here is used for cyclic summation with respect to $x \otimes t^n$, $y \otimes t^m$, $z \otimes t^l$. The first term in the parentheses is

$$\begin{aligned}\langle \alpha_{\check{\mathfrak{g}}}(x \otimes t^n), \langle y \otimes t^m, z \otimes t^l \rangle_{\check{\mathfrak{g}}} \rangle_{\check{\mathfrak{g}}} &= \langle \alpha_{\mathfrak{g}}(x) \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_{\check{\mathfrak{g}}} \rangle_{\check{\mathfrak{g}}} \\ &= \langle \alpha_{\mathfrak{g}}(x), \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}} \otimes t^{n+m+l}\end{aligned}$$

and the second

$$\begin{aligned}\beta_{\check{\mathfrak{g}}} \langle x \otimes t^n, \langle y \otimes t^m, z \otimes t^l \rangle_{\check{\mathfrak{g}}} \rangle_{\check{\mathfrak{g}}} &= \beta_{\check{\mathfrak{g}}}(\langle x, \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}} \otimes t^{n+m+l}) \\ &= (\beta_{\mathfrak{g}} \langle x, \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}}) \otimes t^{n+m+l}.\end{aligned}$$

Adding these terms and then summing up cyclically, using that, according to definition, $\omega_{\check{\mathfrak{g}}}(z \otimes t^l, x \otimes t^n) = \omega_{\mathfrak{g}}(z, x) \otimes \mathrm{id}$, we get

$$\begin{aligned}&\circlearrowleft (\omega_{\mathfrak{g}}(z, x) \otimes \mathrm{id})\big(\langle \alpha_{\mathfrak{g}}(x), \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}} + \beta_{\mathfrak{g}} \langle x, \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}}\big) \otimes t^{n+m+l} \\ &\quad = \Bigg(\underset{x,y,z}{\circlearrowleft} \omega_{\mathfrak{g}}(z, x)\big(\langle \alpha_{\mathfrak{g}}(x), \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}} + \beta_{\mathfrak{g}} \langle x, \langle y, z \rangle_{\mathfrak{g}} \rangle_{\mathfrak{g}}\big) \Bigg) \otimes t^{n+m+l}\end{aligned}$$

and the expression in parentheses is zero since $\mathfrak{g}$ is a qhl-algebra.

We now turn to a large and important class of qhl-algebras associated with twisted derivations, providing new classes of deformations of Lie algebras.

3.1. σ-derivations

In this section, we let $\mathcal{A}$ denote a commutative, associative $\mathbb{k}$-algebra with unity, and let $\mathfrak{D}_\sigma(\mathcal{A})$ denote the set of σ-derivations on $\mathcal{A}$, that is, the set of all $\mathbb{k}$-linear maps $D:\mathcal{A}\to\mathcal{A}$ satisfying the σ-Leibniz rule

$$D(ab)=D(a)b+\sigma(a)D(b).$$

We now fix a homomorphism $\sigma:\mathcal{A}\to\mathcal{A}$, an element $\Delta\in\mathfrak{D}_\sigma(\mathcal{A})$, and an element $\delta\in\mathcal{A}$, assuming that these objects satisfy the following two conditions:

$$\sigma(\mathrm{Ann}(\Delta))\subseteq\mathrm{Ann}(\Delta), \tag{3}$$

$$\Delta(\sigma(a))=\delta\sigma(\Delta(a)),\quad \text{for } a\in\mathcal{A}, \tag{4}$$

where $\mathrm{Ann}(\Delta)=\{a\in\mathcal{A}\mid a\cdot\Delta=0\}$. Let $\mathcal{A}\cdot\Delta=\{a\cdot\Delta\mid a\in\mathcal{A}\}$ denote the cyclic $\mathcal{A}$-submodule of $\mathfrak{D}_\sigma(\mathcal{A})$ generated by Δ and extend σ to $\mathcal{A}\cdot\Delta$ by $\sigma(a\cdot\Delta)=\sigma(a)\cdot\Delta$. We have the following theorem, which introduces a $\mathbb{k}$-algebra structure on $\mathcal{A}\cdot\Delta$.

Theorem 6 (cf. [14]). *If* (3) *holds then the map $\langle\cdot,\cdot\rangle_\sigma$ defined by setting*

$$\langle a\cdot\Delta,b\cdot\Delta\rangle_\sigma=(\sigma(a)\cdot\Delta)\circ(b\cdot\Delta)-(\sigma(b)\cdot\Delta)\circ(a\cdot\Delta),\quad \textit{for } a,b\in\mathcal{A}, \tag{5}$$

where $\circ$ denotes elementwise composition, is a well-defined $\mathbb{k}$-algebra product on the $\mathbb{k}$-linear space $\mathcal{A}\cdot\Delta$, and it satisfies the following identities:

$$\langle a\cdot\Delta,b\cdot\Delta\rangle_\sigma=(\sigma(a)\Delta(b)-\sigma(b)\Delta(a))\cdot\Delta, \tag{6}$$

$$\langle a\cdot\Delta,b\cdot\Delta\rangle_\sigma=-\langle b\cdot\Delta,a\cdot\Delta\rangle_\sigma, \tag{7}$$

for $a,b,c\in\mathcal{A}$. Moreover, if (4) *holds, then*

$$\circlearrowleft_{a,b,c}\big(\langle\sigma(a)\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\rangle_\sigma+\delta\cdot\langle a\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\rangle_\sigma\big)=0. \tag{8}$$

The algebra $\mathcal{A}\cdot\Delta$ in the theorem is a qhl-algebra with $\alpha=\sigma$, $\beta=\delta$ and $\omega=-\mathrm{id}_{\mathcal{A}\cdot\Delta}$.

Remark 7. Let Δ be a non-empty family of commuting σ-derivations on $\mathcal{A}$ closed under composition of maps. Then Δ generates a left $\mathcal{A}$-module $\mathcal{A}\otimes\Delta$ via the rule $b(a\otimes d)=(ba)\otimes d$, where $a,b\in\mathcal{A}$ and $d\in\Delta$. We extend any $d\in\Delta$ from $\mathcal{A}$ to $\mathcal{A}\otimes\Delta$ by the rule $d(a\otimes d')=d(a)\otimes d'+\sigma(a)\otimes dd'$, where dd' denotes (associative) composition

$dd'(a) = d(d'(a))$. For $a \in \mathcal{A}$ and $d \in \Delta$ we can identify $a \otimes d$ and $a \cdot d$ as operators on $\mathcal{A}$ by $a \otimes d(r) = a(d(r))$ for $r \in \mathcal{A}$. Define a product on monomials of $\mathcal{A} \otimes \Delta$ by

$$\langle a \otimes d_1, b \otimes d_2 \rangle_\sigma := \sigma(a) \otimes d_1(b \otimes d_2) - \sigma(b) \otimes d_2(a \otimes d_1),$$

and extend linearly to the whole $\mathcal{A} \otimes \Delta$. Then a simple calculation using the commutativity of $\mathcal{A}$ and Δ shows that

$$\langle a \otimes d_1, b \otimes d_2 \rangle_\sigma = (\sigma(a) d_1(b)) \otimes d_2 - (\sigma(b) d_2(a)) \otimes d_1.$$

Skew-symmetry also follows from this. Note also that if $d_1, d_2 \in \Delta$ then $d_1 - d_2 \in \mathfrak{D}_\sigma(\mathcal{A})$. If Δ is maximal with respect to being commutative then $d_1 - d_2 \in \Delta$. We see that part of the above theorem generalizes to a setting with multiple σ-derivations. However, if there is a nice Jacobi-like identity as in the theorem is uncertain at this moment. The above construction parallels the one given in [25] with the difference that [25] considers the construction in a color Lie algebra setting.

