JOURNAL OF FUNCTIONAL ANALYSIS 70, 80–89 (1987)

# $T$-Property and Nonisomorphic Full Factors of Types II and III

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*Communicated by A. Connes*

Received June 22, 1983

The actions of $T$-groups on von Neumann's hyperfinite algebras are studied. It is proved that there exist factors of type II with different countable fundamental groups and hence, different actions. It is also proved that there exist nonisomorphic full factors of type $\text{III}_1$ with any fixed invariant Sd. © 1987 Academic Press, Inc.

In the last few years a certain advance in the theory of operator algebras has been noted which resulted from the study of nonapproximately finite factors and their groups of automorphisms (see, for instance, [1–4]). To prove the existence of $\text{II}_1$-factors with a countable fundamental group, the property $T$ was used in the basic paper [1]. However, the problem of the existence of $\text{II}_1$-factors with different countable fundamental groups still remains to be solved.

The present paper deals with the actions of $T$-groups on von Neumann's algeras and proves the existence of factors of type II with different countable fundamental groups (see Corollary 1.6) and hence, different actions. The results obtained are used to prove the existence of nonisomorphic full factors of type III with a fixed invariant Sd (see Theorem 2.5).

The paper consists of two sections. Factors of type II are considered in Section 1 and those of type $\text{III}_1$, in Section 2. Factors of type III were studied with the participation of S. L. Gefter.

We give one of the definitions of II-factor fundamental group. Let $M$ be a $\text{II}_1$-factor with a faithful normal (f. norm.) finite trace $\tau$ and let $B$ be a $\text{I}_\infty$-factor with a f. norm. semifinite trace tr. Then $N = M \otimes B$ is a $\text{II}_\infty$-factor with a f. norm. semifinite trace $\mu = \tau \times \text{tr}$. Correlate any $\theta \in \text{Aut}\, N$ with a number $\lambda(\theta) \in \mathbb{R}_+^*$ such that $\mu \cdot \theta = \lambda(\theta)\mu$. The group $\Gamma \subseteq \mathbb{R}_+^*$ generated by $\lambda(\theta)$, $\theta \in \text{Aut}\, N$, is called the fundamental group of the factor $M$ (and $N$).



0022-1236/87 $3.00

## 1

Let $\lambda_i$ $(i=1, 2,...)$ be generators of a countable subgroup $\Gamma \subset \mathbb{R}_+^*$. Consider a von Neumann algebra $(R_\Gamma, \varphi_\Gamma) = \otimes_i (R_{\lambda_i}, \varphi_{\lambda_i})$, where $R_{\lambda_i}$ is an approximately finite factor of type $\mathrm{III}_{\lambda_i}$ (the Powers factor), $\varphi_{\lambda_i}$ is a faithful normal (f. norm) state on $R_{\lambda_i}$, the modular group of which has a period $T_i = 2\pi/\ln \lambda_i$, $\varphi_\Gamma = \prod_i \varphi_{\lambda_i}$. Denote a countable $T$-group e.g., $G = SL(3, \mathbb{Z})$) in which every class of conjugates, except a trivial one, is infinite (a T ICC-group), by $G$. Consider a countable number of factors $M_g$, $g \in G$, where $M_g = R_\Gamma$ for any $g \in G$, and construct a factor $(N_\Gamma, \psi_\Gamma) = \otimes_{g \in G} (M_g, \varphi_g)$ of type $\mathrm{III}_\lambda$, where $\varphi_g = \varphi_\Gamma$ $(0 < \lambda \leqslant 1)$. Each $g \in G$ corresponds to an automorphism $N_\Gamma$ induced by the group shift by the element $g$. Thus, the action $\alpha$ of a group $G$ on $(N_\Gamma, \psi_\Gamma)$ which preserves the state $\psi_\Gamma$ is determined, and the factor $K = N \times \alpha\, G$, which is a cross product of $N$ by $G$ with respect to $\alpha$, can be constructed. Let $\tilde{\psi}_\Gamma$ be a state on $K$ constructed by $\psi_\Gamma$ on $N_\Gamma$ using a standard method. Then it is evident that $K_\psi = N_\psi \times_\alpha G$, where $K_\psi$ is a centralizer of $\tilde{\psi}$ in $K$, and $N_\psi$ is a centralizer of $\psi$ in $N$.

Assume that $N_\psi$ is a GNS-representation of $N_\psi$ by $\psi$, acting in a Hilbert space $H$. Then $K_\psi$ is generated by operators $\pi(a)$ and $\lambda_g$ $(a \in N_\psi,\ g \in G)$, the action of which in a space $l^2(H, G)$ is given by the relationships:

$$(\pi(a)\xi)(g) = \alpha_{g^{-1}}(a)\, \xi(g),$$
$$(\lambda_h \xi)(g) = \xi(h^{-1} g) \qquad (h, g \in G)$$

where $\xi = (\xi(g)) \in l^2(H, G)$, $a \in N_\psi$, $g \in G$.

