On the Stability of Rotating Heavy Chains

J. F. Toland

Fluid Mechanics Research Institute,
University of Essex, Colchester CO4 3SQ, England

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The object of this paper is two-fold. On the one hand we want to develop the
duality theory for critical points which is introduced in [4]. The main result in
this context is that if two critical points are in duality, and one of them is a
local minimiser of its functional, then so is the other. In principle then it is
possible to draw conclusions about the stability of stationary solutions of a
system, not by analysing the potential energy functional, but by examining
an appropriate functional which is dual to it. Our second aim is to illustrate this
idea by means of the specific example of the spinning chain problem. It turns
out that in this case the dual variational problem is much more tractable than the
potential energy functional, and indeed the stability analysis follows at once from
results in the literature, once the duality is taken into account.

1. INTRODUCTION

We begin by giving a description of the physical model which motivated
the duality theory in [4], as well as its extensions in Section 4 of this paper. We
are concerned with the stability of certain steady motions of a heavy inelastic
chain which is made to lie in a plane and to rotate with constant angular velocity
$\mu$. The problem of the existence of steady, planar motions of a heavy, flexible
chain has already been treated exhaustively in a classical article by I. I. Kolodner,
and we shall have much need to consult that work in what follows. It is important
to realise at the outset that the problem which Kolodner treats arises from his
consideration of the motion of a completely flexible chain whereas ours does
not. To clarify the distinction let us fix on a way of describing the physical
situation.

First of all a steadily rotating plane means a plane which rotates about a fixed
vertical axis with constant angular velocity. We will call the motion of a chain
planar if the chain moves on a steadily rotating plane. The planar motion of
a chain will be called stationary if the chain does not move relative to the plane
on which it lies.

We will take the fixed end-point of the chain to coincide with the origin of
a cartesian co-ordinate system. The vector $\mathbf{r}(s, t)$ is then the position vector
of the point on the chain at a distance $s$ from its free end-point (measured along
the chain) at time \( t \), and \( T(s, t) \) will be used to denote the tension in the chain at the same place and time. In this convention \( \mathbf{r}(s, t) = (x(s, t), y(s, t), z(s, t)) \); \( \{(x, y, 0); (x, y) \in \mathbb{R}^2 \} \) denotes a horizontal plane through the fixed end of the chain; the \( z \) co-ordinate is measured positively downwards and \( \mathbf{g} = (0, 0, g), \ g > 0 \) denotes the gravity vector. Thus, if the chain is at rest acted on solely by the forces of gravity and tension it will lie along the positive \( z \)-axis. For the sake of convenience we will suppose that the chain is of unit length, that it is homogeneous and of unit density.

In his classical treatment of this system Kolodner [3] assumed that the dynamics of the chain are governed by the following system of equations:

\[
\begin{align*}
\mathbf{r}_{tt} &= \mathbf{g} + (T\mathbf{r})_s \quad (a) \\
\mathbf{r}_s \cdot \mathbf{r}_s &= 1 \quad (b)
\end{align*}
\]

along with the boundary conditions:

\[
T(0, t) = 0; \quad \mathbf{r}(1, t) = 0. \quad (1.2)
\]

Equation (1.1)(a) is a statement of Newton's law for each particle of the chain and (1.1)(b) reflects the fact that the chain is inextensible and that \( s \) denotes arc-length. By restricting his attention to an investigation of all possible configurations of the chain which are stationary with respect to a plane which rotates with constant angular velocity \( \mu \) about a vertical axis through the fixed end-point (i.e. by taking \( \mathbf{r}(s, t) = (v(s) \cos \mu t, v(s) \sin \mu t, w(s)) \) in (1.1), (1.2)) Kolodner reduced the problem to the following system of ordinary differential equations:

\[
\begin{align*}
(Tv')' + \mu^2v &= 0 \quad (a) \\
(Tw')' + g &= 0 \quad (b) \\
v'^2 + w'^2 &= 1 \quad (c) \\
T(0) = v(1) = w(1) &= 0 \quad (d)
\end{align*}
\]

Here ' denotes differentiation with respect to arc-length.

