ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 155 (2008) 610–621
Topology and its Applications
www.elsevier.com/locate/topol

Extending homeomorphisms from punctured surfaces to handlebodies

Alessia Cattabriga [a,*], Michele Mulazzani [b]

[a] *Department of Mathematics, University of Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy*
[b] *Department of Mathematics, C.I.R.A.M., University of Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy*

Received 26 March 2007; received in revised form 13 December 2007; accepted 14 December 2007

Abstract

Let H_g be a genus g handlebody and $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ be the group of the isotopy classes of orientation preserving homeomorphisms of $\mathrm{T}_g = \partial\mathrm{H}_g$, fixing a given set of $2n$ points. In this paper we find a finite set of generators for $\mathcal{E}_{2n}^g$, the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ consisting of the isotopy classes of homeomorphisms of T_g admitting an extension to the handlebody and keeping fixed the union of n disjoint properly embedded trivial arcs. This result generalizes a previous one obtained by the authors for $n = 1$. The subgroup $\mathcal{E}_{2n}^g$ turns out to be important for the study of knots and links in closed 3-manifolds via (g, n)-decompositions. In fact, the links represented by the isotopy classes belonging to the same left cosets of $\mathcal{E}_{2n}^g$ in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ are equivalent.

MSC: primary 20F38; secondary 57M25

Keywords: (g, b)-decompositions of knots and links; Mapping class groups; Extending homeomorphisms; Handlebodies

1. Introduction and preliminaries

Let H_g be an oriented handlebody of genus $g \geqslant 0$ and $\partial\mathrm{H}_g = \mathrm{T}_g$. Consider a system of n disjoint properly embedded trivial arcs[1] $\mathcal{A} = \{A_1, \dots, A_n\}$ in H_g and let P_{i1}, P_{i2} be the endpoints of the arc A_i, for $i = 1, \dots, n$. We denote with $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ (respectively $\mathrm{MCG}_n(\mathrm{H}_g)$), the group of the isotopy classes of orientation preserving homeomorphisms of T_g (respectively H_g) fixing the set $\mathcal{P}_n = \{P_{i1}, P_{i2} \mid i = 1, \dots, n\}$ (respectively $A_1 \cup \dots \cup A_n$). The group $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ is widely studied and different finite presentations of it are known (see [10,15]). In this article we are interested in studying the subgroup $\mathcal{E}_{2n}^g$ of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$, which is the image of the homomorphism $\mathrm{MCG}_n(\mathrm{H}_g) \to \mathrm{MCG}_{2n}(\mathrm{T}_g)$ induced by restriction. In other words, an element $f \in \mathrm{MCG}_{2n}(\mathrm{T}_g)$ belongs to $\mathcal{E}_{2n}^g$ if it admits an extension to H_g fixing $A_1 \cup \dots \cup A_n$. If $h \in \mathcal{E}_{2n}^g$ we still denote with h an extension of it. The case $n = 0$ is

* Corresponding author.
E-mail addresses: cattabri@dm.unibo.it (A. Cattabriga), mulazza@dm.unibo.it (M. Mulazzani).

[1] A set of mutually disjoint arcs $\{A_1, \dots, A_n\}$ properly embedded in a handlebody H_g is trivial if there exist n mutually disjoint embedded discs, called *trivializing discs*, $D_1, \dots, D_n \subset \mathrm{H}_g$ such that $A_i \cap D_i = A_i \cap \partial D_i = A_i$, $A_i \cap D_j = \emptyset$ and $\partial D_i - A_i \subset \partial\mathrm{H}_g$ for all $i, j = 1, \dots, n$ and $i \neq j$.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.12.010

studied in [18], while in [4] we study the case $n = 1$. Moreover, in [13], the case $g = 0$ is investigated. In this paper we find a finite set of generators for $\mathcal{E}^g_{2n}$ and describe their extension to H_g, for each $n \geqslant 1$ and $g \geqslant 0$.

The main motivation for studying such subgroups lies in their importance in the representation of knots and links via (g,n)-decompositions. The notion of (g,n)-decomposition for links in orientable closed connected 3-manifolds, given by Doll in [7], extends the one of bridge (or plat) decomposition for links in $\mathbf{S}^3$. Roughly speaking, a (g,n)-decomposition of a link $L \subset M$ is the data of a Heegaard surface of genus g in M which cuts the link into two set of n disjoint trivial arcs (the underpasses and the overpasses). It is easy to see that each link admits a (g,n)-decomposition, for suitable g and n with $g \geqslant g_M$ and $n \geqslant n_L$, where g_M denotes the Heegaard genus of M and n_L is the number of components of L.

The use of $(g,1)$-decompositions of knots is revealed to be very fruitful in order to study different topics. For example the strongly-cyclic branched coverings of knots in 3-manifolds are analyzed in [6], while the Alexander polynomial of knots in rational homology spheres is investigated in [14]. The case of knots admitting $(1,1)$-decompositions has been widely studied (see for example [2,3,5,8,9,12,17]). In [11], the Heegaard Floer homology of $(1,1)$-knots is computed, while in [16], the geometry of such knots is determined in terms of the distance of the curve complex associated with their $(1,1)$-decompositions.

In Section 2 we give the definition of (g,n)-decomposition of links and describe its connections with $\mathcal{E}^g_{2n}$. In Section 3 we discuss the genus zero case, while the general case is analyzed in Section 4.

Now we briefly recall the definition of spin of a point along a curve (see [1]). Let P be a point on T_g and c be a simple closed oriented curve on T_g, containing P. Consider a neighborhood N of c, parametrized by coordinates $(y,\theta) \in [-1,1] \times \mathbf{S}^1$, where c is defined by $y = 0$ and P is the point of coordinates $(0,0)$. The *spin* of P about c is the homeomorphism $s_{P,c}$ of T_g obtained extending with the identity the map defined on N by $(y,\theta) \mapsto (y, \theta + \theta')$ where

$$\theta' = \begin{cases} 2\pi(2y+1) & \text{if } -1 \leqslant y \leqslant -1/2, \\ 0 & \text{if } -1/2 \leqslant y \leqslant 1/2, \\ -2\pi(2y-1) & \text{if } 1/2 \leqslant y \leqslant 1. \end{cases}$$

If we denote with t_δ the right-handed Dehn twist along the simple closed curve δ, then $s_{P,c} = t_{c_1}^{-1} t_{c_2}$, where c_1 and c_2 are the curves corresponding to $y = -3/4$ and $y = 3/4$, respectively.

2. (g,n)-decompositions of links

In this section we recall the notion of (g,n)-decompositions of links and describe the connection with $\mathcal{E}^g_{2n}$.

A (g,n)-decomposition for a link L in an orientable closed connected 3-manifold M is the data

$$(M,L) = (\mathrm{H}_g, A_1 \cup \cdots \cup A_n) \cup_\phi (\bar{\mathrm{H}}_g, \bar{A}_1 \cup \cdots \cup \bar{A}_n)$$

where H_g, $\bar{\mathrm{H}}_g$ are two oriented handlebodies of genus g, $\mathcal{A} = \{A_1, \ldots, A_n\}$, $\bar{\mathcal{A}} = \{\bar{A}_1, \ldots, \bar{A}_n\}$ are two systems of n properly embedded trivial arcs in H_g and $\bar{\mathrm{H}}_g$ respectively, and $\phi : (\partial \mathrm{H}_g, \partial A_1 \cup \cdots \cup \partial A_n) \to (\partial \bar{\mathrm{H}}_g, \partial \bar{A}_1 \cup \cdots \cup \partial \bar{A}_n)$ is an attaching (orientation preserving) homeomorphism (see Fig. 1).