Let $B$ be a factor of type $\mathrm{I}_\infty$ acting standardly in a Hilbert space $L^2(B, \mathrm{tr})$, where tr is a f. norm. semifinite trace on $B$. Then $M_\Gamma = K_\psi \otimes B$ is a factor of type $\mathrm{II}_\infty$ with a f. norm. semifinite trace $\tau = \tilde{\psi} \times \mathrm{tr}$ acting standardly in a space $H_\Gamma = L^2(M_\Gamma, \tau)$.

We formulate now our basic result.

THEOREM 1.1. *The subgroup* $A_G \subset \mathrm{Aut}\, M_\Gamma$ *generated algebraically by* $\{\theta \in \mathrm{Aut}\, M_\Gamma : \theta(\lambda_g \otimes I) = \lambda_g \otimes I,\ g \in G\}$ *and* $\mathrm{Int}\, M_\Gamma$ *is open and closed in* $\mathrm{Aut}\, M_\Gamma$.

Note that the topology in $\mathrm{Aut}\, M_\Gamma$ is given by a system of neighbourhoods of unit [5]

$$U_\varphi(\varepsilon) = \{\theta \in \mathrm{Aut}\, M_\Gamma : \|\varphi \cdot \theta - \varphi\| < \varepsilon,\ \varphi \in M_*,\ \varepsilon > 0\}$$

and $\mathrm{Int}\, M_\Gamma$ is a group of inner automorphisms $M_\Gamma$.

LEMMA 1.2. *For $\varepsilon > 0$, $\xi \in H_\Gamma$ and $x \in M_\Gamma$ there exists a neighbourhood of unit $U \subset \operatorname{Aut} M_\Gamma$ such that for $\theta \in U$*

$$\|(\theta(x) - x)\xi\|^2 < \varepsilon \, \|x\| \, (2 + \|x\|).$$

*Proof.* It is clear that if $f(y) = (y\xi, x\xi)$, where $y \in M$, then $f \in (M_\Gamma)_*$. Take

$$U_1 = \{\theta \in \operatorname{Aut} M_\Gamma \colon | f(\theta(y)) - f(y)| < \varepsilon \, \|y\|, \, y \in M\},$$

Then with $y = x$ for $\theta \in U_1$ we have

$$|(\theta(x)\xi, x\xi) - (x\xi, x\xi)| < \varepsilon \, \|x\|. \tag{1}$$

Then, let $U_2 = \{\theta \in \operatorname{Aut} M_\Gamma \colon |(\theta(y)\, \xi, \xi) - (y\xi, \xi)| < \varepsilon \, \|y\|\}$; then with $y = x^*x$ for $\theta \in U_2$ we have

$$|(\theta(x^*x)\xi, \xi) - (x^*x\xi, \xi)| < \varepsilon \, \|x\|^2. \tag{2}$$

And finally, if $U_3 = \{\theta \in \operatorname{Aut} M_\Gamma \colon |(x\xi, \theta(y)\xi) - (x\xi, y\xi)| < \varepsilon \, \|y\|\}$ then with $y = x$ for $\theta \in U_3$

$$|(x\xi, \theta(x)\xi) - (x\xi, x\xi)| < \varepsilon \, \|x\|. \tag{3}$$

Now for $\theta \in \bigcap_{i=1}^{3} U_i$ the statement of the lemma follows from (1)–(3). ∎

LEMMA 1.3. *Let $R(G) = (\lambda_g \otimes I, \, g \in G)''$; then $R(G)' \cap M_\Gamma = I \otimes B$.*

The proof is rather standard and involves the fact that $G$ is a ICC-group. ∎

We prove the theorem. Let $J_\psi$ be a unitary involution in $L_2(K_\psi, \tilde{\psi})$, translating the element $x$ from $K_\psi$ to $x^*$, and let $J_B$ be an involution in $L^2(B, \operatorname{tr})$ such that $J_B x = x^*$ for $x$ from $B$. Then $J = J_\psi \otimes J_B$ is an involution in $L_2(M_\Gamma, \tau)$ with $Jx = x^*$ for a bounded $x$ from $L^2(M_\Gamma, \tau)$. Let $\theta \in \operatorname{Aut} M_\Gamma$; then we consider a unitary representation $\pi_\theta(g) = J(\lambda_g \otimes I)\, J\theta(\lambda_g \otimes I)$ of the group $G$ in a Hilbert space $L^2(M_\Gamma, \tau)$. Consider a vector $\xi = \xi_0 \otimes \eta$ from $H_\Gamma = L^2(M_\Gamma, \tau)$, where

$$\xi_0 = (\xi_0(g)) = \begin{cases} 1 \in L^2(N_\psi, \psi) & \text{for } g = e, \\ 0 & \text{for } g \neq e, \end{cases}$$

and $\eta$ is a vector from $L^2(B, \operatorname{tr})$ such that $J_B \eta = \eta$. The equality $J(\lambda_g^* \otimes I) J\xi = (\lambda_g \otimes I)\xi$ can be easily verified. According to Lemma 1.2, for