If, instead he had sought solutions of (1.1), (1.2) of the form \( \mathbf{r}(s, t) = (v(s, t) \cos \mu t, v(s, t) \sin \mu t, w(s, t)) \) (i.e. solutions moving on a vertical plane which rotates with constant angular velocity \( \mu \)) (1.1), (1.2) would have been reduced to:

\[
\begin{align*}
v_{tt} \cos \mu t - 2v_v \sin \mu t - \mu^2v \cos \mu t &= (Tv_s)_s \cos \mu t \quad (a) \\
v_{tt} \sin \mu t - 2v_v \cos \mu t - \mu^2v \sin \mu t &= (Tv_s)_s \sin \mu t \quad (b) \\
w_{tt} &= g + (Tw_s)_s \quad (c) \\
v_v^2(s, t) + w_w^2(s, t) &= 1 \quad (d) \\
T(0, t) = 0, \quad v(1, t) = w(1, t) &= 0 \quad (e)
\end{align*}
\]
By multiplying (a) by $\sin \mu t$ and (b) by $\cos \mu t$ and subtracting we can conclude that $v_t = 0$. This means that the only solutions of (1.1), (1.2) which represent a chain lying on a steadily rotating plane are those which are stationary with respect to it, and all such solutions are described by (1.3)$_u$.

Thus for a chain whose dynamical behaviour is governed by (1.1), (1.2) no non-stationary planar motion is possible. The stationary planar motions which are defined by solutions of (1.3)$_u$ are special because any infinitesimal perturbation of one of them results in non-planar three dimensional motions of the chain.

Implicit in the supposition that (1.1), (1.2) govern the motion of the chain is the assumption that the chain is completely flexible; and for such a flexible chain it is difficult to envisage how such planar motions could arise in the first place. It is not clear, for example, how the constant angular velocity would be maintained in such a realisation.

From now on we will only consider a system where, far from being exceptional, only planar motions are possible. What we have in mind to do is to study the motion of a chain which is constrained to lie on a steadily rotating plane. As an example of such a situation consider the three dimensional motion of a bicycle-chain, one end of which is rotated at a constant angular velocity, the other end being free. The motion of the whole chain will then be planar, but not necessarily stationary. We will study the stability of stationary solutions of such a “bicycle-chain” problem.

It will be seen in Section 2 that (1.3)$_u$ still describes possible configurations of a bicycle-chain which rotates with constant angular velocity, but these equations are not a reduced version of (1.1), (1.2). The equations which govern the motion of a chain constrained to lie on a steadily rotating plane are presented in the next section.

We adopt the criterion that a stationary solution is stable if it minimizes the potential energy of the chain, and unstable otherwise. Section 3 is devoted to a precise account of the potential energy functional, its domain of definition, and its properties.

It has been observed [4] that the usual treatment [3] of the system (1.3)$_u$ involves the introduction of new dependent and independent variables and a new parameter $\lambda$ which reduce (1.3)$_u$ to a nonlinear singular second order ordinary differential equation

$$u''(s) + \lambda (u(s)^2 + s^2)^{-1/2} u(s) = 0$$

$$u(0) = u'(1) = 0,$$  \hspace{1cm} (1.5)$_\lambda$

the Euler-Lagrange equations of the functional

$$\int_0^1 \left\{ \frac{u'(s)^2}{2\lambda} - (u(s)^2 + s^{1/2}) \right\} ds$$
with the boundary condition that \( u(0) = 0 \). It has been noted in [4] that this functional appears to have no physical interpretation, other than that it is in duality with the chain's potential energy functional.

A novelty of the method of this paper is that the stability question of the chain is decided by analysing, not the potential energy functional, but this functional which is dual to it. More duality theory based on the methods of [4] is needed for this, and such machinery is developed in an abstract setting in Section 4.

In Section 5 we present the existence results for solutions of (1.3)\(_u\). It turns out that by using \( \mu \) as a control parameter (1.3)\(_u\) can be considered as a bifurcation problem, and we will show that the branch of non-trivial solutions which bifurcates from the first eigenvalue of the linearised problem is stable, whereas all other bifurcating branches are not.