Under the assumption that $\mathcal{A}$ is a unique factorization domain there exists $\Delta \in \mathfrak{D}_\sigma(\mathcal{A})$ such that $\mathfrak{D}_\sigma(\mathcal{A}) = \mathcal{A} \cdot \Delta$, and therefore affords a qhl-algebra structure. For more details see [14].

We now apply these ideas for the construction of some explicit examples.

3.1.1. *Non-linearly deformed Witt algebras*

The most general non-zero endomorphism σ on $\mathcal{A} = \Bbbk[t, t^{-1}]$ is one on the form $\sigma(t) = qt^s$ for $s \in \mathbb{Z}$ and $q \in \Bbbk \setminus \{0\}$. With this σ, the left $\mathcal{A}$-module $\mathfrak{D}_\sigma(\mathcal{A})$ can be generated by a single element $D = \eta t^{-k}(1 - qt^{s-1})^{-1}(\mathrm{id} - \sigma)$, for $\eta \in \Bbbk \setminus \{0\}$ and $k \in \mathbb{Z}$. For $q = s = 1$ we define D as $\eta t^{-k} d/dt$. The element δ for this D such that (4) holds is $\delta = q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r$. Equation (3) is clearly still valid. By putting $d_n = -t^n D$ we see that $\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \Bbbk \cdot d_n$ as a $\Bbbk$-space. Using Theorem 6, $\mathfrak{D}_\sigma(\mathcal{A})$ can be made into a qhl-algebra.

Theorem 8 (cf. [14]). *The linear space of σ-derivations on $\mathcal{A}$, $\mathfrak{D}_\sigma(\mathcal{A})$, can be equipped with the skew-symmetric bracket $\langle \cdot,\cdot \rangle_\sigma$ defined on generators by* (5) *as $\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n$ and satisfying defining commutation relations*

$$\langle d_n, d_m \rangle_\sigma = \eta \operatorname{sign}(n-m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$$

for $n, m \geqslant 0$;

$$\langle d_n, d_m \rangle_\sigma = \eta \left(\sum_{l=0}^{-m-1} q^{n+m+l} d_{(m+l)(s-1)+ns+m-k} + \sum_{l=0}^{n-1} q^{m+l} d_{(s-1)l+n+ms-k} \right)$$

for $n \geqslant 0,\ m < 0$;

$$\langle d_n, d_m\rangle_\sigma = \eta\left(\sum_{l=0}^{m-1} q^{n+l} d_{(s-1)l+m+ns-k} + \sum_{l=0}^{-n-1} q^{m+n+l} d_{(n+l)(s-1)+n+ms-k}\right)$$

for $m \geqslant 0,\ n < 0$;

$$\langle d_n, d_m\rangle_\sigma = \eta\,\mathrm{sign}(n-m) \sum_{l=\min(-n,-m)}^{\max(-n,-m)-1} q^{n+m+l} d_{(m+n)s+(s-1)l-k}$$

for $n, m < 0$.

Furthermore, this bracket satisfies the σ*-deformed Jacobi identity*

$$\circlearrowleft_{n,m,l}\left(q^n\langle d_{ns}, \langle d_m, d_l\rangle_\sigma\rangle_\sigma + q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r \langle d_n, \langle d_m, d_l\rangle_\sigma\rangle_\sigma\right) = 0.$$

Example 9. By specifying $k = 0$, $\eta = 1$ and $s = 1$ in the above theorem we get a q-deformed Witt algebra $\mathfrak{D}_\sigma(\mathcal{A})$ with skew-symmetric bracket $\langle\cdot,\cdot\rangle_\sigma$ given on generators d_n, d_m by $q^n d_n d_m - q^m d_m d_n$ and commutation relations

$$q^n d_n d_m - q^m d_m d_n = (\{n\}_q - \{m\}_q) d_{n+m},$$

where $\{n\}_q = (q^n - 1)/(q - 1)$ (see [15]). The deformed Jacobi identity is

$$\circlearrowleft_{n,m,l} (q^n + 1)\langle d_n, \langle d_m, d_l\rangle_\sigma\rangle_\sigma = 0.$$

Note that in this case D is nothing but (t times) the Jackson q-derivative acting on $\mathcal{A}$ and also that we get a hom-Lie algebra with $\delta = 1$ and $\alpha(d_n) = q^n d_n$ associated to the q-deformed Heisenberg algebra [15]. Notice also that by taking $q = 1$ (i.e., $\sigma = \mathrm{id}$) we retain the classical Witt algebra $\mathfrak{d} = \mathfrak{D}_{\mathrm{id}}$.

3.1.2. σ-derivations on $\Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$

Now, we let $\mathcal{A}$ denote the Laurent polynomials in n variables, $\Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and let boldface letters denote $\mathbb{Z}$-vectors, e.g., $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant n$. Also, we let

$$\sigma(z_i) = q_{z_i} z_1^{S_{i,1}} \cdots z_n^{S_{i,n}},$$

for $1 \leqslant i \leqslant n$, $q_{z_i} \in \Bbbk \setminus \{0\}$ and an integer $n \times n$-matrix $[S_{i,j}]$. We choose an element $D \in \mathfrak{D}_\sigma(\mathcal{A})$ given by

$$D = Q z_1^{-G_1} \cdots z_n^{-G_n} (\mathrm{id} - \sigma),$$

for $Q \in \Bbbk \setminus \{0\}$ and $(G_1, \ldots, G_n) \in \mathbb{Z}^n$. This element generates a cyclic $\mathcal{A}$-submodule of $\mathfrak{D}_\sigma(\mathcal{A})$. In addition, we introduce the following notations:

$$\delta_i := S_{1,i}G_1 + S_{2,i}G_2 + \cdots + (S_{i,i} - 1)G_i + \cdots + S_{n,i}G_n, \quad \text{for } 1 \leqslant i \leqslant n,$$

$$\alpha_r(\mathbf{k}) := \sum_{i=1}^{n} S_{i,r}k_i, \qquad d_{\mathbf{k}} := -z_1^{k_1} \cdots z_n^{k_n} D,$$

$$\tau_i' := \alpha_i(\mathbf{k}) + l_i - G_i, \qquad \tau_i'' := \alpha_i(\mathbf{l}) + k_i - G_i.$$

Using Theorem 6, the $\Bbbk$-subvector space of $\mathfrak{D}_\sigma(\mathcal{A})$ spanned by $d_{\mathbf{k}}$ can be endowed with a skew-symmetric bracket defined on generators as

$$\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma = q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} d_{\alpha_1(\mathbf{k}),\ldots,\alpha_n(\mathbf{k})} d_{\mathbf{l}} - q_{z_1}^{l_1} \cdots q_{z_n}^{l_n} d_{\alpha_1(\mathbf{l}),\ldots,\alpha_n(\mathbf{l})} d_{\mathbf{k}}$$

and satisfying relations

$$\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma = Q q_{z_1}^{l_1} \cdots q_{z_n}^{l_n} d_{\tau_1'',\ldots,\tau_n''} - Q q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} d_{\tau_1',\ldots,\tau_n'}.$$

The bracket satisfies the σ-deformed Jacobi identity

$$\circlearrowleft_{\mathbf{k},\mathbf{l},\mathbf{h}} \big(q_{z_1}^{k_1} \cdots q_{z_n}^{k_n} \langle d_{\alpha_1(\mathbf{k}),\ldots,\alpha_n(\mathbf{k})}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\rangle_\sigma + q_{z_1}^{G_1} \cdots q_{z_n}^{G_n} z_1^{\delta_1} \cdots z_n^{\delta_n} \langle d_{\mathbf{k}}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\rangle_\sigma\big) = 0.$$

More details on these deformations can be found in [14].