$\varepsilon > 0$, the vector $\xi \in H_\Gamma$ and $x = \lambda_g \otimes I$ ($g \in K$, $K$ is a finite subset of $G$) there exists a neighbourhood of unity $U$ in Aut $M_\Gamma$ such that for $\theta \in U$

$$\|\theta(\lambda_g \otimes I)\xi - (\lambda_g \otimes I)\xi\| < \varepsilon \qquad (g \in K)$$

or

$$\|\pi_\theta(g)\xi - \xi\| < \varepsilon \qquad (g \in K). \tag{4'}$$

Since $G$ is a $T$-group, it follows that there exists $y \in L^2(M_\Gamma, \tau)$ such that $\pi_\theta(g)y = y (g \in G)$ or

$$\theta(\lambda_g \otimes I)y = J(\lambda_g \otimes I)^* Jy \qquad (g \in G). \tag{4}$$

As will be shown in Lemma 1.4, it follows from (4) that there exists a partial isometry $u \in M_\Gamma$ with the properties:

$$\theta(\lambda_g \otimes I)u = u(\lambda_g \otimes I) \qquad (g \in G). \tag{5}$$

It follows, in particular, that if $q = u^*u$, then $q \in R(G)' \cap M_\Gamma = I \otimes B$ (see Lemma 1.3).

Take $\tilde{M}_\Gamma = M_\Gamma \otimes B$, $\theta_1 = \theta \otimes \mathrm{id}$, $q_1 = q \otimes I$, $u_1 = u \otimes I$ then $q_1$, $I - q_1$ and $p_1 = u_1 u_1^*$, $I - p_1$ are infinite projectors, and in the unitary group $U(\tilde{M}_\Gamma)$ of the factor $\tilde{M}_\Gamma$, therefore, there exists an operator $w_1$ such that $u^* = q_1 w_1$ and

$$(\mathrm{Ad}\, w_1 \cdot \theta_1(\lambda_g \otimes I))q_1 = q_1(\lambda_g \otimes I).$$

Thus, $(\theta_1^{-1} \cdot \mathrm{Ad}\, w_1^*)(q_1) \in I \otimes \tilde{B}$, where $\tilde{B} = B \otimes B$, and then taking the infiniteness of a projector $(\theta_1^{-1} \cdot \mathrm{Ad}\, w_1^*)(q_1)$ into account, we determine that

$$\theta_1 \cdot \mathrm{Ad}\, w_2(\lambda_g \otimes q_1) = \lambda_g \otimes q_1 \qquad (g \in G) \tag{6}$$

for a certain $w_2 \in U(\tilde{M}_\Gamma)$. Take $e_{11} = q_1$, $e_{22} = I - q_1$, and since $e_{11}$ and $e_{22}$ are infinite projectors in $I \otimes \tilde{B}$ then there exists a partial isometry $e_{12} \in I \otimes \tilde{B}$ $(e_{12}^* = e_{21})$ such that $e_{12}e_{21} = e_{11}$ and $e_{22} = e_{21}e_{12}$. It is clear that if $w_3 = e_{11} + e_{21}(\theta_1 \cdot \mathrm{Ad}\, w_2)(e_{12})$, then $\mathrm{Ad}\, w_3 \cdot e_{ii} = e_{ii}$ $(i = 1, 2)$ and $\mathrm{Ad}\, w_3 \cdot (\theta_1 \cdot \mathrm{Ad}\, w_2) \cdot e_{ij} = e_{ij}$ $(i, j = 1, 2)$. Take $w_4 = w_3\theta_1(w_2)$; then

$$\mathrm{Ad}\, w_4 \cdot \theta_1(\lambda_g \otimes q_1) = \lambda_g \otimes q_1 \qquad (g \in G)$$

and

$$\mathrm{Ad}\, w_4 \cdot \theta_1(\lambda_g \otimes I - q_1) = \mathrm{Ad}\, w_4 \cdot \theta_1(e_{21}(\lambda_g \otimes q_1)e_{12}) = e_{21}(\lambda_g \otimes q_1)e_{12}$$
$$= \lambda_g \otimes I - q_1 \qquad (g \in G).$$

