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## Universal curvature identities

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### ABSTRACT

We study scalar and symmetric 2-form valued universal curvature identities. We use this to establish the Gauss–Bonnet theorem using heat equation methods, to give a new proof of a result of Kuz'mina and Labbi concerning the Euler–Lagrange equations of the Gauss–Bonnet integral, and to give a new derivation of the Euh–Park–Sekigawa identity.

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## 1. Introduction and outline of paper

The study of Riemannian geometry relies to a large extent on the examination of curvature and of local curvature invariants of the manifold both for their own sake but also in relationship to other structures (see, for example, [3,4,6,7,15,16,21,22]) – this paper follows in that line of investigation.

### 1.1. Scalar invariants of the metric

Let  $\mathcal{I}_{m,n}$  be the space of scalar invariant local formulas which are homogeneous of order  $n$  in the derivatives of the metric and which are defined in the category of all Riemannian manifolds of dimension  $m$ ; we refer to Section 2 for details. Since  $\mathcal{I}_{m,n} = \{0\}$  if  $n$  is odd, we shall assume  $n$  even henceforth. Such invariants are given by contracting indices in monomials involving the covariant derivatives of the curvature tensor. Let  $R_{ijkl}$  be the components of the curvature tensor relative to a local orthonormal frame  $\{e_1, \dots, e_m\}$  for the tangent bundle of  $M$ . For example, the scalar curvature may be defined by setting:

$$\tau_m := \sum_{i,j=1}^m R_{ijji} \in \mathcal{I}_{m,2}.$$

There is a natural restriction map  $r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}$  given by restricting the summation to range from 1 to  $m-1$  that will be discussed in Section 2. For example, we have that  $r(\tau_m) = \tau_{m-1}$ . Thus the scalar curvature is *universal* and for that reason

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it is not usually subscripted in this fashion. More generally, we have (see, for example, the discussion in [14]) the following universal spanning sets for  $n = 0, 2, 4, 6$ ; we shall suppress the role of the dimension  $m$  to simplify the notation and we shall adopt the *Einstein convention* and sum over repeated indices. Let  $\rho$  be the Ricci tensor and let  $R$  be the full curvature tensor.

**Lemma 1.1.**

- (1)  $\mathcal{I}_{m,0} = \text{Span}\{1\}$ .
- (2)  $\mathcal{I}_{m,2} = \text{Span}\{\tau := R_{ijji}\}$ .
- (3)  $\mathcal{I}_{m,4} = \text{Span}\{\Delta\tau := -R_{ijji;kk}, \tau^2 := R_{ijji}R_{kllk}, |\rho|^2 := R_{ijjk}R_{illk}, |R|^2 := R_{ijkl}R_{ijkl}\}$ .
- (4)  $\mathcal{I}_{m,6} = \text{Span}\{R_{ijji;kkll}, R_{ijji;kR_{mnl;k}}, R_{aija;k}R_{bijb;k}, R_{ajka;n}R_{bjnb;k}, R_{ijkl;n}R_{ijkl;n}, R_{ijji}R_{kllk;nn}, R_{ajka}R_{bjkb;nn}, R_{ajka}R_{bjnb;kn}, R_{ijkl}R_{ijkl;nm}, R_{ijji}R_{kllk}R_{abba}, R_{ijji}R_{ajka}R_{bjkb}, R_{ijji}R_{abcd}R_{abcd}, R_{ajka}R_{bjnb}R_{cknc}, R_{aija}R_{bkkl}R_{ikjl}, R_{ajka}R_{jnli}R_{knli}, R_{ijkn}R_{ijlp}R_{knlp}, R_{ijkn}R_{ilkp}R_{jlnp}\}$ .

Lemma 1.1 follows from Lemma 2.2 (see Section 2) with a bit of work; we shall omit details as we shall not need Lemma 1.1 in what follows and simply present it for the purposes of illustration. The universal scalar invariants given in Lemma 1.1 are linearly independent if  $m \geq n$ . However, they are not linearly independent if  $m = n - 1$  and there is a single additional universal relation amongst these invariants that we may describe as follows. Define the *Pfaffian*  $E_{m,n} \in \mathcal{I}_{m,n}$  for  $n$  even by setting:

$$E_{m,n} := \sum_{i_1, \dots, i_n, j_1, \dots, j_n=1}^m R_{i_1 i_2 j_2 j_1} \cdots R_{i_{n-1} i_n j_n j_{n-1}} g(e^{i_1} \wedge \cdots \wedge e^{i_n}, e^{j_1} \wedge \cdots \wedge e^{j_n}).$$

For example,  $E_{m,2} = 2\tau_m$  is essentially just the scalar curvature. The invariants  $E_{m,n}$  are again *universal*, i.e.

$$E_{m,n} \in \mathcal{I}_{m,n} \quad \text{and} \quad r(E_{m,n}) = E_{m-1,n}.$$

It is also immediate that  $r(E_{m,m}) = 0$  since  $e^{i_1} \wedge \cdots \wedge e^{i_m}$  vanishes on a manifold of dimension  $m - 1$ . Consequently,  $E_{m,m} \in \ker(r : \mathcal{I}_{m,m} \rightarrow \mathcal{I}_{m-1,m})$  and  $E_{m,m}$  provides a universal relation in curvature. Expressing the invariants  $E_{m,2}, E_{m,4}$ , and  $E_{m,6}$  universally in terms of contractions of indices (see, for example, the discussion in [23]) then yields the following relations:

**Lemma 1.2.**

- (1) If  $m = 1$ , then  $0 = R_{ijji}$ .
- (2) If  $m = 3$ , then  $0 = R_{ijji}R_{kllk} - 4R_{aija}R_{bijb} + R_{ijkl}R_{ijkl}$ .
- (3) If  $m = 5$ , then  $0 = R_{ijji}R_{kllk}R_{abba} - 12R_{ijji}R_{aija}R_{bijb} + 3R_{abba}R_{ijkl}R_{ijkl} - 24R_{aija}R_{bkkl}R_{jlik} + 16R_{aija}R_{bjkb}R_{cikc} - 24R_{aija}R_{jkln}R_{lnik} - 2R_{ijkl}R_{klan}R_{anij} + 8R_{kaij}R_{inkl}R_{jlan}$ .