Recall, once again, that this discussion is concerned with "the bicycle-chain problem" and not the completely flexible chain problem which Kolodner uses to introduce his analysis of (1.3)\(_u\). Nonetheless, the stability results for the bicycle-chain problem will be seen to be already implicit in Kolodner’s paper, provided that the duality theory of Section 4 and the Jacobi theory of conjugate points is taken into account.

A direct analysis of the stability question without reference to duality is possible. But the computation of the conjugate points of the Jacobi auxiliary equation is complicated, and we do not present it here. Suffice it to say that we have carried it out, and the duality principle presented here is a manifold simplification of the analysis.

2. THE EQUATIONS OF MOTION AND THE POTENTIAL ENERGY FUNCTIONAL

We consider the motion of an inextensible chain of unit length which is constrained to lie on a steadily rotating vertical plane.

If its fixed end lies at the origin of co-ordinate axes fixed on the rotating plane, \( u(s, t) \) and \( w(s, t) \) represent respectively the distance of the chain from a vertical line and a horizontal plane through its fixed end point. Here, \( s \) denotes arc-length from the free end and \( t \) denotes time.

Since the chain is inextensible we have

\[
\frac{q}{s} v(s, t)^2 + w(s, t)^2 = 1, \quad s \in [0, 1].
\]

If \( T \) denotes the tension in the chain and the angular velocity is \( \mu \) then the motion of the chain is described by

\[
\begin{align*}
\mu^2 v(s, t) - v_{tt}(s, t) + (T(s, t) v_s(s, t))_s &= 0, \\
g - w_{tt}(s, t) + (T(s, t) w_s(s, t))_s &= 0, \\
T(0, t) = v(1, t) = w(1, t) &= 0 \quad \text{for all } t.
\end{align*}
\]
The system \((2.1), (2.2)\) reduces to \((1.3)\) when we seek its time independent solutions. If \((1.3)\) holds then it is easy to see that \(v\) satisfies the boundary value problem:

\[
\begin{align*}
  &\frac{s v'(s)}{(1 - v'(s)^2)^{1/2}}' + \frac{\mu^2}{g} v(s) = 0, \quad (a) \\
  &v(1) = 0. \quad (b)
\end{align*}
\]

which is formally the Euler-Lagrange equation of the functional

\[
\int_0^1 \left[ \frac{\mu^2}{2g} v(s)^2 + s(1 - v'(s)^2)^{1/2} \right] ds + \frac{1}{2}
\]

Since the chain is constrained to lie on a rotating vertical plane it can be considered to lie in a potential field, the resultant force being gravitational, vertically, and centrefugal, horizontally.

Thus the total potential energy of the chain (taking its vertical configuration to have zero potential energy) at time \(t\) is

\[
\mathcal{V}_\mu = -\frac{1}{2}\mu^2 \int_0^1 v(s, t)^2 ds + g \int_0^1 (1 - s - \omega(s, t)) ds
\]

\[
= -\frac{1}{2}\mu^2 \int_0^1 v(s, t)^2 ds - g \int_0^1 s(1 - v'(s, t)^2)^{1/2} ds + g/2,
\]

on account of \((2.1)\). Thus \((2.4)_\mu\) is precisely \(\mathcal{V}_\mu/g\).

If \(\psi\) is a continuously differentiable function on \([0, 1]\) with \(|\psi'(s)| \leq 1\) for all \(s \in [0, 1]\), then \((\psi(s), \omega(s))\) is a possible configuration of an inextensible flexible chain of unit length where \(\omega(s) = \int_s^1 (1 - \psi'(t)^2)^{1/2} dt\). Henceforth we shall describe the configuration of the chain using the function \(\psi\) only, subject to the constraint that \(|\psi'(s)| \leq 1\), and this we can do without loss of generality.

We shall say that a stationary configuration of the chain is stable if the corresponding \(\psi\) is a local minimiser of \(\mathcal{V}_\mu\), and unstable otherwise. Using duality theory we shall be able to give a complete description of all the solutions of \((1.3)\) according to the above criterion.