A link admitting a (g,n)-decomposition will be called a (g,n)-link. It is easy to see that L admits a (g_M,n)-decomposition, for a suitable n, where g_M denotes the Heegaard genus of M. Indeed, given a Heegaard surface S of genus g_M for M, there exists an immersion of L in S with a finite number of singular points which are double points. So, by a slight modification of this immersion near each double point, we are able to embed L in S, except for a finite trivial set of arcs. Moreover, if n_L denotes the number of components of L, by choosing a sufficiently large g, it is always possible to find a (g,n_L)-decomposition of L, since if $n > n_L$ a (g,n)-decomposition determines a $(g+1, n-1)$-decomposition (see Fig. 2).

Let $L \subset M$ be a (g,n)-link with (g,n)-decomposition $(M,L) = (\mathrm{H}_g, A_1 \cup \cdots \cup A_n) \cup_\phi (\bar{\mathrm{H}}_g, \bar{A}_1 \cup \cdots \cup \bar{A}_n)$ and let $\Upsilon : \bar{\mathrm{H}}_g \to \mathrm{H}_g$ be a fixed (orientation preserving) homeomorphism such that $\Upsilon(\bar{A}_i) = A_i$, for $i = 1, \ldots, n$, then $\varphi = \Upsilon_{|\partial \bar{H}_g} \phi$ is an orientation preserving homeomorphism of $(\partial \mathrm{H}_g, \partial A_1 \cup \cdots \cup \partial A_n)$.

Moreover, since two isotopic attaching homeomorphisms produce equivalent (g,n)-links, we have a natural surjective map from $\mathrm{MCG}_{2n}(\mathrm{T}_g)$, the mapping class group of the $2n$-punctured surface of genus g, to the class $\mathcal{L}_{g,n}$ of all (g,n)-links

$$\Theta_{g,n} : \mathrm{MCG}_{2n}(\mathrm{T}_g) \to \mathcal{L}_{g,n}$$

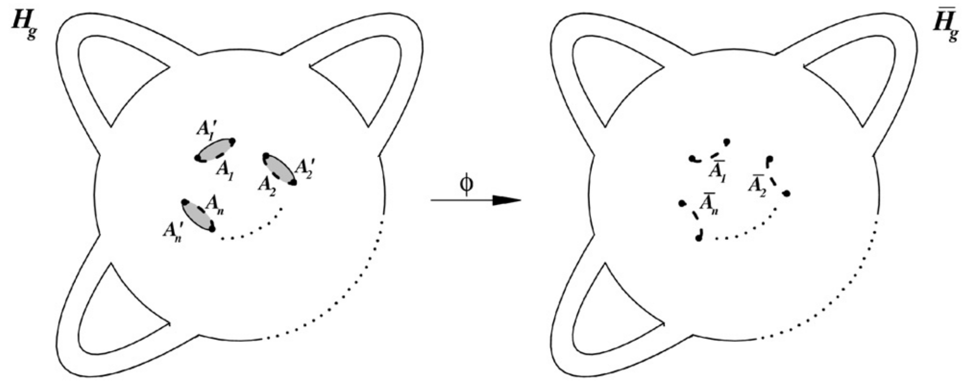

Fig. 1. A (g, n)-decomposition.

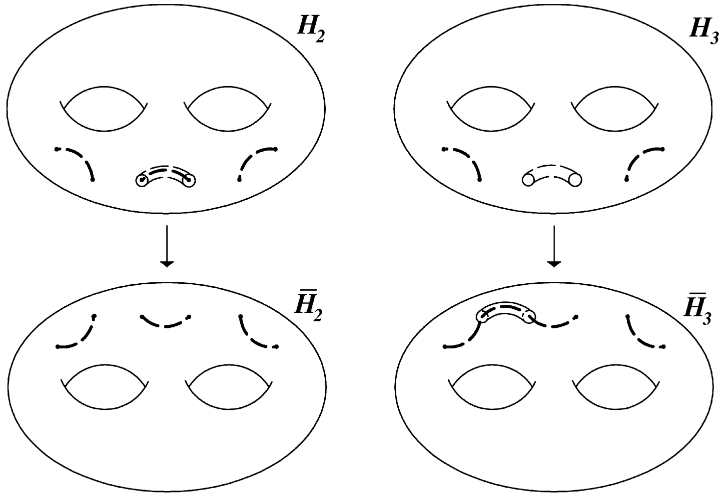

Fig. 2. From a $(2, 3)$-decomposition to a $(3, 2)$-decomposition.

sending $\varphi \in \mathrm{MCG}_{2n}(\mathrm{T}_g)$ to the link $L_\varphi \in \mathcal{L}_{g,n}$, associated to the attaching homeomorphism $\Upsilon^{-1}_{|\partial H_g}\varphi$. Unfortunately $\Theta_{g,n}$ is far away from being injective. Indeed, if we denote with $\mathcal{E}^g_{2n}$ the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ consisting of the elements that admit an extension to H_g, fixing $A_1 \cup \cdots \cup A_n$ as a set, then for each $\varepsilon \in \mathcal{E}^g_{2n}$, the link L_ε is the n-component trivial link in the connected sum of g copies of $\mathbf{S}^2 \times \mathbf{S}^1$. Moreover, for each $\varphi \in \mathrm{MCG}_{2n}(\mathrm{T}_g)$, the links $L_{\varphi\varepsilon}$ and L_φ are equivalent. So we can restrict ourselves to considering the surjective map

$$\Theta'_{g,n} : \mathrm{MCG}_{2n}(\mathrm{T}_g)/\mathcal{E}^g_{2n} \to \mathcal{L}_{g,n},$$

where $\mathrm{MCG}_{2n}(\mathrm{T}_g)/\mathcal{E}^g_{2n}$ denotes the set of left cosets of $\mathcal{E}^g_{2n}$ in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$. So, in this contest, it is important to obtain information about $\mathcal{E}^g_{2n}$. In this paper we find a finite set of generators for this group. In order to do this, let us describe some elements of $\mathcal{E}^g_{2n}$.

Intervals: For $i = 1, \ldots, n$, let D_i be a trivializing disc for A_i. Consider a regular neighborhood N of D_i such that $D_j \cap N = \emptyset$ for $j \neq i$ and parametrize it by $N \cong \mathbf{D}^2 \times [0, 1]$ as in Fig. 3. For $i = 1, \ldots, n$, we denote with ι_i the homeomorphism of H_g obtained extending by the identity the one defined on N by $(\rho, \theta, t) \mapsto (\rho, \theta + \theta', t)$ where

$$\theta' = \begin{cases} \pi & \text{if } 0 \leqslant \rho, t \leqslant 1/2, \\ 2\pi(1-\rho) & \text{if } 1/2 \leqslant \rho \leqslant 1,\ 0 \leqslant t \leqslant 1/2, \\ 2\pi(1-t) & \text{if } 0 \leqslant \rho \leqslant 1/2,\ 1/2 \leqslant t \leqslant 1, \\ 4\pi(1-t)(1-\rho) & \text{if } 1/2 \leqslant \rho, t \leqslant 1. \end{cases} \tag{1}$$

By definition, ι_i exchanges the endpoints of the arc A_i and fixes pointwise the other arcs.

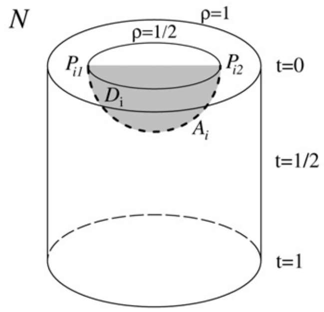

Fig. 3.