In fact, these the only such universal relations of this type [11]:

**Theorem 1.1.**

- (1)  $r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}$  is always surjective.
- (2) If  $n$  is even and if  $m > n$ , then  $r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}$  is bijective.
- (3) Let  $m$  be even. Then  $\ker\{r : \mathcal{I}_{m,m} \rightarrow \mathcal{I}_{m-1,m}\} = E_{m,m} \cdot \mathbb{R}$ .

1.2. Heat trace asymptotics

Theorem 1.1 was originally established to provide a heat equation proof of the Gauss-Bonnet Theorem [11]. We sketch the derivation to illustrate the use of Theorem 1.1. Let  $(M, g)$  be a compact Riemannian manifold. Let  $\Delta_p$  be the Laplacian on  $p$ -forms. The fundamental solution of the heat equation  $e^{-t\Delta_p}$  is of trace class. If  $f \in C^\infty(M)$ , then there is a complete asymptotic series as  $t \downarrow 0$  of the form

$$\text{Tr}_{L^2}(f e^{-t\Delta_p}) \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M f(x) a_{m,n,p}(x, \Delta_p) dv$$

where  $a_{m,n,p} \in \mathcal{I}_{m,n}$  is a local invariant which is homogeneous of order  $n$  in the jets of the metric and where  $dv$  is the Riemannian measure:

$$dv = g dx^1 \cdots dx^m \quad \text{where} \quad g = \sqrt{\det(g_{ij})} \quad \text{and} \quad g_{ij} = g(\partial_{x^i}, \partial_{x^j}).$$

Note that  $a_{m,n,p} = 0$  if  $n$  is odd. We take the super trace and set

$$a_{m,n} := \sum_{p=0}^m (-1)^p a_{m,n,p} \in \mathcal{I}_{m,n}.$$

The cancellation argument of Bott [1] shows that we have a local formula for the Euler–Poincaré characteristic:

$$\chi(M) = \int_M a_{m,m}(x) d\nu.$$

It also follows using suitable product formulas that  $r(a_{m,n}) = 0$  for any  $(m, n)$ . Let  $m$  be even ( $\chi(M) = 0$  if  $m$  is odd). Theorem 1.1 implies that there is a universal constant  $c_m$  so that

$$a_{m,m} = \begin{cases} 0 & \text{if } n < m \\ c_m E_{m,m} & \text{if } n = m \end{cases} \quad \text{and thus } \chi(M) = \int_M c_m E_{m,m}.$$

The constant is easily determined by evaluation on the manifold  $S^2 \times \cdots \times S^2$  and the Gauss–Bonnet formula results. We remark in passing that it is possible to examine  $\ker(r : \mathcal{I}_{m,m+2} \rightarrow \mathcal{I}_{m-1,m+2})$  and thereby evaluate the next term in the heat expansion  $a_{m,m+2}$  [13].

### 1.3. Symmetric 2-tensor valued invariants

Let  $\mathcal{I}_{m,n}^2$  be the space of symmetric 2-form valued invariants which are homogeneous of degree  $n$  in the derivatives of the metric and which are defined in the category of  $m$ -dimensional Riemannian manifolds; again we refer to Section 2 for further details. Let  $\{e_1, \dots, e_k\}$  be a local orthonormal frame for the tangent bundle of  $M$ . If  $\xi$  and  $\eta$  are cotangent vectors, then the symmetric product is denoted by  $\xi \circ \eta := \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi)$ . For example,  $g = e^k \circ e^k$ . One has:

#### Lemma 1.3.

- (1)  $\mathcal{I}_{m,0}^2 = \text{Span}\{e^k \circ e^k\}$ .
- (2)  $\mathcal{I}_{m,2}^2 = \text{Span}\{R_{ijji}e^k \circ e^k, R_{ijki}e^j \circ e^k\}$ .
- (3)  $\mathcal{I}_{m,4}^2 = \text{Span}\{R_{ijji:kk}e^l \circ e^l, R_{kjjl:ii}e^k \circ e^l, R_{ijji:kl}e^k \circ e^l, R_{ijji}R_{klkk}e^n \circ e^n, R_{ijki}R_{ljkl}e^n \circ e^n, R_{ijkl}R_{ijkl}e^n \circ e^n, R_{ijji}R_{klnk}e^l \circ e^n, R_{ikli}R_{jknj}e^l \circ e^n, R_{ijkl}R_{ijkn}e^l \circ e^n, R_{ijjn}R_{kijk}e^l \circ e^n\}$ .

Lemma 1.3 also follows from Lemma 2.2 and again we shall omit details as we shall not need this result in what follows and simply present it for the purposes of illustration.