4. Extensions

Throughout this section we use that exact sequences of linear spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] & E \arrow[r, "\mathrm{pr}"] & L \arrow[r] \arrow[l, bend right, dotted, "s"'] & 0 \end{tikzcd} \tag{9}$$

split in the sense that there is a $\Bbbk$-linear map $s : L \to E$ called a section such that $\mathrm{pr} \circ s = \mathrm{id}_L$. Note that the condition $\mathrm{pr} \circ s = \mathrm{id}_L$ means that the $\Bbbk$-linear section s is injective and so $L \cong s(L)$ (as linear spaces). This, together with the exactness, lets us deduce that $E \cong s(L) \oplus \iota(\mathfrak{a})$ as linear spaces. Hence a basis of E can be chosen such that any $e \in E$ can be decomposed as $e = s(l) + \iota(a)$ for $a \in \mathfrak{a}$ and $l \in L$, that is, we consider $\iota(\mathfrak{a})$ and $s(L)$ as subspaces of E.

Let us from now on assume that L, $\mathfrak{a}$ and E from (9) are qhl-algebras, and that we have a section $s : L \to E$ such that ω_E, ω_L are intertwined with s, meaning that

$$\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ s = s \circ \omega_L(x, y), \tag{10}$$

if $(x, y) \in D_{\omega_L}$ and $(s(x) + \iota(a), s(y) + \iota(b)) \in D_{\omega_E}$. In particular we have,

$$\omega_E(s(x), s(y)) \circ s = s \circ \omega_L(x, y)$$

if a and b are taken to be zero. By definition of a section $\mathrm{pr} \circ s = \mathrm{id}_L$, and since pr is a homomorphism of algebras we have $0 = \mathrm{pr} \circ (\langle s(x), s(y)\rangle_E - s\langle x, y\rangle_L)$ yielding $\langle s(x), s(y)\rangle_E = s\langle x, y\rangle_L + \iota \circ g(x, y)$, where $g : L \times L \to \mathfrak{a}$ is a 2-cocycle-like $\mathbb{k}$-bilinear map which depends in a crucial way on the section s. Thus g is a measure of the deviation of s from satisfying condition (M1). Furthermore, g has to satisfy a generalized skew-symmetry condition on D_{ω_L} by (10):

$$\begin{aligned}
\iota \circ g(x, y) &= \langle s(x), s(y)\rangle_E - s\langle x, y\rangle_L \\
&= \omega_E(s(x), s(y))\langle s(y), s(x)\rangle_E - s \circ \omega_L(x, y)\langle y, x\rangle_L \\
&= \omega_E(s(x), s(y))(\langle s(y), s(x)\rangle_E - s\langle y, x\rangle_L) \\
&= \omega_E(s(x), s(y)) \circ \iota \circ g(y, x)
\end{aligned} \tag{11}$$

for $(x, y) \in D_{\omega_L}$ such that $(s(x), s(y)) \in D_{\omega_E}$.

Definition 10. Denote the set of all maps $L \times L \to \mathfrak{a}$ satisfying (11) by $\mathrm{Alt}^2_\omega(L, \mathfrak{a}; \mathcal{E})$, the ω-alternating $\mathbb{k}$-bilinear maps associated with the extension (9), which we denote by $\mathcal{E}$ to keep notation short.

Remark 11. For Lie algebras $\omega_E(s(x), s(y))$ is just multiplication by -1 and thus by linearity and injectivity of the map ι, the condition (11) reduces to $g(x, y) = -g(y, x)$, which is the classical skew-symmetry, independent on the extension.

By the commutativity of the boxes in (2) we have $\alpha_L \circ \mathrm{pr} = \mathrm{pr} \circ \alpha_E$ which means that $\mathrm{pr} \circ (\alpha_E - s \circ \alpha_L \circ \mathrm{pr}) = 0$ and so

$$\alpha_E = s \circ \alpha_L \circ \mathrm{pr} + \iota \circ f, \tag{12}$$

where $f : E \to \mathfrak{a}$ is a $\mathbb{k}$-linear map. By a similar argument we get

$$\beta_E = s \circ \beta_L \circ \mathrm{pr} + \iota \circ h, \tag{13}$$

for a $\mathbb{k}$-linear $h : E \to \mathfrak{a}$. Obviously, both f and h depends on the section chosen. To simplify notation we do not indicate explicitly this dependence in what follows.

Since any $e \in E$ and $e' \in E$ can be decomposed as $e = s(x) + \iota(a)$ and $e' = s(y) + \iota(b)$ with $x, y \in L$ and $a, b \in \mathfrak{a}$, we have

$$\begin{aligned}
\langle e, e'\rangle_E &= \langle s(x) + \iota(a), s(y) + \iota(b)\rangle_E \\
&= \langle \iota(a), \iota(b)\rangle_E + \langle s(x), \iota(b)\rangle_E + \langle \iota(a), s(y)\rangle_E + \langle s(x), s(y)\rangle_E.
\end{aligned}$$

With $\langle s(x), s(y)\rangle_E = s\langle x, y\rangle_L + \iota \circ g(x, y)$ we can re-write this, noting that by definition, ι is a morphism of algebras:

$$\begin{aligned}\langle e, e'\rangle_E &= \iota\langle a, b\rangle_{\mathfrak{a}} + \langle s(x), \iota(b)\rangle_E + \langle \iota(a), s(y)\rangle_E + s\langle x, y\rangle_L + \iota \circ g(x, y)\\ &= s\langle x, y\rangle_L + \big(\iota\langle a, b\rangle_{\mathfrak{a}} + \langle s(x), \iota(b)\rangle_E + \langle \iota(a), s(y)\rangle_E + \iota \circ g(x, y)\big),\end{aligned}$$

where the expression in parentheses is in $\iota(\mathfrak{a})$ since $\iota(\mathfrak{a})$ is an ideal in E by the exactness. The extension is called inessential if $g \equiv 0$, which is equivalent to viewing L as a subalgebra of E. We consider only central extensions, i.e., extensions satisfying $\iota(\mathfrak{a}) \subseteq \mathrm{Z}(E) := \{e \in E \mid \langle e, E\rangle_E = 0\}$, where $\mathfrak{a}$ is abelian, that is $\langle \mathfrak{a}, \mathfrak{a}\rangle_{\mathfrak{a}} = 0$. This means in particular, by expanding, that $\langle e, e'\rangle_E = s\langle x, y\rangle_L + \iota \circ g(x, y)$.

Theorem 12. *Suppose $(L, \alpha_L, \beta_L, \omega_L)$ and $(\mathfrak{a}, \alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}}, \omega_{\mathfrak{a}})$ are qhl-algebras with $\mathfrak{a}$ abelian and that $(E, \alpha_E, \beta_E, \omega_E)$ is a central extension of $(L, \alpha_L, \beta_L, \omega_L)$ by $(\mathfrak{a}, \alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}}, \omega_{\mathfrak{a}})$. Then for any section $s : L \to E$, satisfying* (10)*, there is an ω-alternating bilinear $g : L \times L \to \mathfrak{a}$ and linear maps $f, h : E \to \mathfrak{a}$ such that $f \circ \iota = \alpha_{\mathfrak{a}}$, $h \circ \iota = \beta_{\mathfrak{a}}$ and the following relations hold*

$$g(\alpha_L(x), \alpha_L(y)) = h \circ \big(s \circ \alpha_L\langle x, y\rangle_L + \iota \circ f\langle s(x), s(y)\rangle_E\big), \tag{14}$$

$$\begin{aligned}&\circlearrowleft \omega_E(s(z) + \iota(c), s(x) + \iota(a)) \circ \big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L)\\ &\quad + \iota \circ h\langle s(x), s\langle y, z\rangle_L\rangle_E\big) = 0,\end{aligned} \tag{15}$$

for all pairs $(x, a), (y, b), (z, c) \in L \times \mathfrak{a}$ such that $(s(z) + \iota(c), s(x) + \iota(a)), (s(x) + \iota(a), s(y) + \iota(b)), (s(y) + \iota(b), s(z) + \iota(c)) \in D_{\omega_E}$, and where $\circlearrowleft$ denotes the cyclic summation $\circlearrowleft_{(x,a),(y,b),(z,c)}$. Moreover, Eq. (15) *is independent of the choice of section s and function h, under the additional assumptions that only sections s, s' satisfying* (10) *and ω_E's such that $\omega_E(s'(x) + \iota(a), s'(y) + \iota(b)) \circ \iota = \omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ \iota$, are considered.*

This last condition is fulfilled, for instance when we, in addition to (10), demand $\omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ \iota = \iota \circ \omega_{\mathfrak{a}}(\iota(a), \iota(b))$ for all sections s.