3. The Potential Energy Functional

To simplify the notation we shall put \(\lambda = \mu^2/g\) and \(V_\lambda = \mathcal{V}_\mu/g^{-1/2}\). Hence the (normalised) potential energy functional \(V_\lambda\) is defined by

\[
V_\lambda(\psi) = -\int_0^1 \left[ \frac{\lambda}{2} \psi(s)^2 + s(1 - \psi'(s)^2)^{1/2} \right] ds
\]
for all \( v \in \mathcal{E} \), where
\[
\mathcal{E} = \{ v \in W^{1,\infty}[0, 1] : v(1) = 0, |v'(s)| \leq 1 \text{ a.e.} \}.
\]

**Definition 3.1.** A critical point of \( V_\lambda \) is an element \( v \in \mathcal{E} \) such that
\[
|v'(s)| \leq 1 - \varepsilon \text{ a.e. for some } \varepsilon > 0, \text{ and } \lim_{t \to 0} (V_\lambda(v + tw) - V_\lambda(v))/t = 0
\]
for all \( w \in \mathcal{K} = \{ w \in W^{1,\infty}[0, 1] : w(1) = 0 \} \).

**Lemma 3.2.** An element \( v \in \mathcal{E} \) is a critical point of \( V_\lambda \) if and only if
\[
\partial H^*(v) \cap \partial G^*(v) \neq \emptyset
\]
where \( G^* \) and \( H^* \) are defined on \( L^2[0, 1] \) as follows:
\[
G^*(v) = \frac{1}{2} \lambda \int_0^1 v(s)^2 \, ds \quad \text{for all } v \in L^2[0, 1],
\]
and
\[
H^*(v) = -\int_0^1 s(1 - v'(s)^2)^{1/2} \, ds \quad \text{if } v \in L^1 \cap \mathcal{E}, \text{ and } H^*(v) = \infty \text{ otherwise.}
\]

Here \( \partial H^* \) and \( \partial G^* \) are subsets of \( L^2 \) and denote the subdifferentials of \( H^* \) and \( G^* \) respectively. (The notion of a subdifferential is recalled at the beginning of the next section).

**Proof.** Suppose \( v \) is a critical point of \( V_\lambda \). Then \( \partial G^*(v) = \{ f \} \) where \( f \) is the bounded linear functional on \( L^2 \) defined by
\[
f(u) = \lambda \int_0^1 u(s) v(s) \, ds \quad \text{for all } u \in L^2.
\]

Now \( H^* \) is a convex functional from \( L^2 \) into \( \mathbb{R} \cup \{+\infty\} \) and so, for each \( t \in ]0, 1[ \), \( u \in L^2 \)
\[
(1 - t) H^*(v) + t H^*(v + u) \geq H^*(v + tu).
\]
But \( H^*(v) \) is finite, and so
\[
H^*(v + u) - H^*(v) \geq (H^*(v + tu) - H^*(v))/t
\]
for all \( t \in ]0, 1[ \). If \( u \in \mathcal{K} \), then \( v + tu \in \mathcal{E} \) for \( t \) sufficiently small and so
\[
H^*(v + u) - H^*(v) \geq \lim_{t \to 0} \sup (H^*(v + tu) - H^*(v))/t.
\]
But \( v \) is a critical point of \( V_\lambda \) and so
\[
\lim_{t \to 0} (H^*(v + tu) - H^*(v))/t = f(u).
\]
So if \( u \in \mathcal{X} \), \( H^*(v + u) - H^*(v) \geq f(u) \). If \( u \notin \mathcal{X} \), then \( H^*(v + u) = \infty \).
Hence in either case
\[
H^*(v + u) - H^*(v) \geq f(u)
\]
for all \( u \in L^2 \) and so \( f \in \partial H^*(v) \).