Therefore,

$$\operatorname{Ad} w_4 \cdot \theta_1(\lambda_g \otimes I) = \lambda_g \otimes I \qquad (g \in G).$$

Since $R(G)' \cap \tilde{M}_\Gamma = I \otimes \tilde{B}$, where $R(G) = (\lambda_g \otimes I, g \in G)''$, then $\operatorname{Ad} w_4 \cdot \theta_1(I \otimes \tilde{B}) = I \otimes \tilde{B}$. Thus, there exists $w_5 \in U(I \otimes \tilde{B})$ such that

$$\operatorname{Ad} w_5 w_4 \cdot \theta_1(I \otimes x) = I \otimes x \qquad (x \in \tilde{B}).$$

It follows that $\operatorname{Ad} w_5 w_4 \cdot \theta_1 \in A_G$, and since $\theta_1 = \theta \otimes \mathrm{id}$, then for any $y \in B$

$$\operatorname{Ad} w_5 \cdot w_4(I \otimes I \otimes y) = \operatorname{Ad} w_5 w_4 \cdot \theta_1(I \otimes I \otimes y) = I \otimes I \otimes y,$$

therefore, $w_5 w_4 \in M_\Gamma \otimes I$. Hence, $\theta \in A_G$ and therefore, $U \subset A_G$, where $U$ is a neighbourhood of unity in $\operatorname{Aut} M_\Gamma$ (see (4')).

This proves that $A_G$ is an open subgroup in $\operatorname{Aut} M_\Gamma$; the fact that $A_G$ is a closed subgroup is proved in a similar, though simpler, way. To complete the proof of Theorem 1.1, the following lemma is to be proved.

LEMMA 1.4. *The validity of* (5) *follows from relation* (4).

*Proof.* Consider the triple $(H_\Gamma, J, D)$, where $H_\Gamma = L^2(M_\Gamma, \tau)$, $J = J_\psi \otimes J_B$ and $D = L^\infty(M_\Gamma, \tau) \cap L^2(M_\Gamma, \tau)$. Then $(H_\Gamma, J, D)$ is considered to be a Hilbert algebra (see Sections 5 in [6]). Let, as in [6], $x^* = Jx$ $(x \in H_\Gamma)$, $L_a b = ab$, $R_a b = ba$ $(a, b \in D)$ then $J L_a J = R_{a^*}$ $(a \in D)$. Take also $L(H_\Gamma) = \{L_a, a \in D\}''$, $R(H_\Gamma) = \{R_a, a \in D\}''$; as is known, $L(H_\Gamma)' = R(H_\Gamma)$. It is clear that any $b \in M_\Gamma$ defines operators $L_b x = bx$ and $R_b x = Jb^*Jx = xb$ $(x \in D)$ which belong to $L(H_\Gamma)$ and $R(H_\Gamma)$, respectively, and $b \to L_b$ is a *-isomorphism of $M_\Gamma$ on $L(H_\Gamma)$.

According to the results from 5.4 [6], any $x \in H_\Gamma$ can be correlated with an operator $L'_x$ in $H_\Gamma$, given on $D \subset H_\Gamma$ by the formula $L'_x a = R_a x$, where $a \in D$. The operator $L'_x$ admits the closure of $L_x$ which is a measurable operator with respect to $L(H_\Gamma)$ adjoint to $L(H_\Gamma)$, with $\tau(L_x^* L_x) = (x, x)$. Besides, if $x = x^*$, then $L_x$ is a self-conjugate operator, and if $x = x_1 + ix_2$, where $x_j^* = x_j$ $(j = 1, 2)$, then $L_x = L_{x_1} + iL_{x_2}$.

Using the operators $L_a$ and $R_b$ $(a, b \in M_\Gamma)$, we rewrite Eq. (4) as

$$L_{\theta(\lambda_g \otimes I)} y = R_{\lambda_g \otimes I} y \qquad (g \in G). \tag{7}$$

If $c \in D$, then $R_c L_{\theta(\lambda_g \otimes I)} y = R_c R_{\lambda_g \otimes I} y$ and $R_c L_{\theta(\lambda_g \otimes I)} y = L_{\theta(\lambda_g \otimes I)} R_c y = L_{\theta(\lambda_g \otimes I)} L'_y c$; similarly, $R_c R_{\lambda_g \otimes I} y = R_{(\lambda_g \otimes I)c}\, y = L'_y(\lambda_g \otimes I)c = L'_y L_{\lambda_g \otimes I} c$. Therefore, there is the equality:

$$L_{\theta(\lambda_g \otimes I)} L'_y c = L'_y L_{\lambda_g \otimes I} c \qquad (g \in G, c \in D).$$