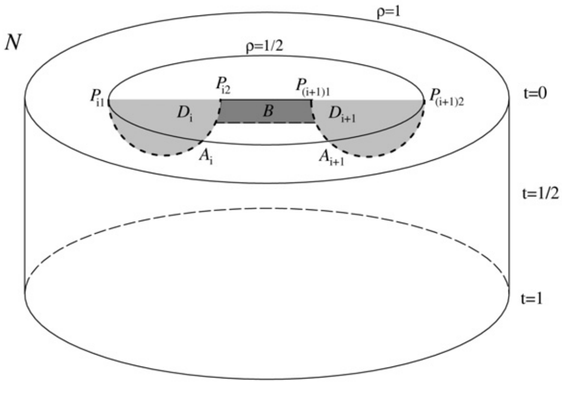

Fig. 4.

Exchanging two arcs: Consider a regular neighborhood N of $D_i \cup B \cup D_{i+1}$, where B is a band connecting D_i and D_{i+1}, such that $D_j \cap N = \emptyset$, for $j \neq i, i+1$, and parametrize it by $N \cong \mathbf{D}^2 \times [0,1]$ as in Fig. 4. For $i = 1, \ldots, n-1$, we denote with $\bar{\lambda}_i$ the homeomorphism of H_g defined as in (1) on N and as the identity outside N. Set $\lambda_i = \iota_{i+1}^{-1} \iota_i^{-1} \bar{\lambda}_i$ exchanges the arcs A_i and A_{i+1} and fixes pointwise the other arcs.

Moreover, let $\mathrm{PMCG}_{2n}(\mathrm{T}_g)$ be the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ consisting of the isotopy classes of the homeomorphisms of T_g pointwise fixing the punctures and set $\bar{\mathcal{E}}_{2n}^g = \mathrm{PMCG}_{2n}(\mathrm{T}_g) \cap \mathcal{E}_{2n}^g$. The next proposition shows that $\mathcal{E}_{2n}^g$ is generated by $\iota_1, \lambda_i, i = 1, \ldots, n-1$, and a set of generators for $\bar{\mathcal{E}}_{2n}^g$.

Proposition 1. *Let Σ_{2n} be the symmetric group on $2n$ letters and denote with Σ'_{2n} the subgroup of Σ_{2n} generated by the transposition $(1\ 2)$ and the permutations $(2i-1\ 2i+1)(2i\ 2i+2)$, for $i = 1, \ldots, n-1$. Then the exact sequence $1 \to \mathrm{PMCG}_{2n}(\mathrm{T}_g) \to \mathrm{MCG}_{2n}(\mathrm{T}_g) \to \Sigma_{2n} \to 1$ restricts to an exact sequence $1 \to \bar{\mathcal{E}}_{2n}^g \to \mathcal{E}_{2n}^g \to \Sigma'_{2n} \to 1$.*

Proof. First of all, we recall that the homomorphism $p : \mathrm{MCG}_{2n}(\mathrm{T}_g) \to \Sigma_{2n}$ is obtained by considering the permutation induced on the punctures by the elements of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$, where the puncture P_{ij} corresponds to the letter $2i + j - 2$, for $i = 1, \ldots, n$ and $j = 1, 2$. Let $\varepsilon \in \mathcal{E}_{2n}^g$, since the extension of ε induces a permutation of the arcs $A_1, \ldots, A_n$, it is easy to see that $p(\varepsilon) \in \Sigma'_{2n}$. Moreover if $p(\varepsilon) = 1$ then an extension of it fixes the arcs, so $\varepsilon \in \bar{\mathcal{E}}_{2n}^g$. In order to complete the proof we only need to check the surjectivity of $\mathcal{E}_{2n}^g \to \Sigma'_{2n}$. This follows by observing that ι_1 maps on $(1\ 2)$ and λ_i maps on $(2i-1\ 2i+1)(2i\ 2i+2)$, for $i = 1, \ldots, n-1$. $\square$

In the next two sections we will find a finite set of generators for $\bar{\mathcal{E}}_{2n}^g$.

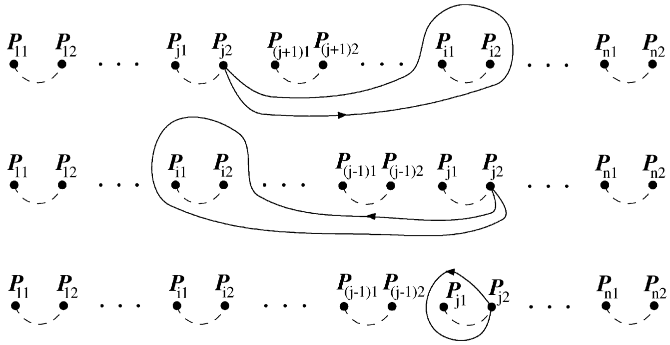

Fig. 5. The spins: s_{ji} with $j < i$, s_{ji} with $j > i$ and s_{jj}.

3. The case of genus zero

We introduce the homeomorphisms whose isotopy classes generate $\bar{\mathcal{E}}^0_{2n}$ and describe their extensions to $\mathbf{B}^3$. For each $i = 1, \ldots, n$, we set $A'_i = D_i \cap \partial\mathbf{B}^3$, where D_i is a trivializing disc for A_i (see Fig. 1).

Spin of a puncture: A simple closed oriented curve $c \subset \partial\mathbf{B}^3$ containing a puncture P_{j2} is called *admissible* if $c \cap A'_i = \emptyset$, $i = 1, \ldots, n$, $i \neq j$ and $c \cap A'_j = P_{j2}$. A spin $s_{P_{j2},c}$ of the puncture P_{j2} along an admissible curve c can be extended to $\mathbf{B}^3$ as follows. We can suppose that a regular neighborhood N of c in $\partial\mathbf{B}^3$ does not intersect the arcs A'_i, for $i \neq j$, and so there exists an embedded ball $\bar{N} = [-1, 1] \times \mathbf{D}^2$ in $\mathbf{B}^3$ such that $[-1, 1] \times \partial\mathbf{D}^2$ is N, $\bar{N} \cap A_i = \emptyset$, for $i \neq j$ and $\bar{N} \cap A_j$ is an arc l transversal to the trivial fibration of $\bar{N}$ in discs. We can extend $s_{P_{j2},c}$ to $\mathbf{B}^3$ by the identity outside $\bar{N}$ and by making the mapping cone from the center of each disc $\{y\} \times \mathbf{D}^2$ inside $\bar{N}$. Obviously such an extension keeps A_i fixed for $i \neq j$. Since two trivial properly embedded arcs in a ball with the same endpoints are ambient isotopic by an isotopy fixing the boundary, then $s_{P_{j2},c}$ is isotopic to a homeomorphism that keeps l and so A_j fixed. Therefore, it admits an extension that keeps all the arcs fixed and, as a consequence, $s_{P_{j2},c} \in \bar{\mathcal{E}}^0_{2n}$.

We denote with s_{ji} the spin of the point P_{j2} about the curve depicted in Fig. 5, for $i, j = 1, \ldots, n$.

Remark 2. Note that if $c = c_1 \cdots c_k$ in $\pi_1(\mathbf{S}^2 - \{P_{j1}, P_{i1}, P_{i2} \mid i = 1, \ldots, n,\ i \neq j\}, P_{j2})$ then $s_{P_{j2},c} = s_{P_{j2},c_k} \cdots s_{P_{j2},c_1}$ in $\mathrm{PMCG}_{2n}(\mathbf{S}^2)$. Moreover if c is admissible, then we can take the c_i's in $\pi_1(\mathbf{S}^2 - (\{P_{j1}\} \bigcup_{i \neq j} A'_i), P_{j2})$ and so $s_{P_{j2},c}$ is a product of the s_{ji}'s.