Restricting the range of summation and setting  $e^j \circ e^k = 0$  if  $j = m$  or if  $k = m$  yields an analogous restriction map  $r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2$ ; the elements given in Lemma 1.3 are universal with respect to restriction. They are linearly independent if  $m > n$ , but there is a single relation if  $m = n$  we may describe as follows. For  $n$  even, define  $T_{m,n}^2 \in \mathcal{I}_{m,n}^2$  by setting:

$$T_{m,n}^2 := \sum_{i_1, \dots, i_{n+1}, j_1, \dots, j_{n+1}=1}^m R_{i_1 i_2 j_2 j_1} \cdots R_{i_{n-1} i_n j_n j_{n-1}} e^{i_{n+1}} \circ e^{j_{n+1}} \times g(e^{i_1} \wedge \cdots \wedge e^{i_{n+1}}, e^{j_1} \wedge \cdots \wedge e^{j_{n+1}}).$$

It is then immediate that  $r(T_{m,n}^2) = T_{m-1,n}^2$  so these elements are again universal. Furthermore, we again have that  $r(T_{m+1,m}^2) = 0$ . This then leads to the identities:

#### Lemma 1.4.

- (1) If  $m = 2$ , then  $0 = R_{ijji}e^k \circ e^k - 2R_{ijki}e^j \circ e^k$ .
- (2) If  $m = 4$ , then  $0 = -\frac{1}{4}(R_{ijji}R_{klkk} - 4R_{ijki}R_{ljkl} + R_{ijki}R_{ijkl})e^n \circ e^n + \{R_{klmi}R_{klmj} - 2R_{knik}R_{lnjl} - 2R_{iklj}R_{nklm} + R_{klkk}R_{nijm}\}e^i \circ e^j$ .

In fact the identities of Lemma 1.4 are the only universal identities of this form if  $m = 2$  or if  $m = 4$ . In Section 2, we will establish the following extension of Theorem 1.1; this is the main new result of this paper:

#### Theorem 1.2.

- (1)  $r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2$  is always surjective.
- (2) If  $n$  is even and if  $m > n + 1$ , then  $r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2$  is bijective.

(3) If  $m$  is even, then  $\ker\{r : \mathcal{I}_{m+1,m}^2 \rightarrow \mathcal{I}_{m,m}^2\} = \mathcal{I}_{m+1,m}^2 \cdot \mathbb{R}$ .

It is worth presenting an example to illustrate the use of Theorem 1.2. Let  $m = 2$ . Then  $T_{3,2}^2 \in \mathcal{I}_{3,2}^2$  is defined by setting:

$$T_{3,2}^2 = \sum_{i_1, i_2, i_3, j_1, j_2, j_3=1}^3 R_{i_1 i_2 j_2 j_1} e^{i_3} \circ e^{j_3} \times g(e^{i_1} \wedge e^{i_2} \wedge e^{i_3}, e^{j_1} \wedge e^{j_2} \wedge e^{j_3}).$$

Then Theorem 1.2 (3) yields the relation:

$$0 = r(T_{3,2}^2) = 2 \sum_{i,j,k=1}^2 R_{ijji} e^k \circ e^k - 4 \sum_{i,j,k=1}^2 R_{kijj} e^i \circ e^j.$$

This implies the following well-known curvature identity on any 2-dimension Riemannian manifold

$$\rho = \frac{1}{2} \tau_2 g.$$

### 1.4. Euler–Lagrange equations

As was the case for Theorem 1.1, Theorem 1.2 is motivated by index theory. Let  $h$  be an arbitrary symmetric 2-tensor field. We form the 1-parameter family of metrics  $g(\varepsilon) := g + \varepsilon h$ . Since  $E_{m,n}$  only involves the first and second derivatives of the metric, the variation only involves the first and second derivatives of  $h$ . We may therefore express

$$\partial_\varepsilon \{E_{m,n}(g(\varepsilon)) dv_{g(\varepsilon)}\} \Big|_{\varepsilon=0} = Q_{ij}^{m,n} h_{ij} + Q_{ijk}^{m,n} h_{ij;k} + Q_{ijkl}^{m,n} h_{ij;kl},$$

where  $h_{ij;k}$  and  $h_{ij;kl}$  give the components of the covariant derivative of  $h$  with respect to the Levi–Civita connection of  $g$  and where we write  $(m, n)$  as a super script on  $Q$  to avoid notational complexity. Let  $Q_{ijk;l}^{m,n}$  and  $Q_{ijkl;uv}^{m,n}$  be the components of the first and second covariant derivatives of these tensors, respectively. Define:

$$S_{m,n}^2 := \{Q_{ij}^{m,n} - Q_{ijk;k}^{m,n} + Q_{ijkl;lk}^{m,n}\} e^i \circ e^j.$$

It is then immediate from the definition that

$$S_{m,n}^2 \in \mathcal{I}_{m,n}^2 \quad \text{and} \quad r(S_{m,n}^2) = S_{m-1,n}^2.$$

This tensor is characterized by the property that if  $(M, g)$  is any compact Riemannian manifold of dimension  $m$ , then we may integrate by parts to see that:

$$\partial_\varepsilon \left\{ \int_M E_{m,n}(g(\varepsilon)) dv_{g(\varepsilon)} \right\} \Big|_{\varepsilon=0} = \int_M S_{m,n,ij}^2 h_{ij} dv(g).$$

The Gauss–Bonnet theorem shows that this vanishes if  $m = n$ . Therefore

$$S_{m+1,m}^2 \in \ker(r : \mathcal{I}_{m+1,m}^2 \rightarrow \mathcal{I}_{m,m}^2) \quad \text{and thus} \quad S_{m+1,m}^2 = d_m T_{m+1,m}^2.$$

In particular, we establish a conjecture of Berger [5] that  $S_{m,n}^2$  involves only the second derivatives of the metric. This result is, of course, not new. It was first established by Kuz'mina [17] and subsequently established using different methods by Labbi [18–20]. It is at the heart of recent work in 4-dimensional geometry [8–10].