To go the other way and show that if \( \partial G^*(v) \cap \partial H^*(v) \neq \emptyset \) then \( v \) is a critical point of \( V_A \) we must rely on the work of [4]. In that paper it is shown that if \( \partial G^*(v) \cap \partial H^*(v) \neq \emptyset \) for some \( v \) in \( L^2 \) then \( v \in H^{1,\infty} \), \( |v'(s)| \leq 1 \) a.e. \( v(1) = 0 \), \( (sv'/|1 - v'^2|)^{1/2} \in W^{1,\infty} \) and
\[
\left( \frac{sv'(s)}{(1 - v'(s)^2)^{1/2}} \right)' + \lambda v(s) = 0 \quad \text{a.e.,} \tag{3.1}
\]
where ' denotes weak differentiation. This last equation implies that \( |v'(s)| \leq (1 - \epsilon) \) for some \( \epsilon > 0 \) and \( v \in C^1[0, 1] \). It will suffice for the rest of the proof to show that for each \( u \in \mathcal{X} \)
\[
\lim_{t \to 0} \frac{(H^*(v + tu) - H^*(v))}{t} = f(u)
\]
where \( f \) is defined above. But this follows immediately from the definition of \( H^* \), and (3.1) above.

This completes the proof of the theorem.

4. SOME DUALITY THEORY

It is now time to recall the duality principle introduced in [4]. Let \( X \) and \( X^* \) be a pair of topological vector spaces in separating duality and let \( \langle \cdot, \cdot \rangle \) determine the duality between them. If \( I: X \to \mathbb{R} \) (\( \mathbb{R} = \mathbb{R} \cup \{\pm \infty\} \)) is a functional then
\begin{enumerate}
  \item (i) \( \text{dom} \ I = \{ x \in X : I(x) < \infty \} \);
  \item (ii) \( \partial I(x) = \{ x^* \in X^* : I(x + h) \geq I(x) + \langle x^*, h \rangle \text{ for all } h \in X \} \);
  \item (iii) \( I^*(x^*) = \sup_{x \in X} \{ \langle x, x^* \rangle - I(x) \} \).
\end{enumerate}

For the basic concepts in convex analysis the reader is referred to [5].

Remark. \( \text{dom} \ I \subset X \) is called the essential domain of \( I \); \( \partial I(x) \subset X^* \) is called the subdifferential of \( I \) at \( x \); and \( I^*: X^* \to \mathbb{R} \) is called the functional conjugate to \( I \).

In everything that follows it is assumed that:

(1) \( (V, V^*) \) and \( (Y, Y^*) \) are two pairs of spaces in separating duality, \( F: V \to \mathbb{R} \), and \( G: Y \to \mathbb{R} \) are both convex, lower semi-continuous functionals on their respective domains and \( A: V \to Y \) is a homeomorphism whose adjoint \( A^*: Y^* \to V^* \) is also a homeomorphism.
On $V$, $J$ is used to denote the functional defined by

$$J_u = G \circ \Lambda u - F u,$$

and $J$ is a mapping from $V$ into $\mathbb{R}$.

On $Y^*$, $J$ is used to denote the functional defined by

$$J_{v^*} = F^* \circ A^* v^* - G^* v^*,$$

and $J$ is a mapping from $Y^*$ into $\mathbb{R}$.

It will be usual to find elements of $V^*$ written as $\ell^* v^*$, $v^* \in Y^*$ and elements of $Y$ written as $\ell u$, $u \in V$. This we do without loss, because $\Lambda$ and $A^*$ are both bijective.

**Theorem 4.1** ([4]). (a) $\inf_{u \in V} \{ J_u \} = \inf_{v^* \in Y^*} \{ J_{v^*} \}$;

(b) if $\partial F(u) \cap \partial (G \circ A)(u) \ni A^* v^*$ then $\partial G^* (v^*) \cap \partial (F^* \circ A^*)(v^*) \ni A u$ and

(i) $J_u = J_{v^*}$;

(ii) $F u + F^* \circ A^* v^* = \langle u, A^* v^* \rangle$;

(iii) $G \circ \Lambda u + G^* v^*$ = $\langle u, A^* v^* \rangle$.

(c) if $u \in V$ is such that $J_u = \inf_{v^* \in Y^*} \{ J_{v^*} \}$ and $\partial F(u) \ni A^* v^*$, then $J_{v^*} = \inf_{v^* \in Y^*} \{ J_{v^*} \}$, $\partial F(u) \cap \partial (G \circ A)(u) \ni A^* v^*$ and the conclusion in (b) holds with $u = u$ and $v^* = v^*$.