If $b_n \to b \in D(L_y)$, where $b_n \in D$ and $L'_y b_n \to L_y b$, then $L_{\lambda_g \otimes I} b_n \to L_{\lambda_g \otimes I} b$ and

$$L'_y(L_{\lambda_g \otimes I} b_n) = L_{\theta(\lambda_g \otimes I)} L'_y b_n \to L_{\theta(\lambda_g \otimes I)} L_y b.$$

Thus,

$$L_y L_{\lambda_g \otimes I} b = L_{\theta(\lambda_g \otimes I)} L_y b \qquad (g \in G,\, b \in D(L_y))$$

i.e., $L_{\lambda_g \otimes I} D(L_y) \subseteq D(L_y)$, but then it follows from the group property that $L_{\lambda_g \otimes I} D(L_y) = D(L_y)$ $(g \in G)$ and therefore,

$$L_y L_{\lambda_g \otimes I} = L_{\theta(\lambda_g \otimes I)} L_y \qquad (g \in G). \tag{8}$$

Apply $J$ to equality (7); then

$$R_{\theta(\lambda_g^* \otimes I)} y^* = L_{\lambda_g^* \otimes I} y^* \qquad (g \in G).$$

Transforming this equality in the same way as (7), we obtain the relation:

$$L'_{y^*} L_{\theta(\lambda_g \otimes I)} c = L_{\lambda_g \otimes I} L'_{y^*} c \qquad (g \in G,\, c \in D)$$

and hence, the relations:

$$\begin{aligned} &L_{y^*} L_{\theta(\lambda_g \otimes I)} b = L_{\lambda_g \otimes I} L_{y^*} b \qquad (g \in G,\, b \in D(L_{y^*})), \\ &L_{\lambda_g \otimes I} D(L_{y^*}) = D(L_{y^*}). \end{aligned} \tag{9}$$

Prove now that $L_{x^*} = (L_x)^*$ for $x \in H_\Gamma$. Indeed, let $x = y + iz$, where $y = y^*$, $z = z^*$; then $L'_{x^*} = L'_y - iL'_z$ and therefore, $L'^*_{x^*} \supseteq L'_{y^*} + iL'_{z^*} = L'_y + iL'_z = L'_x$. Thus, the closure $L_x$ of the operator $L'_x$ is coincident with $L'^*_{x^*}$, i.e., $L_x = L'^*_{x^*}$. Besides, since $L_{x^*}$ is a closure of $L'_{x^*}$, then $L'^*_{x^*} = L^*_{x^*}$ and $L_x = L^*_{x^*}$ and hence, $L^*_x = L^{**}_{x^*} = L_{x^*}$ because of the closed character of $L_{x^*}$. But then (9) can be rewritten in the form:

$$L_{\lambda_g \otimes I} L^*_y = L^*_y L_{\theta(\lambda_g \otimes I)} \qquad (g \in G). \tag{9$'$}$$

Using (8) and (9′), we obtain

$$L_{\lambda_g \otimes I} L^*_y L_y = L^*_y L_{\theta(\lambda_g \otimes I)} L_y = L^*_y L_y L_{\lambda_g \otimes I} \tag{10}$$

for $g \in G$, where $L^*_y L_y$ is a selfconjugate operator which is measurable and adjoint to $L(H_\Gamma)$, in this case $\tau(L^*_y L_y) = (y, y) < \infty$ [6].

Because $L_y$ is closed, we can consider its polar expansion $L_y = U\,|L_y|$, and because $L_y$ and $|L_y|$ are adjoint to $L(H_\Gamma)$ then $U \in L(H_\Gamma)$ and therefore $U = L_u$ for a certain partial isometry $u \in M_\Gamma$. Taking the equality

$|L_y|^2 = L_y^* L_y$ into account, we derive from (8) and (10) the following relation:

$$L_{\theta(\lambda_g \otimes I)} L_u \, |L_y| = L_u \, |L_y| \, L_{\lambda_g \otimes I} = L_u L_{\lambda_g \otimes I} \, |L_y|$$

and therefore $L_{\theta(\lambda_g \otimes I)u} = L_{u(\lambda_g \otimes I)}$ or $\theta(\lambda_g \otimes I)u = u(\lambda_g \otimes I)$ $(g \in G)$. ∎

Now relation (5) is valid and Theorem 1.1 is completely proved. ∎

COROLLARY 1.5. *Let* $\mathrm{Aut}_0 M_\Gamma = \{\theta \in \mathrm{Aut}\, M_\Gamma : \tau \cdot \theta = \tau\}$. *Then* $\mathrm{Aut}\, M_\Gamma / \mathrm{Aut}_0 M_\Gamma$ *is countable and discrete.*

Indeed, the factor-space $\mathrm{Aut}\, M_\Gamma / A_G$ is discrete and countable because $A_G$ is an open subgroup of $\mathrm{Aut}_0\, M_\Gamma$; then the corollary is true. ∎

COROLLARY 1.6. *The fundamental group of the factor* $M_\Gamma$ *(and* $K_\psi$*, too) is countable and involves, as a subgroup, any given countable subgroup* $\Gamma \subset \mathbb{R}_+^*$.