Proof. To simplify notation we put $u := s(x) + \iota(a)$, $v := s(y) + \iota(b)$, $w := s(z) + \iota(c)$. Whenever s is replaced with $\tilde{s}$, for instance, we change accordingly in the substitution, e.g., $\tilde{u} := \tilde{s}(x) + \iota(a)$. First,

$$\begin{aligned}\langle \alpha_E(s(x)), \langle s(y), s(z)\rangle_E\rangle_E &= \langle \alpha_E(s(x)), s\langle y, z\rangle_L + \iota \circ g(y, z)\rangle_E\\ &= \langle s(\alpha_L(x)) + \iota \circ f(s(x)), s\langle y, z\rangle_L + \iota \circ g(y, z)\rangle_E\\ &= \langle s(\alpha_L(x)), s\langle y, z\rangle_L\rangle_E\\ &= s\langle \alpha_L(x), \langle y, z\rangle_L\rangle_L + \iota \circ g(\alpha_L(x), \langle y, z\rangle_L).\end{aligned}$$

In a similar fashion we see that

$$\beta_E\langle s(x), \langle s(y), s(z)\rangle_E\rangle_E = \beta_E \circ s\langle x, \langle y, z\rangle_L\rangle_L + \beta_E \circ \iota \circ g(x, \langle y, z\rangle_L).$$

Observing that by exactness $\beta_E \circ \iota \circ g = \iota \circ h \circ \iota \circ g$, we can re-write the above as

$$\begin{aligned}
&\beta_E \circ s\langle x, \langle y, z\rangle_L\rangle_L + \beta_E \circ \iota \circ g(x, \langle y, z\rangle_L) \\
&\quad = s \circ \beta_L\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ h \circ s\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ h \circ \iota \circ g(x, \langle y, z\rangle_L) \\
&\quad = s \circ \beta_L\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ h \circ (s\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ g(x, \langle y, z\rangle_L)).
\end{aligned}$$

Note that $s\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ g(x, \langle y, z\rangle_L) = \langle s(x), s\langle y, z\rangle_E\rangle_E$. So using this and (10) it follows that (15) is a necessary condition for E to be a qhl-algebra. We now show that (15) is independent of the choice of section s and the map h. Taking another section $\tilde{s}$ with $\mathrm{pr} \circ \tilde{s} = \mathrm{id}_L$ satisfying the intertwining condition $\omega_E(\tilde{s}(x) + \iota(a), \tilde{s}(y) + \iota(b)) \circ \tilde{s} = \tilde{s} \circ \omega_L(x, y)$, we see that $(\tilde{s} - s)(x) = \iota \circ k(x)$, for some linear $k : L \to \mathfrak{a}$, and so $\tilde{s} = s + \iota \circ k$. Hence, since the extension is central, $\iota \circ \tilde{g}(x, y) = \iota \circ g(x, y) - \iota \circ k\langle x, y\rangle_L$. By the injectivity of ι we get $\tilde{g}(x, y) = g(x, y) - k\langle x, y\rangle_L$. Furthermore

$$\begin{aligned}
\iota \circ (\tilde{h} - h)(x) &= (\beta_E - \tilde{s} \circ \beta_L \circ \mathrm{pr} - \beta_E + s \circ \beta_L \circ \mathrm{pr})(x) \\
&= (s - \tilde{s}) \circ \beta_L \circ \mathrm{pr}(x) = -\iota \circ k \circ \beta_L \circ \mathrm{pr}(x)
\end{aligned}$$

giving since ι is an injection $\tilde{h} = h - k \circ \beta_L \circ \mathrm{pr}$. We note two things before we proceed:

$$\omega_E(\tilde{u}, \tilde{v}) \circ s = \omega_E(s(x) + \iota(k(x) + a), s(y) + \iota(k(y) + b)) \circ s = s \circ \omega_L(x, y).$$

From this follows:

$$\begin{aligned}
\omega_E(\tilde{u}, \tilde{v}) \circ \iota \circ k &= \omega_E(\tilde{u}, \tilde{v}) \circ (\tilde{s} - s) = \omega_E(\tilde{u}, \tilde{v}) \circ \tilde{s} - \omega_E(\tilde{u}, \tilde{v}) \circ s \\
&= \tilde{s} \circ \omega_L(x, y) - s \circ \omega_L(x, y) = \iota \circ k \circ \omega_L(x, y).
\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}
&\circlearrowleft \omega_E(\tilde{w}, \tilde{u})\big(\iota \circ \tilde{g}(\alpha_L(x), \langle y, z\rangle_L) + \iota \circ \tilde{h}\langle \tilde{s}(x), \tilde{s}\langle y, z\rangle_L\rangle_E\big) \\
&\quad = \circlearrowleft \omega_E(\tilde{w}, \tilde{u})\big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L) - \iota \circ k\langle \alpha_L(x), \langle y, z\rangle_L\rangle_L \\
&\qquad + \iota \circ h\langle s(x), s\langle y, z\rangle_L\rangle_E - \iota \circ k \circ \beta_L \circ \mathrm{pr}\langle s(x), s\langle y, z\rangle_L\rangle_E\big) \\
&\quad = \circlearrowleft \omega_E(\tilde{w}, \tilde{u})\big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L) + \iota \circ h\langle s(x), s\langle y, z\rangle_L\rangle_E\big) \\
&\qquad - \circlearrowleft \omega_E(\tilde{w}, \tilde{u}) \circ \iota \circ k\big(\langle \alpha_L(x), \langle y, z\rangle_L\rangle_L + \beta_L\langle x, \langle y, z\rangle_L\rangle_L\big) \\
&\quad = \circlearrowleft \omega_E(w, u) \circ \big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L) + \iota \circ h\langle s(x), s\langle y, z\rangle_L\rangle_E\big),
\end{aligned}$$

where we have used that L is a qhl-algebra in addition to (16), and where $\circlearrowleft$ is shorthand for $\circlearrowleft_{(x,a),(y,b),(z,c)}$, thereby proving the claimed independence. The left-hand side of the equality $\langle\alpha_E(s(x)),\alpha_E(s(y))\rangle_E = \beta_E\circ\alpha_E\langle s(x),s(y)\rangle_E$ can be written as $s\langle\alpha_L(x),\alpha_L(y)\rangle_L + \iota\circ g(\alpha_L(x),\alpha_L(y))$ and the right-hand side as

$$s\circ\beta_L\circ\alpha_L\langle x,y\rangle_L + \iota\circ h\circ s\circ\alpha_L\langle x,y\rangle_L + \iota\circ h\circ\iota\circ f\langle s(x),s(y)\rangle_E.$$

After comparing and using the injectivity of ι, we get

$$g(\alpha_L(x),\alpha_L(y)) = h\circ\big(s\circ\alpha_L\langle x,y\rangle_L + \iota\circ f\langle s(x),s(y)\rangle_E\big).$$

Finally $f\circ\iota=\alpha_{\mathfrak a}$ and $h\circ\iota=\beta_{\mathfrak a}$ follows from (12), (13), the commutativity of (2) and the injectivity of ι. The proof is complete. □