**Definition.** If $\partial F(u) \cap \partial (G \circ A)(u) \neq \emptyset$ ($\partial G^* (v^*) \cap \partial (F^* \circ A^*)(v^*) \neq \emptyset$) then $u(v^*)$ is called a critical point of $J$.

The point of Theorem 4.1(b) is that critical points of $J$ and $J$ appear in pairs. A pair $(u, v^*) \in V \times Y^*$ of critical points of $J$ and $J$ are said to be in duality if Theorem 4.1(b) (i), (ii), (iii) hold.

If two critical points of $J$ and $J$ are in duality then it is natural to ask whether they have any characteristics (such as being local extremals or saddle-points in common). The answer is “typically, yes” as we shall see.

**Definition.** Let $X$ and $I$ be as before. An element $x \in \text{dom } I$ is called a local minimiser (resp. local maximiser) of $I$ if there exists a neighbourhood $U$ of $x$ in $X$ such that $I_x \leq I_x$ for all $x \in U$ ($I_x \geq I_x$ for all $x \in U$).

The next definition is due to Browder ([1, p. 38]):

**Definition.** Let $X$, $X^*$ be a pair of spaces in separating duality. Then a mapping $A : X \to 2^{X^*}$ is said to be an upper semi-continuous set-valued mapping if, for each $x \in X$ and each neighbourhood $N$ of $Ax$ there exists a neighbourhood $U$ of $x$ in $X$ such that $A(U) \subseteq N$. 
Remark. In this last definition the possibility that $J(x) = \varnothing$ is not excluded. In our application this is indeed the case.

**Lemma 4.2.** (a) *If $A^*v^* \in \partial F(u)$, then $Ju \geq \hat{J}v^*$.*

(b) *If $w^* \in \partial G(Au)$, then $\hat{J}w^* \geq Ju$.*

**Proof.** (a) By assumption

$$F_u + F^* \circ A^*v^* = \langle u, A^*v^* \rangle,$$

and

$$G \circ Au + G^*w^* \geq \langle u, A^*v^* \rangle.$$ 

Hence $Ju \geq \hat{J}v^*$.

The proof of (b) is similar.

**Theorem 4.3.** Let $(u, y^*) \in V \times Y^*$ be a pair of critical points of $J$ and $\hat{J}$ which are in duality.

(a) *Let $G$ be strictly convex and $\partial G^* : Y^* \rightarrow 2^V$ be set-valued upper semicontinuous. If, in addition, $y$ is a local minimum for $J$, and for some neighbourhood $U$ of $y^*$, $\partial G^*(w^*) \neq \emptyset$ when $w^* \in U \cap \text{dom} \hat{J}$, then $y^*$ is a local minimum for $\hat{J}$.*

(b) *Let $F^*$ be strictly convex and $\partial F : V \rightarrow 2^{Y^*}$ be set-valued upper semicontinuous. If in addition, $y^*$ is a local minimum for $\hat{J}$, and for some neighbourhood $U$ of $y$, $\partial F(u) \neq \emptyset$ when $u \in U \cap \text{dom} J$, then $y$ is a local minimum for $J$.*

**Proof.** (a) Since $G \circ Au + G^*y^* = \langle Au, y^* \rangle$, the strict convexity of $G$ implies that

$$\partial G^*(y^*) = \{Au\}.$$ 

Now $A$ is a homeomorphism and $y$ is a local minimiser of $J$ and so there exists a neighbourhood $N$ of $Au$ in $Y$ such that

$$Ju \leq Ju \quad \text{for all } u \in A^{-1}N.$$ 

But since $\partial G^*$ is set-valued u.s.c. there exists a neighbourhood $U$ of $y^*$ in $Y^*$ such that $\partial G^*(w^*) \subset N$ for all $w^* \in U$. Therefore if $w^* \in U$, then $w^* \in \partial G(Au)$ for some $u \in A^{-1}N$, or $\partial G^*(w^*) = \emptyset$. If $\partial G^*(w^*) = \emptyset$, then $w^* \in \text{dom} \hat{J}$ and $\hat{J}w^* = +\infty$. If $w^* \in \text{dom} \hat{J} \cap U$ then $w^* \in \text{dom} G^* \cap U$ and $w^* \in \partial G(Au)$ for some $u \in A^{-1}N$. Since $G \circ Au$ is therefore finite and $J : V \rightarrow \mathbb{R}$, it is clear that $Ju$ is finite and $u \in \text{dom} J \cap A^{-1}N$. Now Lemma 4.2 implies that

$$\hat{J}y^* = Ju \leq Ju \leq \hat{J}w^*$$

for all $w^* \in \text{dom} \hat{J} \cap U$. We have shown that $y^*$ is a local minimiser for $J$.