### 1.5. Outline of the paper

In Section 2, we shall define the spaces  $\mathcal{I}_{m,n}$  and  $\mathcal{I}_{m,n}^2$ . We shall discuss the restriction map and derive its elementary properties. We review the first theorem of H. Weyl [24] on the invariants of the orthogonal group. These are used in Lemma 2.3 to show that  $r$  is surjective; this establishes Assertion (1) of Theorem 1.1 and of Theorem 1.2. We will continue our study and complete the proof of Assertion (2) of Theorem 1.1 and of Theorem 1.2 in Lemma 2.5. We then use the second theorem of H. Weyl on the invariants of the orthogonal group to establish Assertion (2) of Theorem 1.1 and of Theorem 1.2.

We remark the generalization of Theorem 1.1 [12] to the complex setting yields a heat equation proof of the Riemann–Roch theorem for Kähler manifolds; it would be interesting to know if there is a suitable generalization of Theorem 1.2 to the Kähler setting that could be used to study the associated Euler–Lagrange equations for the Chern numbers.

## 2. Invariance theory

In this section, we review the basic results of invariance theory that we shall need. We work non-classically in Section 2.1 and use the derivatives of the metric rather than the Riemann curvature tensor to define the space  $\mathcal{I}_{m,n}$  of scalar invariant local formulas and the space  $\mathcal{I}_{m,n}^2$  of symmetric 2-tensor valued invariant local formulas which are homogeneous of degree  $n$  in the jets of the metric in the category of  $m$ -dimensional Riemannian manifolds. In Section 2.2, we give a more classical treatment using the Riemann curvature tensor. In Section 2.3 we review the first theorem of invariants of H. Weyl [24]. In Section 2.4, we discuss the restriction map and establish in Lemma 2.3 that  $r$  is surjective. In Lemma 2.5 we show that  $\ker(r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}) = \{0\}$  if  $m > n$  (resp. that  $\ker(r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2) = \{0\}$  if  $m > n + 1$ ). We also derive some results in the limiting case  $m = n$  (resp.  $m = n + 1$ ) that will be useful subsequently. In Section 2.5 we recall H. Weyl's second theorem of invariants; this result is used in Section 2.6 to complete the proof of Theorem 1.1 and in Section 2.7 to complete the proof of Theorem 1.2. This approach is a bit different from that used in [11] and is, we believe, more instructive.

### 2.1. Local scalar invariants of the metric

We follow the discussion in [11] to establish Theorem 1.1. Let  $\delta_i^j$  and  $\delta_{ij}$  be the Kronecker symbols:

$$\delta_i^j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}.$$

Fix a dimension  $m$ . Let  $\alpha = (a_1, \dots, a_m)$  be a non-trivial multi-index where the  $a_i = \alpha(i)$  are non-negative integers not all of which vanish. Introduce formal variables

$$\{g_{ij} = g_{ji}, g^{ij} = g^{ji}, g, g_{ij/\alpha} = g_{ji/\alpha}\} \quad \text{for } 1 \leq i, j \leq m.$$

Let  $\mathcal{Q}_m$  be the free commutative unital  $\mathbb{R}$  algebra generated by these variables where we impose the obvious relationships:

$$\sum_{k=1}^m g_{ik} g^{jk} = \delta_i^j \quad \text{and} \quad \det(g_{ij}) = g^2;$$

$\mathcal{Q}_m$  is the algebra of local formulae in the derivatives of the metric. Given a system of local coordinates  $\vec{x} = (x^1, \dots, x^m)$  defined near a point  $P$  of a Riemannian manifold  $(M, g)$ , let  $\partial_{x^i} := \frac{\partial}{\partial x^i}$ . It will also be convenient to introduce the following notation for the first and second derivatives of the metric:

$$g_{ij/k} := \partial_{x^k} g_{ij} \quad \text{and} \quad g_{ij/kl} := \partial_{x^k} \partial_{x^l} g_{ij}.$$

If  $Q \in \mathcal{Q}_m$ , then we shall define  $Q(\vec{x}, g, P) \in \mathbb{R}$  by substitution setting:

$$\begin{aligned} g_{ij}(\vec{x}, g, P) &:= g(\partial_{x^i}, \partial_{x^j})(P), & g^{ij}(\vec{x}, g, P) &:= g(dx^i, dx^j)(P), \\ g(\vec{x}, g, P) &:= \det\{g_{ij}(\vec{x}, g, P)\}^{1/2}, & g_{ij/\alpha}(\vec{x}, g, P) &:= \partial_{x^1}^{a_1} \cdots \partial_{x^m}^{a_m} g_{ij}(\vec{x}, g, P). \end{aligned}$$

We say that  $Q$  is invariant if  $Q(\vec{x}, g, P)$  is independent of the coordinate system  $\vec{x}$  for every possible such  $(M, g, P)$ ; we denote this common value by  $Q(g, P)$  and let  $\mathcal{I}_m$  be the vector space of all such invariant local formulae.

We define the weight of  $g_{ij/\alpha}$  to be  $|\alpha| := a_1 + \dots + a_m$  and the weight of  $\{g_{ij}, g^{ij}, g\}$  to be zero. Let  $\mathcal{I}_{m,n} \subset \mathcal{I}_m$  be the space of invariant local formulas which are weighted homogeneous of order  $n$ . One can use dimensional analysis to establish [11] that:

**Lemma 2.1.** *Let  $Q \in \mathcal{I}_m$ . Then  $Q \in \mathcal{I}_{m,n}$  if and only if  $Q(c^2 g, P) = c^{-n} Q(g, P)$  for all  $0 \neq c \in \mathbb{R}$  and all  $(M, g, P)$ .*

As a consequence of Lemma 2.1, we may decompose  $\mathcal{I}_m = \bigoplus_n \mathcal{I}_{m,n}$  as the graded direct sum of the formulae which are weighted homogeneous of degree  $n$ . Furthermore, by taking  $c = -1$ , we see that  $\mathcal{I}_{m,n} = \{0\}$  if  $n$  is odd and we shall restrict to the case  $n$  even henceforth.