Slide of an arc: Consider a simple closed oriented curve $c \subset \mathbf{S}^2 - \mathcal{P}_n$ such that $c \cap A'_j$ is a single point $P \neq P_{j1}, P_{j2}$. The slide $S_{j,c}$ of the arc A_j along the curve c is defined as the spin of the point P along the curve c, where we require that the parametrization of the regular neighborhood $N \cong [-1, 1] \times \mathbf{S}^1$ of c is chosen such that A'_j is parametrized by $[-1/2, 1/2] \times \{0\}$, and so is kept fixed by $S_{j,c}$. Note that $S_{j,c}$ is isotopic to the product of spins $s_{P_{j1},c_1} s_{P_{j2},c_2}$, where c_1 and c_2 are the curves parametrized by $\{-1/2\} \times \mathbf{S}^1$ and $\{1/2\} \times \mathbf{S}^1$. In order to extend $S_{j,c}$ to the whole ball, let us consider an embedded solid torus $\tilde{N} = [-1, 1] \times \mathbf{S}^1 \times [0, 1]$ in $\mathbf{B}^3$ such that $N = [-1, 1] \times \mathbf{S}^1 \times \{0\}$, $\tilde{N} \cap A_i = \emptyset$, for $i = 1, \ldots, n$, $i \neq j$, and $\tilde{N} \cap A_j \subset [-1, 1] \times \mathbf{S}^1 \times [0, 1/2]$. Then we can extend $S_{j,c}$ to $\mathbf{B}^3$ by the identity outside $\tilde{N}$, and in $\tilde{N}$ by the map

$$(y, \theta, t) \mapsto \begin{cases} (S_{j,c}(y, \theta), t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ (y, \theta + \theta', t) & \text{if } 1/2 \leqslant t \leqslant 1, \end{cases}$$

with

$$\theta' = \begin{cases} -4\pi(2 - 2t)(y + 1) + 2\pi & \text{if } -1 \leqslant y \leqslant -1/2, \\ 2\pi(2t - 1) & \text{if } -1/2 \leqslant y \leqslant 1/2, \\ 4\pi(2 - 2t)(y - 1) + 2\pi & \text{if } 1/2 \leqslant y \leqslant 1. \end{cases}$$

It is clear that $S_{j,c}$ fixes all the arcs and so belongs to $\bar{\mathcal{E}}^0_{2n}$.

We denote with S_{ji} and S'_{ji} the slides of the arc A_j along the curves depicted in Fig. 6(a) and (b), respectively, for $i, j = 1, \ldots, n$, $i \neq j$.

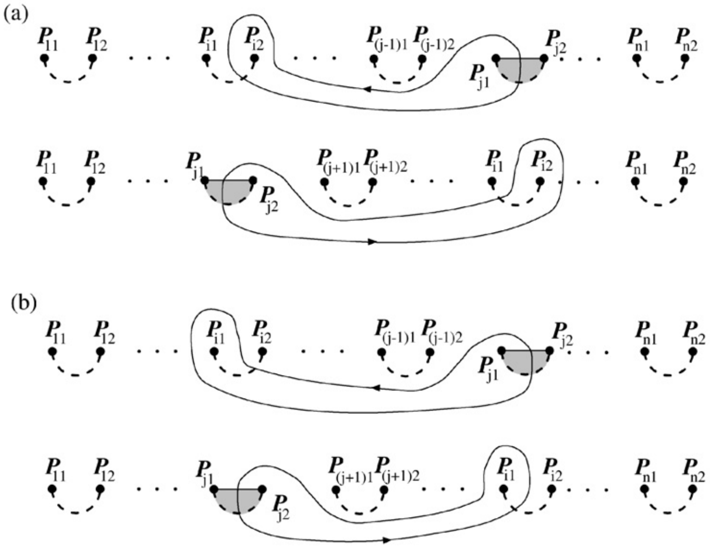

Fig. 6. Slides S_{ji} and S'_{ji}.

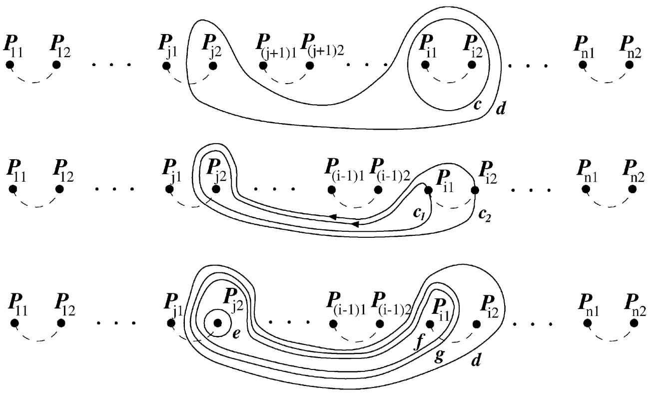

Fig. 7.

Lemma 3. *The following relation holds $s_{ji} = S_{ij}s_{ii}^{-1}$ for each $i, j = 1, \ldots, n$ and $i \neq j$.*

Proof. Let $j < i$. Referring to Fig. 7 we have that $s_{ji} = t_c^{-1}t_d = s_{ii}^{-1}t_d$. Moreover, $S_{ij} = s_{P_{i1},c_1}s_{P_{i2},c_2} = t_e^{-1}t_f t_g^{-1}t_d = t_d$, where the last equality holds since g and f are isotopic curves and t_e is trivial. So $s_{ji} = s_{ii}^{-1}S_{ij}$. The proof in the case $j > i$ is completely analogous. □

Proposition 4. *(See [4].) Consider the homomorphism $j_{g,n,m} : \mathrm{PMCG}_n(\mathrm{T}_g) \to \mathrm{PMCG}_m(\mathrm{T}_g)$ induced by the inclusion, with $1 \leqslant m < n$ for $g \geqslant 1$ and $3 \leqslant m < n$ for $g = 0$. Then $\ker j_{g,n,m} \cong \pi_1(F_{m,n-m}(\mathrm{T}_g))$, where $F_{m,n-m}(\mathrm{T}_g)$ denotes the configuration space of $n - m$ points in T_g with m punctures.*

The next result describes a set of generators for the group $\bar{\mathcal{E}}_{2n}^0$.

Theorem 5. *The subgroup $\bar{\mathcal{E}}_{2n}^0$ of $\mathrm{PMCG}_{2n}(\mathbf{S}^2)$ is generated by spins s_{jj} of P_{j2} along the simple closed oriented curve depicted in Fig. 5 and slides S_{ji}, S'_{ji} of the arc A_j along the curves depicted in Fig. 6(a) and (b), respectively, for $i, j = 1, \ldots, n$, $i \neq j$.*

Proof. Let $\mathcal{G}_{2n}^0$ be the subgroup of $\bar{\mathcal{E}}_{2n}^0$ generated by the following elements:

(a) spins $s_{P_{j2},c}$ of P_{j2} along admissible curves on $\mathbf{S}^2$, $j=1,\dots,n$;
(b) slides $S_{j,c}$ of the arc A_j along simple closed oriented curves, in $\mathbf{S}^2-\mathcal{P}_n$ intersecting A'_j in a single point different from P_{j1} and P_{j2}, $j=1,\dots,n$.

We will prove that $\mathcal{G}^0_{2n}=\bar{\mathcal{E}}^0_{2n}$ by induction on n, and that $\mathcal{G}^0_{2n}$ is generated by s_{ji} with $i,j=1,\dots,n$, $i\leqslant j$ and S_{ji},S'_{ji}, with $i,j=1,\dots,n$, $j\neq i$.

For $n=1$ the group $\bar{\mathcal{E}}^0_2$ is trivial since $\mathrm{PMCG}_2(\mathbf{S}^2)$ is trivial, so there is nothing to prove. In order to prove the inductive step we need the following lemma.