(b) The second half of this theorem follows by duality. We can interchange the roles of $J$, $\hat{J}$, $F$, $G^*$, $G$, $F^*$, $V$, $Y^*$, and $V$, $Y^*$. This completes the proof of the theorem.
THEOREM 4.4. If all the hypotheses of Theorem 4.3(a) and (b) hold and \((u, v^*)\) is a pair of critical points of \(J\) and \(\tilde{J}\) which are in duality then \(v^*\) is the unique critical point of \(\tilde{J}\) which is in duality with \(u\) and vice versa.

Furthermore, \(u\) is a local minimiser for \(J\) if and only if \(v^*\) is a local minimiser for \(\tilde{J}\).

Proof. In view of Theorem 4.3 we need only prove the uniqueness result in the first sentence of the theorem. Since \(G \circ A_u + G^* v^* = \langle A_u, v^* \rangle\) and \(G\) is strictly convex it is clear that \(A_u\) is the unique element of \(\partial G^*(v^*)\), and so \(u\) is the unique critical point of \(J\) which is in duality with \(v^*\). A similar argument to prove the converse completes the proof of the theorem.

Remark. In an obvious way it is possible to make mild assumptions on \(F\) and \(G^*\) which ensure that there is a correspondence, in general, between local maxima of \(J\) and \(\tilde{J}\). By exclusion there would then be a correspondence between the critical points which are neither local maxima nor local minima, i.e., the saddle-points of \(J\) and \(\tilde{J}\).

Finally let us specialise to the case where \(V\) and \(Y\) are Banach spaces and \(V^*\) and \(Y^*\) are their topological dual spaces. As always \(A: V \rightarrow Y\) is a homeomorphism. To avoid ambiguity we will use \(\| \cdot \|\) to denote the norm in \(V\) and \(V^*\) and \(| \cdot |\) the norm in \(Y\) and \(Y^*\). There now follows a special version of Theorem 4.4 in the Banach space setting.

THEOREM 4.5. Let \(V\) and \(Y\) be Banach spaces and suppose that all the hypotheses of Theorem 4.4 hold and that \(\partial G\) is set-valued u.s.c.

If \(u\) is a critical point of \(J\) suppose that there exists an increasing function \(c: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) and \(\epsilon > 0\) such that

\[
J(u + h) - J(u) \geq c(\|h\|)
\]

for all \(h \in V\), with \(\|h\| < \epsilon\).

Then if \(v^*\) is the critical point of \(\tilde{J}\) in duality with \(u\), there exists \(\delta > 0\) and a strictly increasing function \(d: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that whenever \(v^* + k^* \in \text{dom} \tilde{J}\), we have

\[
\tilde{J}(v^* + k^*) - \tilde{J}(v^*) \geq d(|k^*|), \quad \text{for all } k^* \text{ such that } |k^*| \leq \delta.
\]

Proof. In the proof of the Theorem 4.3(a) it is shown that there exists \(\delta > 0\) such that if \(v^* + k^* \in \text{dom} \tilde{J}\) and \(|k^*| < \delta\) then \(v^* + k^* \in \partial G(A(u + h))\) for some \(h \in V\) with \(\|h\| < \epsilon\). Therefore if \(|k^*| < \delta\) and \(v^* + k^* \in \text{dom} \tilde{J}\), by Lemma 4.2

\[
\tilde{J}(v^*) = J(u) \leq J(u + h) \leq \tilde{J}(v^* + k^*)
\]

and so

\[
\tilde{J}(v^* + k^*) - \tilde{J}(v^*) \geq c(\|h\|).
\]
Now suppose that $|k^*| = \delta' < \delta$. Since $G$ is strictly convex, $\partial G^*(v^* + k^*)$ is a singleton set if $v^* + k^* \in \text{dom} \tilde{f}$, and $|k^*| < \delta$. But $\partial G$ is set-valued u.s.c. and so there exists $\rho(\delta') > 0$ such that if $\|h\| < \rho(\delta')$ then $\partial G(A(u + h)) \subset \{v^* + k^* : |k^*| < \delta'\}$.