Next, we consider a local formula

$$Q = \sum_{i,j=1}^m Q_{ij} dx^i \circ dx^j$$

where the  $Q_{ij} \in \mathcal{Q}_m$ . Evaluation is defined as above and we say  $Q$  is invariant if  $Q(\vec{x}, g, P)$  is independent of  $\vec{x}$  for all  $(g, P)$ . We let  $\mathcal{I}_m^2$  be the space of all such invariant local formulas. The obvious generalization of Lemma 2.1 permits us to decompose  $\mathcal{I}_m^2 = \bigoplus_n \mathcal{I}_{m,n}^2$  where  $\mathcal{I}_{m,n}^2$  consists of those invariant local formulas which are homogeneous of degree  $n$  in the jets of the metric. Again,  $\mathcal{I}_{m,n}^2 = \{0\}$  if  $n$  is odd.

### 2.2. The Riemann curvature tensor

Although convenient for our subsequent purposes, the definition of local invariants given in Section 2.1 is non-classical and it is worth making contact with the more standard approach. Let  $\nabla$  be the Levi-Civita connection of a Riemannian manifold  $(M, g)$ . The associated *Christoffel symbols* are defined in a system of local coordinates by setting:

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}^k \partial_{x^k} \quad \text{where } \Gamma_{ij}^k := \frac{1}{2} g^{kl} (\partial_{x^i} g_{jl} + \partial_{x^j} g_{il} - \partial_{x^l} g_{ij}).$$

The Riemann curvature tensor  $R_{ijk}^l$ , the Ricci tensor  $\rho$ , the scalar curvature  $\tau$ , the norm  $|\rho|^2$  of the Ricci tensor, and the norm  $|R|^2$  of  $R$  are then given by:

$$\begin{aligned} R_{ijk}^l &:= \partial_{x^i} \Gamma_{jk}^l - \partial_{x^j} \Gamma_{ik}^l + \Gamma_{in}^l \Gamma_{jk}^n - \Gamma_{jn}^l \Gamma_{ik}^n, \\ \rho_{jk} &:= R_{ijk}^i, \quad \tau := g^{i_1 j_1} \rho_{i_1 j_1}, \quad |\rho|^2 := g^{i_1 j_1} g^{i_2 j_2} \rho_{i_1 i_2} \rho_{j_1 j_2}, \\ |R|^2 &:= g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g_{i_4 j_4} R_{i_1 i_2 i_3}^{i_4} R_{j_1 j_2 j_3}^{j_4}. \end{aligned} \tag{2.a}$$

Again, we really should subscript to indicate the dependence on the dimension  $m$  explicitly in the Einstein summations but we will omit this additional notational complexity in the interests of brevity as the formulas are universal and no confusion will result from this notational imprecision. Since  $\Gamma$  has weight 1 and  $\partial_{x^i} \Gamma$  has weight 2, we see that  $R$  has weight 2. Consequently,

$$\tau \in \mathcal{I}_{m,2}, \quad \tau^2 \in \mathcal{I}_{m,4}, \quad |\rho|^2 \in \mathcal{I}_{m,4}, \quad |R|^2 \in \mathcal{I}_{m,4}.$$

We let “;” denote multiple covariant differentiation. If  $\Delta$  is the scalar Laplacian, we have

$$\Delta \tau = -g^{ij} \tau_{;ij} \in \mathcal{I}_{m,4}.$$

### 2.3. H. Weyl’s Theorem of invariants

Let  $V$  be a finite-dimensional vector space which is equipped with a positive definite bilinear form  $\langle \cdot, \cdot \rangle$  of signature  $(p, q)$ . Let  $\mathcal{O}$  be the associated orthogonal group. We say that  $\psi : \otimes^k V^* \rightarrow \mathbb{R}$  is a *linear orthogonal invariant* if  $\psi$  is a linear map and if

$$\psi(\Theta \cdot w) = \psi(w) \quad \forall \Theta \in \mathcal{O}, \forall w \in \otimes^k V^*.$$

We can construct such maps as follows. Let  $k = 2\ell$  and let  $\pi \in \text{Perm}(2\ell)$  be a permutation of the integers from 1 to  $2\ell$ . Define

$$\psi_\pi(v^1, \dots, v^{2\ell}) := \langle v^{\pi(1)}, v^{\pi(2)} \rangle \dots \langle v^{\pi(2\ell-1)}, v^{\pi(2\ell)} \rangle. \tag{2.b}$$

We show  $\psi_\pi$  is an orthogonal invariant by computing

$$\begin{aligned} \psi_\pi(\Theta v^1, \dots, \Theta v^{2\ell}) &= \langle \Theta v^{\pi(1)}, \Theta v^{\pi(2)} \rangle \dots \langle \Theta v^{\pi(2\ell-1)}, \Theta v^{\pi(2\ell)} \rangle = \langle v^{\pi(1)}, v^{\pi(2)} \rangle \dots \langle v^{\pi(2\ell-1)}, v^{\pi(2\ell)} \rangle \\ &= \psi_\pi(v^1, \dots, v^{2\ell}). \end{aligned}$$

Since  $\psi_\pi$  is a multi-linear map, it extends naturally to a linear orthogonal invariant mapping  $\otimes^{2\ell} V$  to  $\mathbb{R}$ . We refer to [24] (see Theorem 2.9.A on page 53) for the proof of the following result:

**Theorem 2.1.** *The space of linear orthogonal invariants of  $\otimes^{2k} V^*$  is spanned by the maps  $\psi_\pi$  of Eq. (2.b).*