Lemma 6. *Given an element $h\in\bar{\mathcal{E}}^0_{2n}$ there exists an element $h'\in\mathcal{G}^0_{2n}$ such that $h'h(A'_n)=A'_n$.*

Proof. Let $\tilde{A}_0=h(A'_n)$ and $\tilde{D}_0=h(D_n)$. Suppose that $\tilde{A}_0\cap A'_i=\emptyset$ for $i=1,\dots,n-1$. Then the closed curve $c=\tilde{A}_0\cup A'_n$ based on P_{n2} is homotopic rel P_{n2} to the product of simple closed curves $c_1\cdots c_k$ in $\mathbf{S}^2-\bigcup_{i=1}^{n-1}A'_i$ and each c_j is a curve of type (a). Moreover $s_{P_{n2},c_k}\cdots s_{P_{n2},c_1}(\tilde{A}_0)=A'_n$, so in this case $h'=s_{P_{n2},c_k}\cdots s_{P_{n2},c_1}$.

Let $j\in\{1,\dots,n-1\}$ such that $\tilde{A}_0\cap A'_j\neq\emptyset$. Up to a small deformation we can suppose that D_j and $\tilde{D}_0$ intersect transversally and so the connected components of the intersection are arcs or circles. Let us consider the circular components. By an innermost argument, it is possible to choose one of them, let us say C, such that the union of the discs bounded by C on D_j and $\tilde{D}_0$ is a sphere S that intersects $D_j\cap\tilde{D}_0$ only in C. Obviously S does not contain any of the arcs A_i and then, by an isotopy, the intersection C can be removed. Iterating the procedure we can remove all the circular intersections. Now, let us consider the arcs. Since $h(A_n)=A_n$, then $h(A_n)\cap A_j=\emptyset$. So the endpoints of the arcs in $D_j\cap\tilde{D}_0$ are points of $\tilde{A}_0\cap A'_j$. By an innermost argument, there exists an arc B_0 that determines a disc both in $\tilde{D}_0$ and in D_j, whose union is a disc $\bar{D}_0$, properly embedded in the ball, that intersects $\tilde{D}_0\cap D_j$ only in B_0. Let Δ_0 be the connected component of $\mathbf{B}^3-\bar{D}_0$ that does not contain A_j. If none of the A_i for $i=1,\dots,n-1$ is contained in Δ_0 then, by isotopy, the intersection B_0 can be removed.

Otherwise, let A_i be contained in Δ_0. Referring to Fig. 8, choose a simple oriented closed curve c on $\mathbf{S}^2$ such that $c\cap A'_i$ is a single point and $c\cap\tilde{A}_0$ is a single point in $\Delta_0\cap\mathbf{S}^2$ (the dashed disc in the top part of the figure represents $\Delta_0\cap\mathbf{S}^2$). Consider the slide $S_{i,c}$ of the arc A_i along the curve c and let $h_1=S_{i,c}h$. The figure illustrates the image of $\tilde{A}_0$ after the application of the homeomorphism $S_{i,c}$ and, subsequently, of a homeomorphism isotopic to the identity. If we set $\tilde{A}_1=S_{i,c}(\tilde{A}_0)=h_1(A'_n)$ and $\tilde{D}_1=S_{i,c}(\tilde{D}_0)=h_1(D_n)$, we have that $\tilde{D}_1\cap D_j$ has the same number of connected components of $\tilde{D}_0\cap D_j$, and $\tilde{A}_1\cap A'_k=\tilde{A}_0\cap A'_k$, for each $k=1,\dots,n-1$. Moreover, by the same argument used before, it is possible to find an arc B_1 that determines a disc both in $\tilde{D}_1$ and in D_j, whose union is a disc $\bar{D}_1$, properly embedded in the ball, that intersects $\tilde{D}_1\cap D_j$ only in B_1. If we denote with Δ_1 the connected component of $\mathbf{B}^3-\bar{D}_1$ that does not contain A_j (the dashed disc in the bottom part of the figure represents $\Delta_1\cap\mathbf{S}^2$) then $(\bigcup_{i=1}^n A_i)\cap\Delta_1=(\bigcup_{i=1}^n A_i)\cap\Delta_0-A_i$. So, after k-steps and composing h with the opportune slides, we obtain the case of $(\bigcup_{i=1}^n A_i)\cap\Delta_k=\emptyset$ and so as before we can remove the intersection arc B_k. Since the intersections of $\tilde{D}_i\cap D_j$ are finitely many, we can remove them all and so return to the case in which $\tilde{A}_k\cap A'_j=\emptyset$.

Since during the process we do not increase the number of intersections of $\tilde{A}_k$ with the other arcs A'_h by applying it to the other arcs, we return to the case $\tilde{A}_i\cap A'_k=\emptyset$ for each $k=1,\dots,n-1$ already considered. □

Continuation of the proof of Theorem 5. By the previous lemma it is enough to consider the subgroup $\tilde{\mathcal{E}}^0_{2n}$ of $\bar{\mathcal{E}}^0_{2n}$, consisting of the elements fixing A'_n. Let $h\in\tilde{\mathcal{E}}^0_{2n}$, by hypothesis $h(A_n)=A_n$, so we can assume that h fixes the whole disc D_n, as well as a regular neighborhood N of it. By contracting N to the point P_{n2} we obtain a surjective map $i_1:\tilde{\mathcal{E}}^0_{2n}\to E^0_{2n-1}$, where E^0_{2n-1} is the group of elements of $\mathrm{PMCG}_{2n-1}(\mathbf{S}^2)$ that extend to the ball $\mathbf{B}^3$ fixing $A_1,\dots,A_{n-1}$. Moreover, the surjective homomorphism $j_{0,2n-1,2n-2}:\mathrm{PMCG}_{2n-1}(\mathbf{S}^2)\to\mathrm{PMCG}_{2n-2}(\mathbf{S}^2)$ of Proposition 4 restricts to a surjective homomorphism $i_2:E^0_{2n-1}\to\bar{\mathcal{E}}^0_{2n-2}$. When $n=2$, the kernel of i_2 is trivial since $\mathrm{PMCG}_3(\mathbf{S}^2)=\mathrm{PMCG}_2(\mathbf{S}^2)=1$. Otherwise, when $n>2$, by [1, pp. 158–160] and [4], $\ker i_2$ is generated by $A=\{s_{P_{n2},\gamma_i}\mid i=1,\dots,2n-3\}$, where the loops γ_i are generators of $\pi_1(\mathbf{S}^2-\mathcal{P}_{n-1},P_{n2})$. By the induction hypothesis, $\bar{\mathcal{E}}^0_{2n-2}$ is generated by $B=\{s_{P_{j2},c}\mid c\cap A'_i=\emptyset,\ c\cap A'_j=P_{j2},\ i,j=1,\dots,n-1,\ i\neq j\}$ and $C=\{S_{j,c}\mid c\subset\mathbf{S}^2-\mathcal{P}_{n-1},\ \sharp(c\cap A'_j)=1,\ c\cap A'_j\neq P_{j1},P_{j2}j=1,\dots,n-1\}$. So a complete set of generators for $\tilde{\mathcal{E}}^0_{2n}$ is given by the generators