Example 13. By taking $\beta_L=\mathrm{id}_L$, $\beta_E=\mathrm{id}_E$, $\beta_{\mathfrak a}=\mathrm{id}_{\mathfrak a}$ and $\omega_L(x,y)v_L=-1\cdot v_L$ for all $x,y,v_L\in L$, $\omega_E(e,e')v_E=-1\cdot v_E$ for all $e,e',v_E\in E$ (we have $D_{\omega_L}=L\times L$ and $D_{\omega_E}=E\times E$ here), that is if we consider only hom-Lie algebras, we recover the results from [14]. To see this consider first (15). The assumption that $\beta_E=\mathrm{id}_E$ and $\beta_L=\mathrm{id}_L$ implies

$$\iota\circ h=\mathrm{id}_E - s\circ\mathrm{pr}, \tag{17}$$

and hence by exactness

$$\begin{aligned}
&\iota\circ h\circ\big(s\langle x,\langle y,z\rangle_L\rangle_L + \iota\circ g(x,\langle y,z\rangle_L)\big)\\
&\quad= (\mathrm{id}_E - s\circ\mathrm{pr})\circ\big(s\langle x,\langle y,z\rangle_L\rangle_L + \iota\circ g(x,\langle y,z\rangle_L)\big)\\
&\quad= s\langle x,\langle y,z\rangle_L\rangle_L - s\langle x,\langle y,z\rangle_L\rangle_L + \iota\circ g(x,\langle y,z\rangle_L) = \iota\circ g(x,\langle y,z\rangle_L).
\end{aligned}$$

This means that (15) can be re-written using that ι is an injective qhl-algebra morphism as

$$\circlearrowleft_{x,y,z} g\big((\mathrm{id}_L+\alpha_L)(x),\langle y,z\rangle_L\big)=0,$$

obtained in [14]. In the same manner, using (17) and the injectivity of ι, Eq. (14) reduces to $g(\alpha_L(x),\alpha_L(y)) = f\langle s(x),s(y)\rangle_E$.

Note that (17) can be written as $h\circ\iota|_{\mathfrak a}=\mathrm{id}_{\mathfrak a}$ and $h\circ s|_L=0$. Indeed, we can decompose any $e\in E$ as $e=s(x)+\iota(a)$ and so

$$\iota\circ h(s(x)+\iota(a)) = (\mathrm{id}_E - s\circ\mathrm{pr})(s(x)+\iota(a)) = 0+\iota(a)=\iota(a).$$

Since ι is an injection this gives $h(s(x)+\iota(a))=\iota(a)$. Restricting even further to (color) Lie algebras and thus having $\alpha_L=\alpha_E=\mathrm{id}$, we have that f satisfies a similar condition $f\circ\iota|_{\mathfrak a}=\mathrm{id}_{\mathfrak a}$ and $f\circ s|_L=0$.

Example 14. Consider the following short exact sequence of color Lie algebras L, E and $\mathfrak{a}$ with the same Γ-grading and commutation factor ε

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} E \xrightarrow{\mathrm{pr}} L \longrightarrow 0$$

with $\iota(\mathfrak{a})$ central in E. This setup is a special case of the construction of Scheunert and Zhang [30] and Scheunert [29], special in the sense that we consider central extensions and not just abelian. We shall show that our construction encompasses the one in [29,30] for central extensions. Let us first briefly recall Scheunert and Zhang's construction (as given in [29]). In their setup the above sequence becomes

$$0 \longrightarrow \bigoplus_{\gamma\in\Gamma} \mathfrak{a}_\gamma \xrightarrow{\iota} \bigoplus_{\gamma\in\Gamma} E_\gamma \xrightarrow{\mathrm{pr}} \bigoplus_{\gamma\in\Gamma} L_\gamma \longrightarrow 0.$$

Note that this means that ι and pr are color Lie algebra homomorphisms and this in turn implies that they are homogeneous of degree zero. Take a section $s : L \to E$ which is homogeneous of degree zero, that is, $s(L_\gamma) \subseteq E_\gamma$ for all $\gamma \in \Gamma$. With this data the Scheunert–Zhang 2-cocycle condition can be expressed as $\circlearrowleft_{x,y,z} \varepsilon(\gamma_z, \gamma_x) g(x, \langle y, z\rangle) = 0$, for homogeneous elements x, y, z and where $\gamma_x, \gamma_y, \gamma_z$ are the graded degrees of x, y, z respectively.

Putting the above in a qhl-algebra setting means letting ω play the role of the commutation factor ε, where ω is then defined on homogeneous elements,

$$D_\omega = D_\varepsilon = \Big(\bigcup_{\gamma\in\Gamma} L_\gamma\Big) \times \Big(\bigcup_{\gamma\in\Gamma} L_\gamma\Big),$$

and dependent only on the graded degree of these elements. The set $\mathrm{Alt}^2_\varepsilon(L, \mathfrak{a}; \mathcal{E})$ includes all g coming from the "defect"-relation $\iota \circ g(x, y) = \langle s(x), s(y)\rangle_E - s\langle x, y\rangle_L$, for s a homogeneous section of degree zero. Hence all such g's are also homogeneous of degree zero. Noting that $h \circ \iota|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$ and $h \circ s|_L = 0$ from the Example 13, the relation (15) now becomes,

$$\begin{aligned}
&\circlearrowleft \varepsilon(w, u) \circ \big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L) + \iota \circ h \circ (s\langle x, \langle y, z\rangle_L\rangle_L + \iota \circ g(x, \langle y, z\rangle_L))\big) \\
&\quad = 2 \circlearrowleft \varepsilon(w, u) \circ \iota \circ g(x, \langle y, z\rangle_L) = 0,
\end{aligned}$$

where $w = s(z) + \iota(c)$, $u = s(x) + \iota(a)$, $v = s(y) + \iota(b)$ and $\circlearrowleft$ indicates the cyclic summation $\circlearrowleft_{u,v,w}$. This implies that $\circlearrowleft_{x,y,z} \varepsilon(z, x) g(x, \langle y, z\rangle_L) = 0$, for homogeneous elements, which is the Scheunert–Zhang 2-cocycle condition for central extensions [29].

4.1. Equivalence between extensions

Let $\varphi : E \to E'$ be a weak qhl-algebra morphism satisfying condition (M4). We call two extensions E and E' weakly equivalent or a weak equivalence if the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] \arrow[d, equal] & E \arrow[r, "\mathrm{pr}"] \arrow[d, "\varphi"'] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota'"] & E' \arrow[r, "\mathrm{pr}'"] & L \arrow[r] & 0 \end{tikzcd} \tag{18}$$

commutes. Similarly one defines strong equivalence as a diagram with the map $E \to E'$ being a strong morphism. That φ is automatically an isomorphism of linear spaces follows from the 5-lemma.

Definition 15. The set of weak equivalence classes of extensions of L by $\mathfrak{a}$ is denoted by $E(L, \mathfrak{a})$.

Remark 16. In the case of Lie algebras, or generally, color Lie algebras, weak and strong extensions coincide since weak and strong morphisms do.

First we observe that, for central extensions, the same calculation leading up to (11) also shows that

$$\iota \circ g(x, y) = \omega_E(s(x) + \iota(a), s(y) + \iota(b)) \circ \iota \circ g(y, x) \tag{19}$$

for any $a, b \in \mathfrak{a}$. We pick sections $s : L \to E$ and $s' : L \to E'$ satisfying $\mathrm{pr} \circ s = \mathrm{id}_L = \mathrm{pr}' \circ s'$ such that (10) holds for s' and s. Then there is a $g' \in \mathrm{Alt}^2_\omega(L, \mathfrak{a}; \mathcal{E}')$ associated with the extension $\mathcal{E}'$ of L by $\mathfrak{a}$ such that

$$\langle s'(x), s'(y)\rangle_{E'} = s'\langle x, y\rangle_L + \iota' \circ g'(x, y).$$