In geodetic polar coordinates, we set  $g_{ij}(P) = \delta_{ij}$  and  $g_{ij/k}(P) = 0$ ; the remaining derivatives of the metric can be expressed in terms of the covariant derivatives of the curvature tensor at  $P$ . The following result [2] is then a direct consequence of Theorem 2.1; the extension from scalar to symmetric 2-form valued invariants is immediate. Lemma 1.1 and Lemma 1.3 follow directly the following Lemma after using the curvature identities to eliminate redundancies and we refer the reader to those results to illustrate exactly what is meant by Lemma 2.2:

**Lemma 2.2.** *All scalar invariants and all symmetric 2-form valued invariants which are given by a local formula in the derivatives of the metric and which are homogeneous of order  $n$  arise by contracting indices in pairs in monomial expressions of weight  $n$  in the covariant derivatives of the curvature tensor.*

2.4. The restriction map

Let  $(N, g_N)$  be a Riemannian manifold of dimension  $m - 1$ . Let  $M = N \times S^1$  and let  $g_M = g_N + d\theta^2$  where  $\theta$  is the usual periodic parameter on the circle. Let  $\theta_0$  be the basepoint of the circle; since  $(S^1, d\theta^2)$  is a homogeneous space, the choice of the basepoint plays no role. If  $y \in N$ , we let  $i(y) := (y, \theta_0) \in M$ . If  $Q \in \mathcal{I}_{m,n}$  or if  $Q \in \mathcal{I}_{m,n}^2$ , then we set

$$r(Q)(g_N, y) := i^*Q(g_M, i(y)); \tag{2.c}$$

(we have to restrict this tensor to  $N \times \{\theta_0\}$ ). This defines natural maps

$$r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n} \quad \text{and} \quad r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2.$$

Assertion (1) of Theorem 1.1 and of Theorem 1.2 will follow from:

**Lemma 2.3.** *We have  $r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n} \rightarrow 0$  and  $r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2 \rightarrow 0$ .*

**Proof.** By Lemma 2.2, all local invariants are given in terms of contractions of indices of various monomials of weight  $n$  in the covariant derivatives of the curvature tensor. Instead of letting the indices range from 1 to  $m$  in the contractions of indices which define  $Q$ , we let the indices range from 1 to  $m - 1$  in defining  $r(Q)$  since the metric is flat in the last direction. Thus, for example, as noted above we have:

$$\tau_m := \sum_{i,j=1}^m R_{ijji} \quad \text{then} \quad r(\tau_m) = \tau_{m-1} = \sum_{i,j=1}^{m-1} R_{ijji}.$$

This is, of course, implicit in the notation that we used in Eq. (2.a) in defining the scalar curvature in the first instance. The dimension  $m$  appears implicitly in the range of summation and the formula is “universal” over all dimensions in that respect, i.e.  $r(\tau_m) = \tau_{m-1}$ . Thus we usually don’t subscript but simply talk of the scalar curvature  $\tau$  without mentioning the underlying dimension  $m$ . We may choose a spanning set for  $\mathcal{I}_{m-1,n}$  or  $\mathcal{I}_{m-1,n}^2$  similar to those given in Lemma 1.1 and in Lemma 1.3 which involves contracting indices in covariant derivatives of the curvature tensor. The desired lift to  $\mathcal{I}_{m,n}$  or to  $\mathcal{I}_{m,n}^2$  is then obtained by letting the indices range from 1 to  $m$  instead of from 1 to  $m - 1$ . This lift is, of course, not unique and is exactly measured by  $\ker(r)$  which gives the universal relations satisfied in dimension  $m - 1$  which are not satisfied in dimension  $m$ . □

We used the tensor calculus to show that  $r$  is surjective. We now return to the non-invariant formulation to continue our study. We may always restrict to coordinate systems  $\vec{x}$  which are normalized at the point  $P$  so that

$$g_{ij}(\vec{x}, g, P) = \delta_{ij} \quad \text{and} \quad g_{ij/k}(\vec{x}, g, P) = 0. \tag{2.d}$$

We let  $\tilde{Q}_m := \mathbb{R}[g_{ij/\alpha}]_{|\alpha| \geq 2}$  be the polynomial algebra in the jets of the metric of order at least 2. One can use a partition of unity and Taylor series to derive the following result:

**Lemma 2.4.** *If  $0 \neq Q \in \tilde{Q}_m$ , then there exists  $(\vec{x}, g, P)$  so that  $\vec{x}$  satisfies the normalizations of Eq. (2.d) and so that  $Q(\vec{x}, g, P) \neq 0$ .*

We note that Lemma 2.4 is **not** true if we work with the Riemann curvature tensor. There are “hidden” and non-obvious relations that do not follow from the usual  $\mathbb{Z}_2$  symmetries and the generalized Bianchi identities that are dimension specific – that is the whole point, of course, of the relations given in Lemma 1.2 and in Lemma 1.4. And it is Lemma 2.4 that will be crucial in our discussion.

Let  $A = g_{i_1 j_1 / \alpha_1} \cdots g_{i_\ell j_\ell / \alpha_\ell}$  be a monomial of  $\tilde{Q}_m$ . We define

$$\text{deg}_k(A) := \delta_{i_1, k} + \delta_{j_1, k} + \alpha_1(k) + \cdots + \delta_{i_\ell, k} + \delta_{j_\ell, k} + \alpha_\ell(k)$$

to be the number of times that the index  $k$  appears in  $A$ . We extend this notion to the context of symmetric 2-form valued invariants by defining:

$$\text{deg}_k(A dx^{i_{\ell+1}} \circ dx^{j_{\ell+1}}) := \text{deg}_k(A) + \delta_{i_{\ell+1}, k} + \delta_{j_{\ell+1}, k}.$$