According to the construction of $K_\psi \subset K$ and if the relation $\psi(p_1)/\psi(p_2) \in \Gamma$ is fulfilled for projectors $p_1$ and $p_2$ from $K_\psi$, there exists a partial isometry, $u$ in $K$, such that $u^*u = p_1$, $uu^* = p_2$ but then $(K_\psi)_{p_1} = u^*(K_\psi)_{p_2} u$ and $(K_\psi)_{p_1} \sim (K_\psi)_{p_2}$. ∎[1]

COROLLARY 1.7. *The* $T(ICC)$*-group has a continuum of nonequivalent actions onto hyperfinite factors of type* $II_1$ *(compare with* [8]*).*

## 2

We apply the above results to investigation of type III-full factors with an almost periodic weight. Note that the factor $M$ is called a full factor if the group $\mathrm{Int}\, M$ is closed; the equivalent definitions are given in Theorem 3.1 [7]. The factor $M$ is considered to have an almost periodic weight $\varphi$ (state), if $\varphi$ is a f. norm. semifinite weight (state) on $M$ and a corresponding modular operator $\Delta_\varphi$ is diagonal: $\Delta_\varphi = \sum_{\lambda > 0} \lambda E_\lambda$, where the projectors $E_\lambda$ give a partition of $I$.

To study full factors of type $III_1$, the following invariant was introduced in [7]:

$$\mathrm{Sd}(M) = \bigcap_{\varphi} \text{ a point spectrum } \Delta_\varphi, \quad \varphi \text{ an almost periodic weight on } M.$$

[1] Another proof is based on Lemma 4.3 [7].

It has been proved that for the full factor $M$ with $\mathrm{Sd}(M)=\Gamma\neq\mathbb{R}_+^*$ the following conditions are equivalent: (a) $\psi$ is an $\Gamma$-almost periodic weight on $M$; (b) a point spectrum $\Delta_\psi=\mathrm{Sd}\,M$; (c) $M'_\psi\cap M=\mathbb{C}$; (d) $M_\psi$ is a factor (see Lemma 4.8 in [7]).

Construct now examples of full factors of type III. Let $\lambda_i$ ($i=1, 2,...$) be generators of the group $\Gamma\subset\mathbb{R}_+^*$, $\mathbb{N}=N_1\cup N_2$ be a partition of $\mathbb{N}$ into two infinite subsets. Denote a subgroup $\Gamma$ with generatricies $\lambda_i$ ($i\in N_1$) by $\Gamma_1$, and a subgroup with generatricies $\lambda_i$ ($i\in N_2$) by $\Gamma_2$. Consider a factor $N=N_{\Gamma_1}\otimes(N_{\Gamma_2})_{\psi_{\Gamma_2}}$ of type III with a state $\psi=\psi_{\Gamma_1}\times\psi_{\Gamma_2}$, where the symbols from Section 1 were used. Then each $g\in G$ corresponds to an automorphism induced by a group shift onto $N$. Thus, the action $\alpha$ of group $G$ on $(N,\psi)$ is given which preserves $\psi$, and the factor $K(\Gamma,\Gamma_1)=N\times_\alpha G$ which is a crossed product of $N$ by $G$ with respect to $\alpha$ can be constructed. A state on $K(\Gamma,\Gamma_1)$ constructed by $\psi$ on $N$ using a standard method is denoted by $\psi(\Gamma,\Gamma_1)$, the factor $K$ with the state $\tilde{\psi}$, constructed in Section 1, by $K(\Gamma)$ and $\tilde{\psi}$ by $\psi(\Gamma)$.

It follows from the construction that $K(\Gamma,\Gamma_1)_{\psi(\Gamma,\Gamma_1)}$ is a type $\mathrm{II}_1$-factor and

$$K(\Gamma,\Gamma_1)_{\psi(\Gamma,\Gamma_1)}\sim K(\Gamma)_{\psi(\Gamma)}, \tag{11}$$

$$\mathrm{Spec}\,\Delta_{\psi(\Gamma)}=\Gamma, \qquad \mathrm{Spec}\,\Delta_{\psi(\Gamma,\Gamma_1)}=\Gamma_1. \tag{12}$$

LEMMA 2.1. *Let $\lambda_g$ $(g\in G)$ be unitary operators in $K(\Gamma,\Gamma_1)$ corresponding to a shift on $g\in G$. If $R(G)=(\lambda_g, g\in G)''$, then $R(G)'\cap K(\Gamma,\Gamma_1)=\mathbb{C}$.*

The proof is rather traditional and follows both from the definition of the actions $\alpha$ of $G$ on $N$ and the fact that $G$ is a ICC-group.