Given a map $\varphi : E \to E'$ such that the diagram (18) commutes means in particular that $\mathrm{pr}' \circ \varphi = \mathrm{id}_L \circ \mathrm{pr}$ and so $\mathrm{pr}' \circ \varphi(s(x)) = x$, which gives us that $0 = \mathrm{pr}'(s'(x) - \varphi(s(x)))$. Hence $s'(x) = \varphi \circ s(x) + \iota' \circ \xi(x)$ for some $\Bbbk$-linear $\xi : L \to \mathfrak{a}$. Taking $x, y \in L$ we have, using the centrality,

$$\iota' \circ g'(x, y) = \langle s'(x), s'(y)\rangle_{E'} - s'\langle x, y\rangle_{E'} = \varphi \circ \iota \circ g(x, y) - \iota' \circ \xi(\langle x, y\rangle_L)$$

and since $\varphi \circ \iota = \iota'$ by (18) we get $\iota' \circ g'(x, y) = \iota' \circ g(x, y) - \iota' \circ \xi\langle x, y\rangle_L$ or

$$g'(x, y) = g(x, y) - \xi\langle x, y\rangle_L, \tag{20}$$

by the injectivity of ι'. We need to check that this is compatible with (11). For brevity we put $u := s(x) + \iota(a)$ and $v := s(y) + \iota(b)$. The computation:

$$\begin{aligned} \iota' \circ \xi\langle x, y\rangle_L &= \iota' \circ \xi \circ \omega_L(x, y)\langle y, x\rangle_L = (s' - \varphi \circ s) \circ \omega_L(x, y)\langle y, x\rangle_L \\ &= \big(\omega_{E'}(s'(x), s'(y)) \circ s' - \varphi \circ \omega_E(u, v) \circ s\big)\langle y, x\rangle_L \\ &= \big(\omega_{E'}(s'(x), s'(y)) \circ s' - \omega_{E'}(\varphi(u), \varphi(v)) \circ \varphi \circ s\big)\langle y, x\rangle \end{aligned}$$

$$
\begin{aligned}
&= [\text{Put } a := \xi(x),\ b := \xi(y) \text{ using } \iota' = \varphi \circ \iota,\ s' = \varphi \circ s + \iota' \circ \xi] \\
&= \big(\omega_{E'}(s'(x), s'(y)) \circ s' - \omega_{E'}(s'(x), s'(y)) \circ \varphi \circ s\big)\langle y, x\rangle_L \\
&= \omega_{E'}(s'(x), s'(y)) \circ \iota' \circ \xi \langle y, x\rangle_L,
\end{aligned}
$$

in combination with:

$$
\begin{aligned}
\iota' \circ g(x, y) = \varphi \circ \iota \circ g(x, y) &= \varphi \circ \omega_E(s(x), s(y)) \circ \iota \circ g(y, x) \\
&= \omega_{E'}(\varphi \circ s(x) + \iota'(a), \varphi \circ s(y) + \iota'(b)) \circ \varphi \circ \iota \circ g(y, x) \\
&= \omega_{E'}(s'(x), s'(y)) \circ \iota' \circ g(y, x),
\end{aligned}
$$

where we have used (19) and that $a = \xi(x)$, $b = \xi(y)$, shows the desired compatibility. We can view $\xi\langle x, y\rangle_L$ as a "2-coboundary" thus motivating the following definition.

Definition 17. The set of all 2-cocycle-like maps modulo 2-coboundary-like maps with respect to a weak isomorphism is denoted by $H^2_\omega(L, \mathfrak{a}; \mathcal{E})$.

Remark 18. One can show that (20) reduces to its classical and colored counterparts with the natural specifications [22].

Now, given two extensions E and E' of L by $\mathfrak{a}$, subject to the condition $g'(x, y) = g(x, y) - \xi\langle x, y\rangle_L$, can we construct a weak equivalence, that is, a weak isomorphism making (18) commute? Observe that this forces some kind of relation between ω_E and $\omega_{E'}$. We can view E and E' as $E = s(L) \oplus \iota(\mathfrak{a})$ and $E' = s'(L) \oplus \iota'(\mathfrak{a})$ respectively, since sequences on the form (9) are split. This means that any element $e \in E$ can be decomposed as $e = s(l) + \iota(a)$ for $a \in \mathfrak{a}$ and $l \in L$. We define a map $\varphi : E \to E'$ by $\varphi(s(l) + \iota(a)) := s'(l) + \iota'(a - \xi(l))$ and assume that condition (M4) is satisfied with respect to this map. We will show that this is a weak isomorphism of qhl-algebras. That it is surjective is clear. Suppose that $s'(l) + \iota'(a - \xi(l)) = s'(\tilde{l}) + \iota'(\tilde{a} - \xi(\tilde{l}))$. This is equivalent to $s'(l - \tilde{l}) + \iota'(a - \tilde{a} + \xi(l - \tilde{l})) = 0$ and so injectivity follows from the injectivity of ι' and s'. To have a weak equivalence we must check $\langle\varphi(x), \varphi(y)\rangle_{E'} = \varphi\langle x, y\rangle_E$ but this is easy and left to the reader. Hence,

Theorem 19. *With definitions and notations as above, there is a one-to-one correspondence between elements of $E(L, \mathfrak{a})$ and elements of $H^2_\omega(L, \mathfrak{a}; \mathcal{E})$.*

Rephrased, the theorem says that there is a one-to-one correspondence between weak equivalence classes of central extensions of L by an abelian $\mathfrak{a}$ and bilinear maps g transforming according to (20) under weak isomorphisms. If we are seeking strong equivalence we also have to condition and check the intertwining conditions $\varphi \circ \alpha_E = \alpha_{E'} \circ \varphi$ and $\varphi \circ \beta_E = \beta_{E'} \circ \varphi$ in addition to condition (M4). One convinces oneself that it is necessary that $\alpha_{\mathfrak{a}} \circ \xi = \xi \circ \alpha_L$, $f \circ s = f' \circ s'$ and $\beta_{\mathfrak{a}} \circ \xi = \xi \circ \beta_L$, $h \circ s = h' \circ s'$.

4.2. Existence of extensions

So far we have shown how the 2-cocycle-like bilinear maps g (that is, elements $g \in \mathrm{Alt}^2_{\omega}(L, \mathfrak{a}; \mathcal{E})$ such that (15) holds) satisfying (20) corresponds in a one-to-one fashion to weak equivalence classes of extensions. We now address the natural question of existence.

Put $E := L \oplus \mathfrak{a}$ and choose the canonical section $s : L \to E$, $x \mapsto (x, 0)$, defining pr and ι to be the natural projection and inclusion, respectively, i.e., $\mathrm{pr} : E \to L$, $\mathrm{pr}(x, a) = x$ and $\iota : \mathfrak{a} \to E$, $\iota(a) = (0, a)$. We also define ω_E by $\omega_E((x, a), (y, b)) \circ s = s \circ \omega_L(x, y)$ and $\omega_E((x, a), (y, b)) \circ \iota = \iota \circ \omega_{\mathfrak{a}}(a, b)$ for $(x, y) \in D_{\omega_L}$, $(a, b) \in D_{\omega_{\mathfrak{a}}}$ and $((x, a), (y, b)) \in D_{\omega_E}$. Furthermore we put $\alpha_E(x, a) := (\alpha_L(x), f(x, a))$ and $\beta_E(x, a) := (\beta_L(x), h(x, a))$, with f and h as in the theorem to be stated now.

Theorem 20. *Suppose L and $\mathfrak{a}$ are quasi-hom-Lie algebras with $\mathfrak{a}$ abelian and put $E := L \oplus \mathfrak{a}$. Then for every bilinear g satisfying* (15) *and every pair of linear maps $f, h : L \oplus \mathfrak{a} \to \mathfrak{a}$ such that*

$$f(0, a) = \alpha_{\mathfrak{a}}(a) \quad \text{and} \quad h(0, a) = \beta_{\mathfrak{a}}(a) \quad \text{for } a \in \mathfrak{a}, \tag{21}$$

$$g(\alpha_L(x), \alpha_L(y)) = h\big(\alpha_L\langle x, y\rangle_L, f(\langle x, y\rangle_L, g(x, y))\big), \tag{22}$$

$$\begin{aligned} \circlearrowleft_{(x,a),(y,b),(z,c)} \quad & \omega_E((z, c), (x, a)) \circ \big(\iota \circ g(\alpha_L(x), \langle y, z\rangle_L) \\ & + \iota \circ h(\langle x, \langle y, z\rangle_L\rangle_L, g(x, \langle y, z\rangle_L))\big) = 0, \end{aligned} \tag{23}$$

for $x, y, z \in L$ and $((z, c), (x, a)), ((x, a), (y, b)), ((y, b), (z, c)) \in D_{\omega_E}$, the linear direct sum E with morphisms α_E, β_E, ω_E given above and product given by $\langle (x, a), (y, b)\rangle_E := (\langle x, y\rangle_L, g(x, y))$ is a quasi-hom-Lie algebra central extension of L by $\mathfrak{a}$.