Set  $r_1(A) = A$  if  $\text{deg}_m(A) = 0$  and  $r_1(A) = 0$  if  $\text{deg}_m(A) > 0$  to define a polynomial map  $r_1 : \tilde{Q}_m \rightarrow \tilde{Q}_{m-1}$ . Assertion (2) of Theorem 1.1 and Assertion (2) of Theorem 1.2 will follow Lemma 2.3 and from:

**Lemma 2.5.**

- (1) *If  $Q \in \mathcal{I}_{m,n}$  or if  $Q \in \mathcal{I}_{m,n}^2$ , then  $r_1(Q) = r(Q)$ .*
- (2) *If  $Q \in \mathcal{I}_{m,n} \cap \ker(r)$  or if  $Q \in \mathcal{I}_{m,n}^2 \cap \ker(r)$ , then  $\text{deg}_k(A) \geq 2$  for  $1 \leq k \leq m$  for every monomial  $A$  of  $Q$ .*

- (3) If  $m > n$ , then  $\ker(r : \mathcal{I}_{m,n} \rightarrow \mathcal{I}_{m-1,n}) = \{0\}$ .
- (4) If  $m = n$ , if  $Q \in \ker(r) \cap \mathcal{I}_{m,n}$ , and if  $A$  is a monomial of  $Q$ , then  $\deg_k(A) = 2$  and  $|\alpha_a| = 2$  for  $1 \leq k \leq m$  and  $1 \leq a \leq \ell$ .
- (5) If  $m > n + 1$ , then  $\ker(r : \mathcal{I}_{m,n}^2 \rightarrow \mathcal{I}_{m-1,n}^2) = \{0\}$ .
- (6) If  $m = n + 1$  if  $Q \in \ker(r) \cap \mathcal{I}_{m,n}^2$ , and if  $A$  is a monomial of  $Q$ , then  $\deg_k(A) = 2$  and  $|\alpha_a| = 2$  for  $1 \leq k \leq m$  and  $1 \leq a \leq \ell$ .

**Proof.** Assertion (1) gives an algebraic reformulation of the geometric definition given in Eq. (2.c) and is immediate from that definition; the metric on  $N \times S^1$  is flat in the final direction; we also set  $e^i \circ e^j = 0$  if either  $i$  or  $j$  is the final index as we have to restrict the tensor to the submanifold.

Let  $r(Q) = 0$ . By Lemma 2.4, we may identify the local formula defined by  $Q$  with the polynomial  $Q \in \mathcal{Q}_m$ . It then follows that  $\deg_m(A) > 0$  for every monomial  $A$  of  $Q$ . Let  $y = (x^1, \dots, x^{m-1}, -x^m)$ , we see  $\deg_m(A)$  is even and hence  $\deg_m(A) \geq 2$ . Since  $Q$  is invariant under coordinate permutations, Assertion (2) follows.

Let  $0 \neq Q \in \mathcal{I}_{m,n} \cap \ker(r)$ . Let  $A = g_{i_1 j_1 / \alpha_1} \cdots g_{i_\ell j_\ell / \alpha_\ell}$  be a monomial of  $Q$ . Since  $|\alpha_a| \geq 2$ , we have

$$2\ell \leq \sum_{a=1}^{\ell} |\alpha_a| = n. \tag{2.e}$$

By Assertion (2) we have  $\deg_k(A) \geq 2$  for every  $k$ . Thus

$$2m \leq \sum_{1 \leq k \leq m} \deg_k(A) = \sum_{a=1}^{\ell} \sum_{k=1}^m \{\delta_{i_a, k} + \delta_{j_a, k} + \alpha_a(k)\} = \sum_{a=1}^{\ell} \{1 + 1 + |\alpha_a|\} = 2\ell + n \leq n + n = 2n. \tag{2.f}$$

This shows that  $m \leq n$  and proves Assertion (3). Furthermore, if  $m = n$ , all the inequalities in Eq. (2.e) and in Eq. (2.f) must have been equalities; this establishes Assertion (4).

Similarly let  $0 \neq Q \in \mathcal{I}_{m,n}^2 \cap \ker(r)$  and let  $A$  be a monomial of  $Q$ . Express

$$A = g_{i_1 j_1 / \alpha_1} \cdots g_{i_\ell j_\ell / \alpha_\ell} dx^{i_\ell+1} \circ dx^{j_\ell+1}.$$

We estimate similarly:

$$2\ell \leq \sum_{a=1}^{\ell} |\alpha_a| = n, \tag{2.g}$$

$$2m \leq \sum_{k=1}^m \deg_k(A) = \sum_{a=1}^{\ell} \sum_{k=1}^m \{\delta_{i_a, k} + \delta_{j_a, k} + \alpha_a(k)\} + 2 = \sum_{a=1}^{\ell} \{1 + 1 + |\alpha_a|\} + 2 = 2\ell + n + 2 \leq 2n + 2. \tag{2.h}$$

Again, this is not possible if  $m > n + 1$  which establishes Assertion (5). If  $m = n + 1$ , all the equalities must have been equalities and the desired result follows.  $\square$

### 2.5. H. Weyl's second theorem

Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space of dimension  $m$ . A typical relation among scalar products is the following which involves  $m + 1$  vectors  $\{v^0, \dots, v^m\}$  and  $m + 1$  vectors  $\{w^0, \dots, w^m\}$ . One necessarily has:

$$\det \begin{pmatrix} \langle v^0, w^0 \rangle & \langle v^0, w^1 \rangle & \dots & \langle v^0, w^m \rangle \\ \langle v^1, w^0 \rangle & \langle v^1, w^1 \rangle & \dots & \langle v^1, w^m \rangle \\ \dots & \dots & \dots & \dots \\ \langle v^m, w^0 \rangle & \langle v^m, w^1 \rangle & \dots & \langle v^m, w^m \rangle \end{pmatrix} = 0. \tag{2.i}$$

One also has [24] (see Theorem 2.17.A page 75).