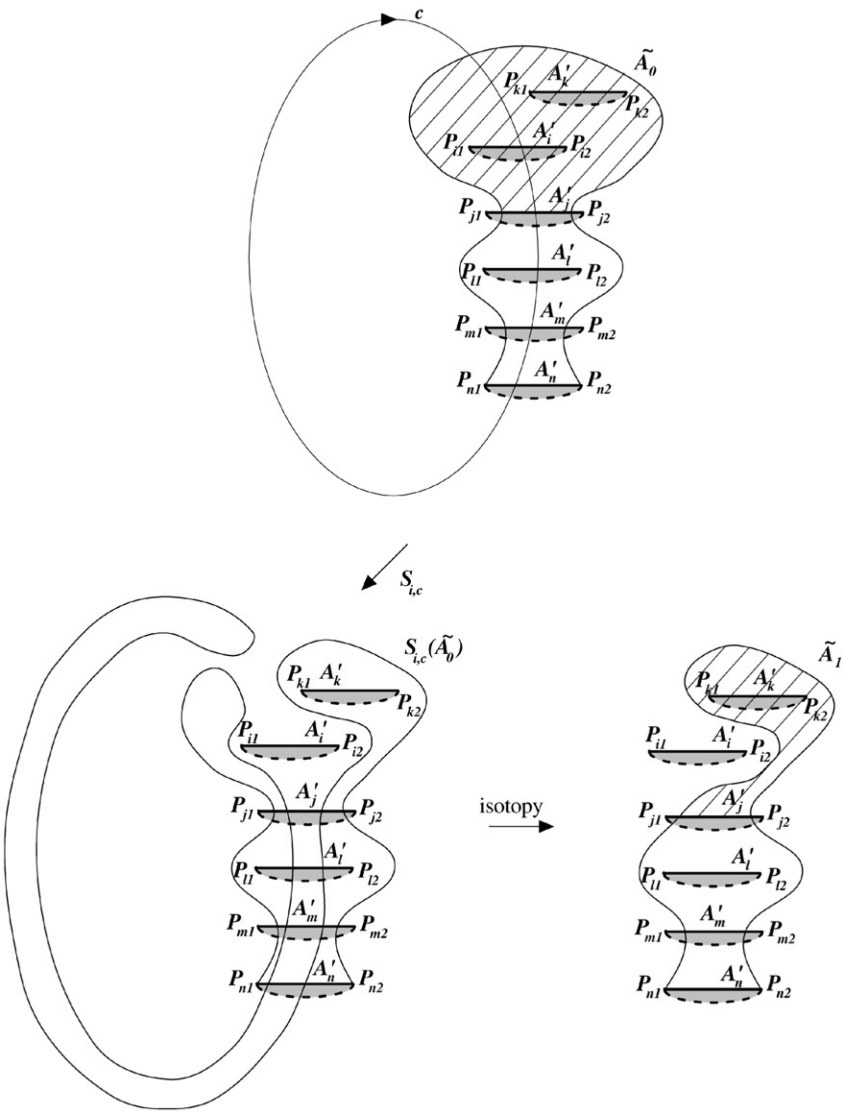

Fig. 8.

of $\ker i_1$, the lifts of elements in A via i_1, and the lifts of elements in B and C via $i_2 i_1$. The kernel of i_1 is generated by s_{nn} and so it is contained in $\mathcal{G}^0_{2n}$. A spin $s_{P_{n2},\gamma_i} \in A$ lifts to a slide of the arc A_n along a suitable curve, and so belongs to $\mathcal{G}^0_{2n}$. An element $s_{P_{j2},c}$ in B lifts to a spin still based on P_{j2} along the lifting of c. Moreover, since we can suppose that c avoids P_{n2}, its lifting does not intersect A'_n, and so belongs to $\mathcal{G}^0_{2n}$. In the same way the lifting of each element in C belongs to $\mathcal{G}^0_{2n}$.

Now we find a finite set of generators for $\mathcal{G}^0_{2n}$. From Remark 2 and Lemma 3, it follows that elements of type (a) are generated by s_{jj} and S_{ji} for $i, j = 1, \ldots, n$, $i \neq j$. Note that a slide S of the arc A_j fixes A'_j. Consider the analogue of the map i_1 obtained by contracting a regular neighborhood of A'_j to P_{j2}. The kernel of the map is s_{jj}, while the image of S is a spin s based on P_{j2} along a curve in $\mathbf{S}^2 - \{P_{j1}, P_{i1}, P_{i2} \mid i = 1, \ldots, n,\ i \neq j\}$. Since s decomposes into a product of spins along a set of curves whose homotopy classes generate $\pi_1(\mathbf{S}^2 - \{P_{j1}, P_{i1}, P_{i2} \mid i = 1, \ldots, n,\ i \neq j\}, P_{j2})$ and the lifting of each spin is one of the S_{ji}'s or the S'_{ji}'s, then S is a product of the S_{ji}'s and S'_{ji}'s. $\square$

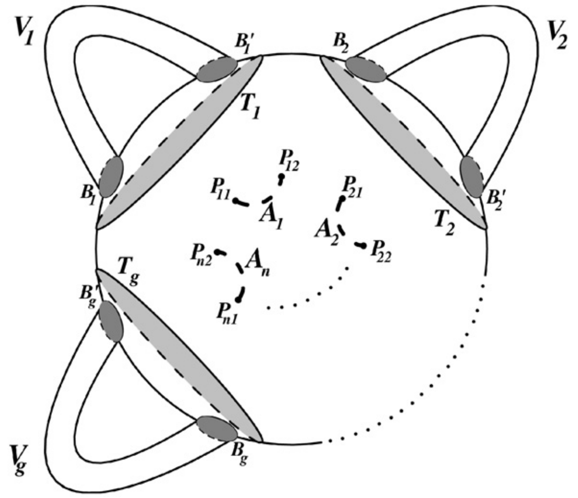

Fig. 9. A model for H_g.

Corollary 7. *The subgroup* $\mathcal{E}^0_{2n}$ *of* $\mathrm{MCG}_{2n}(\mathbf{S}^2)$ *is generated by* $\iota_1, \lambda_k, s_{11}, S_{12}, S'_{12}$, *for* $k = 1, \dots, n-1$.

Proof. By Theorem 5 and Proposition 1 the group $\mathcal{E}^0_{2n}$ is generated by ι_1, λ_k, s_{jj}, S_{ji}, S'_{ji}, for $j, i = 1 \dots, n$, $i \neq j$ and $k = 1, \dots, n-1$. Set $\Lambda_i = \lambda_1^{-1} \cdots \lambda_{i-1}^{-1}$, then the statement follows from the relations $s_{ii} = \Lambda_i^{-1} s_{11} \Lambda_i$, $S_{ji} = (\lambda_1 \Lambda_i \Lambda_j)^{-1} S_{12} \lambda_1 \Lambda_i \Lambda_j$ if $j < i$ and $S_{ji} = (\Lambda_j \Lambda_i)^{-1} S_{12} \Lambda_j \Lambda_i$ if $j > i$. $\square$

We note that the sets of generators obtained for $\mathcal{E}^0_{2n}$ and $\bar{\mathcal{E}}^0_{2n}$ is considerably smaller than the ones obtained in [13].

4. The general case

We are now ready to analyze the general case. We introduce the homeomorphisms whose isotopy classes generate $\bar{\mathcal{E}}^g_{2n}$ and describe their extension to H_g.

Semitwist of a handle: Let T_i be the 2-cell depicted in Fig. 9 for $i = 1, \dots, g$. Then T_i cuts away from H_g a solid torus K_i, containing the ith handle V_i. We denote with ω_i the homeomorphism of H_g which is a counterclockwise rotation of π radians of K_i along T_i on K_i and the identity outside a regular neighborhood of K_i, for $i = 1, \dots, g$. As usual, we still denote with ω_i its restriction to T_g. Note that ω_i^2 is isotopic to the Dehn twist along ∂T_i.

Twist of a meridian disk: Referring to Fig. 9, let $\beta_i = \partial B_i$, for $i = 1, \dots, g$. We denote with τ_i the right-handed Dehn twist along β_i. Note that τ_i admits an extension to H_g, whose effect is to give a complete twist to the ith meridian disk B_i.