Proof. First note that the definition of the bracket can be written in the usual form $\langle s(x), s(y)\rangle_E = s\langle x, y\rangle_L + \iota \circ g(x, y)$. This gives

$$\begin{aligned} \langle (x, a), (y, b)\rangle_E &= (\langle x, y\rangle_L, g(x, y)) = s\langle x, y\rangle_L + \iota \circ g(x, y) \\ &= s \circ \omega_L(x, y)\langle y, x\rangle_L + \omega_E(s(x), s(y)) \circ \iota \circ g(y, x) \\ &= \omega_E(s(x), s(y)) \circ s\langle y, x\rangle_L + \omega_E(s(x), s(y)) \circ \iota \circ g(y, x) \\ &= \omega_E(s(x), s(y)) \circ (s\langle y, x\rangle_L + \iota \circ g(y, x)) \end{aligned}$$

which amounts to $\langle (x, a), (y, b)\rangle_E = \omega_E(s(x), s(y))\langle (y, b), (x, a)\rangle_E$. That α_E satisfies the β-twisting condition follows from

$$\begin{aligned} \langle \alpha_E(x, a), \alpha_E(y, b)\rangle_E &= \langle (\alpha_L(x), f(x, a)), (\alpha_L(y), f(y, b))\rangle_E \\ &= \big(\langle \alpha_L(x), \alpha_L(y)\rangle_L, g(\alpha_L(x), \alpha_L(y))\big), \\ \beta_E \circ \alpha_E\langle (x, a), (y, b)\rangle_E &= \beta_E \circ \alpha_E(\langle x, y\rangle_L, g(x, y)) \end{aligned}$$

$$
\begin{aligned}
&= \beta_E(\alpha_L\langle x,y\rangle_L, f(\langle x,y\rangle_L, g(x,y)))\\
&= \bigl(\beta_L\circ\alpha_L\langle x,y\rangle_L, h\bigl(\alpha_L\langle x,y\rangle_L, f(\langle x,y\rangle_L, g(x,y))\bigr)\bigr)
\end{aligned}
$$

using (22). The condition for the qhl-Jacobi identity to hold is obtained by adding

$$
\begin{aligned}
\langle\alpha_E(x,a), \langle(y,b),(z,c)\rangle_E\rangle_E &= \langle(\alpha_L(x), f(x,a)), (\langle y,z\rangle_L, g(y,z))\rangle_E\\
&= (\langle\alpha_L(x), \langle y,z\rangle_L\rangle_L, g(\alpha_L(x), \langle y,z\rangle_L))
\end{aligned}
$$

to

$$
\begin{aligned}
\beta_E\langle(x,a), \langle(y,b),(z,c)\rangle_L\rangle_L &= \beta_E\langle(x,a), (\langle y,z\rangle_L, g(y,z))\rangle_E\\
&= \beta_E(\langle x,\langle y,z\rangle_L\rangle_L, g(x,\langle y,z\rangle_L))\\
&= \bigl(\beta_L\langle x,\langle y,z\rangle_L\rangle_L, h(\langle x,\langle y,z\rangle_L\rangle_L, g(x,\langle y,z\rangle_L))\bigr)
\end{aligned}
$$

composing the result with $\omega_E((z,c),(x,a))$, performing cyclic summation and using that L is a qhl-algebra. That the diagram (2) has exact rows is obvious from the definition of ι and pr. Moreover, using (21), it is easy to show that they are also qhl-algebra morphisms, thereby proving the theorem. $\square$

Example 21. With the notations and definitions leading up to the above theorem we pick the canonical section $x \overset{s}{\mapsto} (x,0)$ and the canonical injection $a \overset{\iota}{\mapsto} (0,a)$. Define a bracket on E by $\langle\cdot,\cdot\rangle_E := (\langle\cdot,\cdot\rangle_L, g(\cdot,\cdot))$ for some bilinear $g: L\times L\to\mathfrak{a}$. Note that $\langle\cdot,\cdot\rangle_E$ is compatible with the map s. Finding f and h such that $f(0,a)=\alpha_{\mathfrak{a}}(a)$ and $h(0,a)=\beta_{\mathfrak{a}}(a)$ equips E with the structure of a qhl-algebra. We now make the general ansatz $\iota\circ f(l,a) = (0,\alpha_{\mathfrak{a}}(a)+F(l))$ and also $\iota\circ h(l,a) = (0,\beta_{\mathfrak{a}}(a)+H(l))$, for $F,H: L\to\mathfrak{a}$ linear. A simple calculation shows that α_E and β_E can be defined by $\alpha_E(l,a) = (\alpha_L(l), \alpha_{\mathfrak{a}}(a)+F(l))$ and $\beta_E(l,a) = (\beta_L(l), \beta_{\mathfrak{a}}(a)+H(l))$. With this one obtains the qhl-Jacobi identity

$$
\circlearrowleft \omega_{\mathfrak{a}}(c,a)\bigl(g(\alpha_L(x), \langle y,z\rangle_L) + \beta_{\mathfrak{a}}\circ g(x,\langle y,z\rangle_L) + H\langle x,\langle y,z\rangle_L\rangle_L\bigr) = 0,
$$

where $\circlearrowleft$ is shorthand for $\circlearrowleft_{(x,a),(y,b),(z,c)}$. In addition we must also have

$$
\iota\circ g(\alpha_L(x),\alpha_L(y)) = \bigl(0, \beta_{\mathfrak{a}}\circ f\langle s(x),s(y)\rangle_E + H\circ\alpha_L\langle x,y\rangle_L\bigr).
$$

Example 22 (*Example* 13 *continued*). Taking L and $\mathfrak{a}$ to be hom-Lie algebras, that is $h\circ\iota|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$ and $h\circ s|_L = 0$, we get Theorem 7 from [14].

Example 23 (*Example* 14 *continued*). Consider two color Lie algebras L and $\mathfrak{a}$ with the same grading group Γ and the same commutation factor ε. The vector space $E = \bigoplus_{\gamma\in\Gamma} E_\gamma = \bigoplus_{\gamma\in\Gamma}(L_\gamma\oplus\mathfrak{a}_\gamma) = L\oplus\mathfrak{a}$, is clearly Γ-graded. We know from Theorem 9 and the deduction preceding it that we can endow this with a color structure as follows. From Examples 13 and 14 we see that $f\circ\iota|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$, $f\circ s|_L = 0$ and $h\circ\iota|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$,

$h \circ s|_L = 0$ and so (21) is true. Take $s : x \mapsto (x, 0)$ and define the product on E by $\langle (x,a),(y,b)\rangle_E := (\langle x,y\rangle_L, g(x,y))$ for some $g \in \mathrm{Alt}^2_\varepsilon(L, \mathfrak{a}; \mathcal{E})$. That (23) is satisfied we saw already in Example 14. Note that (22) becomes tautological. Hence we have a color central extension of L by $\mathfrak{a}$. Now E is a color Lie algebra central extension of L by $\mathfrak{a}$. Note, however, that we have not constructed an explicit extension. What we have done is constructing an extension given $g \in \mathrm{Alt}^2_\varepsilon(L, \mathfrak{a}; \mathcal{E})$ satisfying (15) or rather its colored restriction. The existence of such g is not guaranteed in general. See Scheunert [29, Proposition 5.1] for a result that emphasizes this. In our setting this proposition implies that $H^2_\varepsilon(L, \mathfrak{a}; \mathcal{E}) = \{0\}$ and so there are no non-trivial central extensions. The actual construction of extensions, qualifying to finding 2-cocycles, is a highly non-trivial task. Specializing the above to one-dimensional central extensions with $\mathfrak{a} = \mathbb{k}$, we first note that $\mathbb{k}$ comes with a natural Γ-grading given by $\mathbb{k} = \bigoplus_{\gamma\in\Gamma} K_\gamma$, where $K_0 = \mathbb{k}$, $K_\gamma = \{0\}$, for $\gamma \neq 0$. Then there is a product on $E = L \oplus \mathbb{k}$ defined by $\langle (x,a),(y,b)\rangle_E := (\langle x,y\rangle_L, g(x,y))$, where $g : L \times L \to \mathbb{k}$ is the $\mathbb{k}$-valued 2-cocycle.