LEMMA 2.2. *$K(\Gamma,\Gamma_1)$ are full factors.*

*Proof.* We prove the fullness of $K(\Gamma)$; the proof for $K(\Gamma.\Gamma_1)$ is the same. Let $\xi_0$ be a cyclic separating vector for $\psi(\Gamma)=\psi$, i.e., $\psi(x)=(x\xi_0,\xi_0)$ for $x\in K(\Gamma)$, and $J$ be a unitary involution corresponding to $\xi_0$.

To prove the fullness of $K(\Gamma)$, it sufficies to show that $K(\Gamma)$ does not contain nontrivial central sequences (see Theorem 3.1(d) and Proposition 2.8 ($\gamma$) [7]). The norm bounded sequence $(x_n)_{n=1}^\infty$ of operators from $K(\Gamma)$ is called central if $s^*\text{-}\lim_{n\to\infty}[x_n,x]=0$ for any $x\in K(\Gamma)$.

Let $H(\Gamma)$ be a Hilbert space in which operators from $K(\Gamma)$ act. Then $g\to\rho(g)=\lambda_g J\lambda_g J$ determines the unitary representation of $G$ in $H(\Gamma)$. Note that since $\lambda_g\in K(\Gamma)_\psi$ $(g\in G)$ then $\rho(g)\xi_0=\lambda_g J\lambda_g\xi_0=\lambda_g\lambda_g^*\xi_0=\xi_0$ and therefore we can consider the restriction $\pi(g)$ of representation $\rho(g)$ onto a subspace $H_\pi$ orthogonal to $\xi_0$. Let $(x_n)_{n=1}^\infty$ be a central sequence in $K(\Gamma)$ such that $\xi_n=x_n\xi_0\in H_\pi$, i.e., $\psi(x_n)=0$. Since $\pi(g)\xi_n=\rho(g)\,x_n\xi_0=$

$\lambda_g x_n \lambda_g^* \xi_0$ then $\|\pi(g)\xi_n - \xi_n\| \to 0$ for any $g \in G$. Because of the $T$-property there exists $\xi \in H(\Gamma)$ such that $\pi(g)\xi = \xi$ $(g \in G)$ and $\xi \perp \xi_0$.

Since $\Delta_\psi = \sum_{\lambda \in \Gamma} \lambda E_\lambda$, then $\xi = \sum_{\lambda \in \Gamma} E_\lambda \xi$. We can show that the operator $\rho(g)$ $(g \in G)$ commutes with $\Delta_\psi^{it}$ $(t \in \mathbb{R})$. Indeed, $\Delta_\psi^{it} \rho(g) x\xi_0 = \Delta_\psi^{it}(\lambda_g x \lambda_g^*)\xi_0 = \lambda_g \sigma_t^\psi(x)\, \lambda_g^* \xi_0 = \rho(g)\, \Delta_\psi^{it} x\xi_0$. Hence, $\rho(g)$ (and thereby $\pi(g)$) commutes with $E_\lambda$. But then it follows from $\pi(g)\xi = \xi$ $(g \in G)$ that there exists $\lambda \in \Gamma$ such that $E_\lambda \xi \neq 0$ and $\pi(g)\, E_\lambda \xi = E_\lambda \xi$ $(g \in G)$. To be more exact, we assume that $0 < \lambda < 1$. The case with $\lambda \geqslant 1$ is considered in the same way. Then there exists a partial isometry $u_\lambda \in K(\Gamma)$ such that $u_\lambda^* u_\lambda = I$, $\sigma_t^\psi(u_\lambda) = \lambda^{it} u_\lambda$ $(t \in \mathbb{R})$. Note that $S\rho(g)\, x\xi_0 = \rho(g)\, Sx\xi_0$, hence $\rho(g)$ and thereby $\pi(g)$, commutes with the unitary involution $J$. But then $\xi_\lambda = JE_\lambda \xi$ is an eigenvector for $\Delta_\psi$ with an eigenvalue $\lambda^{-1}$, and in this case $\pi(g)\xi_\lambda = \xi_\lambda$ $(g \in G)$. Thus, $u_\lambda \xi_\lambda \in E_1 H(\Gamma) = [K_\psi \xi_0]$, where $[K_\psi \xi_0]$ is a closure of the linear space $K_\psi \xi_0$ in $H(\Gamma)$. But since $K_\psi$ is a $\mathrm{II}_1$-factor, then $u_\lambda \xi_\lambda = A\xi_0$, where $A$ is a closed measurable operator adjoint to $K_\psi$. Therefore, $\xi_\lambda = u_\lambda^* A \xi_0$ and $\lambda_g u_\lambda^* A = u_\lambda^* A \lambda_g$. In this case, the same arguments as were used to prove Lemma 1.4, show that there exists a partial isometry $v_\lambda \in K(\Gamma)$ such that $\sigma_t^\psi(v_\lambda) = \lambda^{it} v_\lambda$ $(t \in \mathbb{R})$ and $\lambda_g v_\lambda = v_\lambda \lambda_g (g \in G)$. Since $R(G)' \cap K(\Gamma) = \mathbb{C}$ (see Lemma 2.1) then $v_\lambda \in \mathbb{C}$, but no such case is possible. ∎