**Theorem 2.2.** Every relation among scalar products is an algebraic consequence of the relations given above in Eq. (2.i).

### 2.6. Proof of Theorem 1.1

Let  $m = 2\bar{m}$  be even. We introduce formal variables  $g_{ij/kl} \in S^2 \otimes S^2$  for  $1 \leq i, j, k, l \leq m$ . If  $Q \in \ker(r : \mathcal{I}_{m,m} \rightarrow \mathcal{I}_{m-1,m})$ , then we have shown that in Lemma 2.5 that  $Q$  can be regarded as a polynomial of degree  $\bar{m}$  in  $\mathbb{R}[g_{ij/kl}]$ . Let  $S^2$  denote the space of symmetric 2 tensors. Since  $g_{ij/kl} \in S^2 \otimes S^2$ , we can regard  $Q$  as a linear orthogonal invariant on  $\otimes^{\bar{m}}\{S^2 \otimes S^2\}$ . Such an orthogonal invariant extends naturally to the full tensor algebra to be zero on the orthogonal complement of  $\otimes^{\bar{m}}\{S^2 \otimes S^2\}$  and hence H. Weyl's theorem applies where the dimension of the underlying vector space is  $m - 1$  not  $m$ . Since the restriction of  $Q$  to the lower-dimensional setting vanishes, we can apply Theorem 2.2 to express  $Q$  as a linear combination of invariants of the form



$$A_\sigma = g_{i_1 i_2 / i_3 i_4} \cdots g_{i_{2m-3} i_{2m-2} / i_{2m-1} i_{2m}} \times g(dx^{i_{\sigma_1}} \wedge dx^{i_{\sigma_2}} \wedge \cdots \wedge dx^{i_{\sigma_m}}, dx^{i_{\sigma_{m+1}}} \wedge \cdots \wedge dx^{i_{\sigma_{2m}}})$$

where  $\sigma$  is a permutation of  $\{1, \dots, 2m\}$ . If  $i_1 = i_{\sigma_a}$  for some index  $a$  with  $1 \leq a \leq m$ , then necessarily  $i_2 = i_{\sigma_b}$  for some index  $b$  with  $m+1 \leq b \leq 2m$  since  $g_{i_1 i_2 / i_3 i_4}$  is symmetric in the indices  $\{i_1, i_2\}$  where as the wedge product is anti-symmetric. By permuting the indices  $\{i_1, i_2\}$  if necessary, we may therefore assume  $i_1 = \sigma_{a_1}$  and  $i_2 = \sigma_{b_1}$  for  $1 \leq a_1 \leq m$  and  $m+1 \leq b_1 \leq 2m$ . This implies we can write

$$A_\sigma = g_{i_1 j_1 / i_2 j_2} \cdots g_{i_{m-1} j_{m-1} / i_m j_m} \times g(dx^{i_{\rho_1}} \wedge \cdots \wedge dx^{i_{\rho_m}}, dx^{j_{\rho_1}} \wedge \cdots \wedge dx^{j_{\rho_m}})$$

where  $\rho$  and  $\varrho$  are permutations of  $m$  indices. Reordering the factors then yields

$$A_\sigma = \pm g_{i_1 j_1 / i_2 j_2} \cdots g_{i_{m-1} j_{m-1} / i_m j_m} \times g(dx^{i_1} \wedge \cdots \wedge dx^{i_m}, dx^{j_1} \wedge \cdots \wedge dx^{j_m}).$$

This shows  $\dim\{\ker(r : \mathcal{I}_{m,m} \rightarrow \mathcal{I}_{m-1,m})\} \leq 1$ . Since  $r(E_{m,m}) = 0$  and  $E_{m,m}$  is non-trivial, Assertion (3) of Theorem 1.1 follows.  $\square$

## 2.7. Proof of Theorem 1.2

The proof of Theorem 1.2 (3) is essentially the same. The crucial feature is, of course, that we have eliminated the higher order jets of the metric and only have to deal with second derivatives. The dimension of the underlying vector space is now  $m = 2\bar{m}$  rather than  $m-1$ . Let  $Q \in \mathcal{I}_{m+1,m}^2$ . We can express  $Q = Q_{uv} dx^u \circ dx^v$  where  $Q_{uv} \in \mathbb{R}[g_{ij/kl}]$  is homogeneous of degree  $\bar{m}$ . Since  $r(Q) = 0$ , we may express  $Q$  as a linear combination of invariants of the form:

$$A_\sigma = g_{i_1 i_2 / i_3 i_4} \cdots g_{i_{2m-3} i_{2m-2} / i_{2m-1} i_{2m}} dx^{i_{2m+1}} \circ dx^{i_{2m+2}} \times g(dx^{i_{\sigma_1}} \wedge dx^{i_{\sigma_2}} \wedge \cdots \wedge dx^{i_{\sigma_{m+1}}}, dx^{i_{\sigma_{m+2}}} \wedge \cdots \wedge dx^{i_{\sigma_{2m+2}}}).$$

The same symmetry argument used to establish Theorem 1.1 then shows in fact we are dealing with

$$A_\sigma = \pm g_{i_1 j_1 / i_2 j_2} \cdots g_{i_{m-1} j_{m-1} / i_m j_m} dx^{i_{m+1}} \circ dx^{j_{m+1}} \times g(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{m+1}}, dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{m+1}}).$$

Again, this shows  $\dim\{\ker(r : \mathcal{I}_{m+1,m}^2 \rightarrow \mathcal{I}_{m,m}^2)\} \leq 1$ . The desired result then follows as  $T_{m+1,m}^2 \in \ker(r : \mathcal{I}_{m+1,m}^2 \rightarrow \mathcal{I}_{m,m}^2)$  is non-trivial.  $\square$

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