Exchanging two handles: Let C_i be the properly embedded 2-cell, depicted in Fig. 10, for $i = 1, \dots, g-1$. Then C_i cuts away from H_g a handlebody K'_i of genus two, containing the ith and the $(i+1)$st handles. Let $\bar{\rho}_i$ be the homeomorphism of H_g which exchanges V_i and V_{i+1} by a counterclockwise rotation of π radians along C_i and is the identity outside a regular neighborhood of K'_i. We set $\rho_i = \omega_i^{-1} \omega_{i+1}^{-1} \bar{\rho}_i$ for $i = 1, \dots, g-1$. Moreover, for $i < j$ we set $\rho_{ij} = \rho_i \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1} \rho_{j-2}^{-1} \cdots \rho_{i+1}^{-1} \rho_i^{-1}$. Obviously ρ_{ij} exchanges the ith handle with the jth handle and keeps fixed the other handles.

Slides of a meridian disc: Referring to Fig. 9, let Z_i and Z'_i be the centers of the properly embedded meridian discs B_i and B'_i in H_g, respectively. Moreover, denote with H^i_g the genus $(g-1)$ handlebody obtained by removing the ith handle V_i from H_g. A simple closed oriented curve e on $\partial \mathrm{H}^i_g - \mathcal{P}_n$ containing Z_i, with $B'_i \cap e = \emptyset$, will be called an *i-loop*. If we require that in the parametrization of a regular neighborhood $N \cong [-1, 1] \times \mathbf{S}^1$ of e the disk B_i is contained in $[-1/2, 1/2] \times \mathbf{S}^1$ and $B'_i \cap N = \emptyset$, then the spin of Z_i along e keeps B_i and B'_i fixed, and so can be

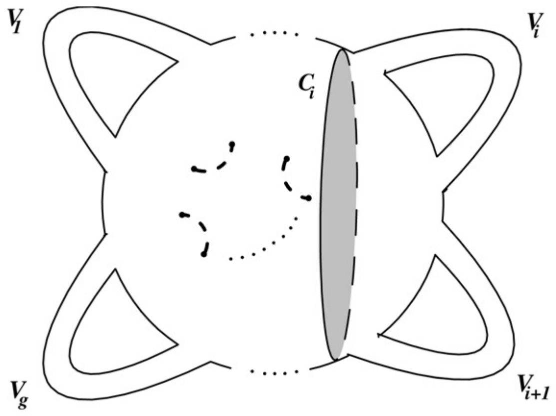

Fig. 10. Exchanging two handles.

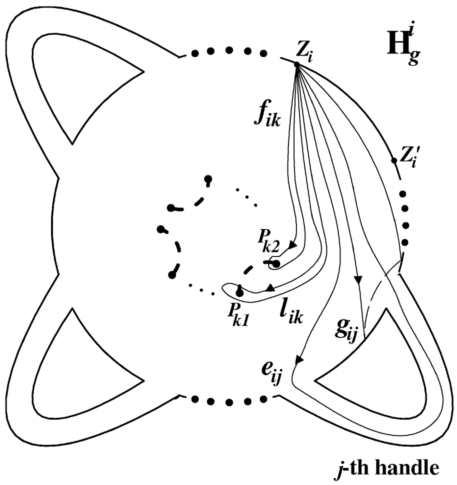

Fig. 11. The loops $e_{ij}, g_{ij}, f_{ik}, l_{ik}$, on H_g^i.

extended to V_i by the identity and to H_g^i in the same way as the extension of a slide of an arc (see p. 614). We call this homeomorphism of H_g, as well as its restriction to T_g, a *slide of* B_i along the ith loop e and denote it with $\sigma_{i,e}$. In a completely analogous way we can define an i'-loop e' and a slide of B_i' along e' and denote it with $\sigma'_{i',e'}$. Note that $\sigma'_{i',e'} = \omega_i^{-1}\sigma_{i,e}\omega_i$, where $e = \omega_i(e')$. We set $\theta_{ij} = \sigma_{i,e_{ij}}$, $\eta_{ij} = \sigma_{i,g_{ij}}$, $\xi_{ik} = \sigma_{i,f_{ik}}$, $\zeta_{ik} = \sigma_{i,l_{ik}}$, where $e_{ij}, g_{ij}, f_{ik}, l_{ik}$ are the oriented curves depicted in Fig. 11, with $i, j = 1, \dots, g$, $i \neq j$, and $k = 1, \dots, n$.

Remark 8. By [18, Lemma 3.6] if the i-loop e is homotopic to the product of i-loops $e_1 \cdots e_n$ on $\partial\mathrm{H}_g^i - (\mathcal{P}_n \cup \{Z_i'\})$ rel Z_i, then $\sigma_{i,e}$ is isotopic to $\sigma_{i,e_n} \cdots \sigma_{i,e_1}$ modulo τ_i. Since the loops $e_{ij}, g_{ij}, f_{ik}, l_{ik}$, for $k = 1, \dots, n$ and $j = 1, \dots g$, with $i \neq j$, are a free set of generators for $\pi_1(\partial\mathrm{H}_g^i - (\mathcal{P}_n \cup \{Z_i'\}), Z_i)$, then $\theta_{ij}, \eta_{ij}, \xi_{ik}, \zeta_{ik}, \tau_i$, with $k = 1, \dots, n$, $j = 1, \dots, g$ and $i \neq j$, generate all the slides of the ith meridian disc B_i.

Now we are ready to describe a finite set of generators for $\bar{\mathcal{E}}_{2n}^g$. In the following statement, the maps s_{kk}, S_{kl}, S_{kl}' are the ones introduced in Section 3, where the defining curves (see Figs. 5 and 6) are contained in a disc embedded in $\partial\mathrm{H}_g - \partial(V_1 \cup \cdots \cup V_g)$.

Theorem 9. *The subgroup* $\bar{\mathcal{E}}_{2n}^g$ *of* $\mathrm{PMCG}_{2n}(\mathrm{T}_g)$ *is generated by* τ_1, ω_1, ρ_i, θ_{12}, η_{12}, ξ_{1k}, ζ_{1k}, s_{kk}, S_{kl}, S_{kl}', *with* $i = 1, \dots, g$ *and* $k, l = 1, \dots, n$, $l \neq k$.

Proof. Let $\mathcal{G}^g_{2n}$ be the subgroup of $\mathrm{PMCG}_{2n}(\mathrm{T}_g)$ generated by $\tau_i, \omega_i, \rho_i, \rho_{im}, \theta_{ij}, \eta_{ij}, \xi_{ik}, \zeta_{ik}, \sigma_{i,e}, \sigma'_{i,e'}, s_{kh}, S_{kl}, S'_{kl}$, where e is an i-loop, e' is an i'-loop, $i, j = 1, \ldots, g$, $i \neq j$, $i < m$ and $k, h, l = 1, \ldots, n$, $h \leqslant k$, $l \neq k$.

We will prove that $\mathcal{G}^g_{2n} = \bar{\mathcal{E}}^g_{2n}$, by induction on g, and that $\mathcal{G}^g_{2n}$ is generated by $\tau_1, \omega_1, \rho_i, \theta_{12}, \eta_{12}, \xi_{1k}, \zeta_{1k}, s_{kh}, S_{kl}, S'_{kl}$, with $i = 1, \ldots, g$ and $k, h, l = 1, \ldots n$, $h \leqslant k$, $l \neq k$.

The case $g = 0$ is proved in Theorem 5.

Now we prove the inductive step. Let $g > 0$ and denote with $\tilde{\mathcal{E}}^g_{2n}$ the subgroup of $\bar{\mathcal{E}}^g_{2n}$ consisting of the isotopy classes of the homeomorphisms that are the identity on the boundary of the gth handle V_g. By the same arguments as for the proof of [18, Lemma 4.4], we have that, for each $h \in \bar{\mathcal{E}}^g_{2n}$, there exists an element $h' \in \mathcal{G}^g_{2n}$ such that $h'h$ is the identity on a meridian disk of the gth handle and so, up to isotopy, on all the handle. Therefore $h'h \in \tilde{\mathcal{E}}^g_{2n}$ and so it is enough to show that $\tilde{\mathcal{E}}^g_{2n} \subseteq \mathcal{G}^g_{2n}$.