Corollary 2.3. $\mathrm{Sd}(K(\Gamma, \Gamma_1)) = \Gamma_1$.

Indeed, since $K(\Gamma, \Gamma_1)_{\psi(\Gamma, \Gamma_1)}$ is a $\mathrm{II}_1$-factor (see (11)) and $\mathrm{Spec}\, \Delta_{\psi(\Gamma, \Gamma_1)} = \Gamma_1$ (see (12)). The corollary follows from the fullness of $K(\Gamma, \Gamma_1)$, relations (11), (12) and Theorem 4.1 [7]. ∎

*Note* 2.4. According to the above corollary, the full factors of type III $K(\Gamma, \Gamma_1)$ and $K(\Gamma, \Gamma_2)$, where $\Gamma_1 \neq \Gamma_2$, are nonisomorphic because their invariants Sd are different; in this case, however, $K(\Gamma, \Gamma_1)_{\psi(\Gamma, \Gamma_1)} \sim K(\Gamma, \Gamma_2)_{\psi(\Gamma, \Gamma_2)}$ (see (12)). The theorem given below demonstrates that there exist nonisomorphic full III-factors with the same invariant Sd which is any countable nontrivial subgroup $\Gamma_1 \subset \mathbb{R}_+^*$, the factors being different due to the fact that their centralizers of $\Gamma_1$-almost periodic states are nonisomorphic.

Theorem 2.5. *There exists a continuum of nonsiomorphic full factors of type* $\mathrm{III}_1$, *the invariants* Sd *of which are coincident.*

*Proof.* According to the results from Section 1, for the factor $K(\Gamma_1)$ there exists a countable subgroup $\Gamma_2 \subset \mathbb{R}_+^* (\Gamma_1 \subset \Gamma_2)$ such that the $\mathrm{II}_1$-factors $K(\Gamma_1)_{\psi(\Gamma_1)}$ and $K(\Gamma_2)_{\psi(\Gamma_2)}$ have different fundamental subgroups. Construct the factors $K(\Gamma_1, \Gamma')$ and $K(\Gamma_2, \Gamma')$, where $\Gamma' \subset \Gamma_1 \cap \Gamma_2$. It is evident that the factor $K(\Gamma_1, \Gamma')_{\psi(\Gamma_1,\Gamma')}$ is not isomorphic to $K(\Gamma_2, \Gamma')_{\psi(\Gamma_2,\Gamma')}$ though Sd $K(\Gamma_i, \Gamma') = \Gamma'$ $(i = 1, 2)$ according to Corollary 2.3.

Assume that $K(\Gamma_1, \Gamma') \sim K(\Gamma_2, \Gamma')$; then $K(\Gamma_1, \Gamma') \otimes B \sim K(\Gamma_2, \Gamma') \otimes B$, where $B$ is a $\mathrm{I}_\infty$-factor. Consider the weights $\psi_i = \psi(\Gamma_i, \Gamma') \otimes \mathrm{tr}$ $(i = 1, 2)$ on $K(\Gamma_i, \Gamma') \otimes B$ $(i = 1, 2)$, respectively, where tr is a f. norm. semifinite trace on $B$. It is evident that $\psi_1$ and $\psi_2$ are $\Gamma'$-almost periodic weight on $K(\Gamma_1, \Gamma') \otimes B$ with $\psi_1(I) = \psi_2(I) = \infty$. According to Theorem 4.7(2) [7], $\psi_2(\cdot) = \alpha\psi_1(u \cdot u^*)$, where $\alpha \in \mathbb{R}_+^*$ and $u$ is unitary from $K(\Gamma_1, \Gamma') \otimes B$. It follows that the centralizers of weights $\psi_1$ and $\psi_2$ are to be spatially isomorphic. But no such case is possible because $(K(\Gamma_i, \Gamma') \otimes B)_{\psi_i} = K(\Gamma_i, \Gamma')_{\psi(\Gamma_i,\Gamma')} \otimes B$ $(i = 1, 2)$. But the latter factors of type II have different fundamental groups. ∎

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