Example 24 (*Example* 9 *continued*). The classical Witt algebra $\mathfrak{d}$ has a unique one-dimensional central extension in the category of Lie algebras called the Virasoro algebra [10]. When q is not a root of unity, our q-deformation of $\mathfrak{d}$ in Example 9, being a hom-Lie algebra, has a central extension Vir_q in the category of hom-Lie algebras. This is defined as the algebra with linear basis $\{d_n \mid n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ subject to relations

$$\langle \mathrm{Vir}_q, \mathbf{c}\rangle = \langle \mathbf{c}, \mathrm{Vir}_q\rangle = 0,$$

$$\langle d_n, d_m\rangle = (\{n\}_q - \{m\}_q)d_{n+m} + \delta_{n+m,0}\frac{q^{-m}}{6(1+q^m)}\{m-1\}_q\{m\}_q\{m+1\}_q\mathbf{c},$$

with associated map α_{Vir_q} defined by $\alpha_{\mathrm{Vir}_q}(d_n) = q^n d_n$ and $\alpha_{\mathrm{Vir}_q}(\mathbf{c}) = \mathbf{c}$. The Jacobi identity for Vir_q is the same as the one given in Example 9. For the proof of these assertions, see [14].

It would be of interest to develop a theory for quasi-hom-Lie algebra extensions of one qhl-algebra by another qhl-algebra, and apply it to get qhl-algebra extensions of the Virasoro algebra by a Heisenberg algebra [17].

4.3. *Central extensions of the* (α, β, ω)*-deformed loop algebra*

Form the vector space $\hat{\mathfrak{g}} = \check{\mathfrak{g}} \oplus \mathbb{k}\cdot\mathbf{c}$ with a "central element" $\mathbf{c}$ and take the section $s : \check{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $x \otimes t^n \mapsto (x \otimes t^n, 0)$. Define a $\mathbf{c}$-centralizing bilinear product $\langle\cdot,\cdot\rangle_{\hat{\mathfrak{g}}}$ on $\hat{\mathfrak{g}}$ by

$$\langle x\otimes t^n + a\cdot\mathbf{c}, y\otimes t^m + b\cdot\mathbf{c}\rangle_{\hat{\mathfrak{g}}} = \langle x,y\rangle_{\mathfrak{g}} \otimes t^{n+m} + g(x\otimes t^n, y\otimes t^m)\cdot\mathbf{c},$$

for a 2-cocycle-like bilinear map $g : \check{\mathfrak{g}} \times \check{\mathfrak{g}} \to \mathbb{k}$. Define, in addition to this, $\alpha_{\hat{\mathfrak{g}}}(x\otimes t^n + a\cdot\mathbf{c}) := \alpha_{\check{\mathfrak{g}}}(x\otimes t^n) + a\cdot\mathbf{c}$, $\beta_{\hat{\mathfrak{g}}}(x\otimes t^n + a\cdot\mathbf{c}) := \beta_{\check{\mathfrak{g}}}(x\otimes t^n) + a\cdot\mathbf{c}$ and $\omega_{\hat{\mathfrak{g}}}(x\otimes t^n + a\cdot\mathbf{c}, y\otimes t^m + b\cdot\mathbf{c}) := \omega_{\check{\mathfrak{g}}}(x\otimes t^n, y\otimes t^m) + \mathrm{id}$. By straightforward computations it easy to check the ω-skew symmetry of $\langle\cdot,\cdot\rangle_{\hat{\mathfrak{g}}}$ and β-twisting of $\alpha_{\hat{\mathfrak{g}}}$ with the above definitions.

Also, a necessary condition that a 1-dimensional "central extension" $\hat{\mathfrak{g}}$ of $\check{\mathfrak{g}}$ can be given the structure of a qhl-algebra follows from demanding a qhl-Jacobi identity. This condition can be written as

$$\circlearrowleft_{(x,n),(y,m),(z,l)} g\big((\alpha_{\mathfrak{g}} + \mathrm{id}_{\mathfrak{g}})(x) \otimes t^n, \langle y, z\rangle_{\mathfrak{g}} \otimes t^{m+l}\big) = 0. \tag{24}$$

For the full computations see [22].

Now to do this a little more explicit and more in tune with the classical Lie algebra case [10] we construct the product on $\hat{\mathfrak{g}}$ a bit differently. Assume $\omega_{\mathfrak{g}} = \omega_{\check{\mathfrak{g}}} = \omega_{\hat{\mathfrak{g}}} = -1$, that is, that the product is skew-symmetric, we take a σ-derivation D on $\mathbb{k}[t, t^{-1}]$, where σ is the map $t \mapsto qt$, for $q \in \mathbb{k}^*$, the multiplicative group of non-zero elements of $\mathbb{k}$.[1] Explicitly we can take (see Theorem 8) $D = \eta t^{-k}(1-q)^{-1}(\mathrm{id} - \sigma)$ leading to $D(t^n) = \eta\{n\}_q t^{n-k}$. Take a bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}$ and factor the 2-cocycle-like bilinear map g as $g(x \otimes t^n, y \otimes t^m) = B(x, y) \cdot (D(t^n) \cdot t^m)_0$, where the notation $(f)_0$ is the zeroth term in the Laurent polynomial f or, put differently, t times the residue $\mathrm{Res}(f)$. The above trick to factor the 2-cocycle (in the Lie algebra case) as B times a "residue" is apparently due to Kac and Moody from their seminal papers where they introduced what is now known as Kac–Moody algebras, [18] and [24], respectively. This means that $(D(t^n) \cdot t^m)_0 = \eta\{n\}_q \delta_{n+m,k}$. Calculating the 2-cocycle-like condition (24) now leads to

$$\circlearrowleft_{(x,n),(y,m),(z,l)} (\eta \cdot \{n\}_q \cdot \delta_{n+m+l,k}) \cdot B\big((\alpha_{\mathfrak{g}} + \mathrm{id})(x), \langle y, z\rangle_{\mathfrak{g}}\big) = 0,$$

and for $\alpha_{\mathfrak{g}} = \mathrm{id}$, $\eta = 1$, $k = 0$ and $q = 1$ we retrieve the classical 2-cocycle discovered by Kac and Moody. Notice, however, that in the Lie algebra case it is assumed that B is symmetric and $\mathfrak{g}$-invariant, this leading to a nice 2-cocycle identity unlike the one we have here. What we thus obtained by the preceding factorization is a $(\alpha, \beta, -1)_q$-deformed, one-dimensional central extension of the (Lie) loop algebra, where the q-subscript is meant to indicate that we have q-deformed the derivation on the Laurent polynomial as well as the underlying algebra.

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[1] One would be tempted to try a more general σ-derivation with $\sigma(t) = qt^s$ as in Section 3.1.1, but the following construction seems only to work with $s = 1$.

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Further reading

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