Let $f : \mathrm{T}_g \to \mathrm{T}_g$ be a homeomorphism fixing ∂V_g pointwise and whose isotopy class belongs to $\tilde{\mathcal{E}}^g_{2n}$. By cutting out the gth handle, and capping the resulting holes with the two disks B_g and B'_g, we can identify f with a homeomorphism f' of T_{g-1}, such that $f'_{|_{B_g \cup B'_g}} = \mathrm{Id}$ and $f' = f$ on $\mathrm{T}_{g-1} - (B_g \cup B'_g)$. Moreover, by shrinking B_g and B'_g to their centers Z_g and Z'_g, the map f' becomes a map $\tilde{f}$ of T_{g-1} fixing Z_g and Z'_g. In order to simplify the notation, we set $P_{2n+1} = Z_g$ and $P_{2n+2} = Z'_g$. Obviously, $\tilde{f}$ extends to H_{g-1} fixing $A_1 \cup \cdots \cup A_n$ pointwise. So we obtain a surjective map $i_1 : \tilde{\mathcal{E}}^g_{2n} \to E^{g-1}_{2n+2}$, where E^{g-1}_{2n+2} is the subgroup of $\mathrm{PMCG}_{2n+2}(\mathrm{T}_{g-1})$ consisting of the elements which extend to the handlebody fixing $A_1 \cup \cdots \cup A_n$ pointwise. Moreover, the surjective homomorphism $j_{g-1,2n+2,2n} : \mathrm{PMCG}_{2n+2}(\mathrm{T}_{g-1}) \to \mathrm{PMCG}_{2n}(\mathrm{T}_{g-1})$ of Proposition 4 restricts to a surjective homomorphism $i_2 : E^{g-1}_{2n+2} \to \bar{\mathcal{E}}^{g-1}_{2n}$. So, a set of generators of $\tilde{\mathcal{E}}^g_{2n}$ is given by the generators of $\ker i_1$, the lift of the generators of $\ker i_2$ via i_1 and the lift of the generators of $\bar{\mathcal{E}}^{g-1}_{2n}$ via $i_2 i_1$. The kernel of i_1 is generated by τ_g, and so belongs to $\mathcal{G}^g_{2n}$. By [1, pp. 158–160] and [4] $\ker i_2$ is generated by spins of Z_g and Z'_g about appropriate loops not containing $\mathcal{P}_n$, lifting to slides of B_g and B'_g on T_g, which are elements of $\mathcal{G}^g_{2n}$. Moreover, by the induction hypothesis $\bar{\mathcal{E}}^{g-1}_{2n} = \mathcal{G}^{g-1}_{2n}$. Since we can suppose that the generators of $\mathcal{G}^{g-1}_{2n}$ keep Z_g and Z'_g fixed, they lift to elements of $\mathcal{G}^g_{2n}$.

Now we prove that $\tau_1, \omega_1, \rho_i, \theta_{12}, \eta_{12}, \xi_{1k}, \zeta_{1k}, s_{kh}, S_{kl}, S'_{kl}$, with $i = 1, \ldots, g$ and $k, h, l = 1, \ldots n$, $h \leqslant k$, $l \neq k$ generate $\mathcal{G}^g_{2n}$. By Remark 8, and since, as already observed, $\sigma'_{i',e'} = \omega_i^{-1} \sigma_{i,e} \omega_i$, where $e = \omega_i(e')$, then the elements $\theta_{ij}, \eta_{ij}, \xi_{ik}, \zeta_{ik}, \tau_i, \omega_i$ for $k = 1, \ldots, n$ and $i, j = 1, \ldots, g$, with $i \neq j$, generate all the slides $\sigma_{i,e}$ and $\sigma'_{i',e}$. Moreover we have $\theta_{ij} = \rho_{ji}^{-1} \omega_i^{-2} \theta_{ji} \omega_i^2 \rho_{ji}$ if $i > j$ and $\theta_{ij} = \rho_{1i} \rho_{2j} \theta_{12} \rho_{2j}^{-1} \rho_{1i}^{-1}$, if $i < j$. The same relations hold for the other slides of a meridian disk. To end the proof it is enough to observe that $\tau_i = \rho_{1i} \tau_1 \rho_{1i}^{-1}$, $\omega_i = \rho_{1i} \omega_1 \rho_{1i}^{-1}$ and that, by definition, ρ_{ij} is a product of ρ_i's. $\square$

As a consequence of this theorem and Proposition 1 we obtain the following result.

Theorem 10. *The subgroup $\mathcal{E}^g_{2n}$ of* $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ *is generated by $\iota_1, \lambda_k, \tau_1, \omega_1, \rho_i, \theta_{12}, \eta_{12}, \xi_{1k}, \zeta_{1k}, s_{11}, S_{12}, S'_{12}$, with $i = 1, \ldots, g$, $k = 1, \ldots, n-1$.*

References

[1] J.S. Birman, Braids, Links, and Mapping Class Groups, Princeton Univ. Press, Princeton, NJ, 1974.

[2] A. Cattabriga, M. Mulazzani, Strongly-cyclic branched coverings of (1, 1)-knots and cyclic presentations of groups, Math. Proc. Cambridge Philos. Soc. 135 (2003) 137–146.

[3] A. Cattabriga, M. Mulazzani, (1, 1)-knots via the mapping class group of the twice punctured torus, Adv. Geom. 4 (2004) 263–277.

[4] A. Cattabriga, M. Mulazzani, Extending homeomorphisms from 2-punctured surfaces to handlebodies, Kobe J. Math. 24 (2007) 11–20.

[5] D.H. Choi, K.H. Ko, Parametrizations of 1-bridge torus knots, J. Knot Theory Ramifications 12 (2003) 463–491.

[6] P. Cristofori, M. Mulazzani, A. Vesnin, Strongly-cyclic branched coverings of knots via $(g, 1)$-decompositions, math.GT/0402393, Acta Math. Hungar. 116 (2007) 163–176.

[7] H. Doll, A generalized bridge number for links in 3-manifold, Math. Ann. 294 (1992) 701–717.

[8] M. Eudave-Muñoz, Incompressible surfaces and (1, 1)-knots, J. Knot Theory Ramifications 15 (2006) 935–948.

[9] H. Goda, C. Hayashi, H.-J. Song, A criterion for satellite 1-genus 1-bridge knots, Proc. Amer. Math. Soc. 132 (2004) 3449–3456.

[10] S. Gervais, A finite presentation of the mapping class group of a punctured surface, Topology 40 (2001) 703–725.

[11] H. Goda, H. Matsuda, T. Morifuji, Knot Floer homology of (1, 1)-knots, Geom. Dedicata 112 (2005) 197–214.

[12] C. Hayashi, 1-genus 1-bridge splittings for knots, Osaka J. Math. 41 (2004) 371–426.

[13] H.M. Hayashi, Generators for two groups related to the braid group, Pacific J. Math. 59 (1975) 475–486.
[14] Y. Koda, Strongly-cyclic branched coverings and the Alexander polynomial of knots in rational homology spheres, Math. Proc. Cambridge Philos. Soc. 142 (2007) 259–268.
[15] C. Labruère, L. Paris, Presentations of the punctured mapping class groups in terms of Artin groups, Algeb. Geom. Topol. 1 (2001) 73–114.
[16] T. Saito, Genus one 1-bridge knots as viewed from the curve complex, Osaka J. Math. 41 (2004) 427–454.
[17] M. Sakuma, The topology, geometry and algebra of unknotting tunnels. Knot theory and its applications, Chaos Solitons Fractals 9 (1998) 739–748.
[18] S. Suzuki, On homeomorphisms of a 3-dimensional handlebody, Canad. J. Math. 29 (1977) 111–124.