Therefore if $|k^*| = \delta'$ and $v^* + k^* \in \text{dom} \tilde{f}$,

$$\partial G^*(v^* + k^*) \subset \{A(u + h) : \|h\| \geq \rho(\delta')\}.$$

Consequently

$$\tilde{f}(v^* + k^*) - \tilde{f}(v^*) \geq c(\rho(\delta')) = c(\rho(|k^*|))$$

if $|k^*| = \delta'$, $v^* + k^* \in \text{dom} \tilde{f}$. This completes the proof of the theorem.

5. The Stability Analysis

In this last section we resume our discussion of the normalised potential energy functional introduced in Section 3. We intend to analyse the critical points of $V_\lambda$, but we begin with a precise result about the existence of critical points.

**Theorem 5.1.** There exists a sequence $\{\lambda_n\}$ of real numbers ($\lambda_n = (\sigma_n/2)^2$, $\sigma_n$ being the $n$th zero of $J_0$) with the following properties:

(a) If $\lambda \in [\lambda_n, \lambda_{n+1}]$ there exists exactly $n$ non-zero critical points, $v_1(\lambda), \ldots, v_n(\lambda)$, of $V_\lambda$ each of which lies in $C^3[0, 1] \cap C^2[0, 1]$. Furthermore $v_\lambda(\lambda)$, $v = 1, \ldots, n$, has exactly $(v - 1)$ isolated zeros in $[0, 1]$;

(b) the zero function is a critical point of $V_\lambda$ for all values of $\lambda > 0$ and when $\lambda \in [0, \lambda_1]$ it is the only critical point of $V_\lambda$;

(c) every critical point of $V_\lambda$ for $\lambda \geq 0$ satisfies the differential equation (3.1).

In order to prove this result it is sufficient to use the duality theory of [4] in the context of this problem. So we make the following identification. Let $V = \{u \in W^{1,2}[0, 1] : u(0) = 0\}$ and let $Y = L^2[0, 1]$. Since $V$ and $Y$ are Hilbert spaces we can identify $V^*$ with $V$ and $Y^*$ with $Y$. If $\| \|$ denotes the norm on $Y^*$, then we will use $\| \|$ to denote the norm on $V$ where $\|u\| = |u|$. We shall define $F, G_\lambda$ and $A$ as follows:

$$G_\lambda(v) = \int_0^1 \frac{v(s)^2}{2\lambda} \, ds \quad \text{for all } v \in L^2[0, 1],$$

$$F(u) = \int_0^1 (u(s)^2 + s^2)^{1/2} \, ds \quad \text{for all } u \in V,$$

1 That this norm is equivalent to the $W^{1,2}$ norm on $V$ is a consequence of Poincaré's inequality which holds because of the zero end-condition satisfied by elements of $V$. 


and $Au = u' \in Y$ for all $u \in V$, where $'$ denotes weak differentiation. Then by $J_\lambda$ we mean the functional $G_\lambda \circ A - F$ on $V$.

A calculation carried out in [4] shows that for $v \in Y^* = Y$,
\[
F^* \circ A^*v = \begin{cases} 
-\int_0^1 s(1 - v'(s)^2)^{1/2} \, ds & \text{if } v \in \mathcal{E} \\
+\infty & \text{otherwise}
\end{cases}, \quad \text{and} \quad G_\lambda^*v = \int_0^1 \frac{\lambda v(s)^2}{2} \, ds
\]

Thus, in the notation of Section 3, $H^* = F^* \circ A^*$, and now it is clear that the connection between $J_\lambda$ and $V_\lambda$ is one of duality—in the notation of Section 4—$V_\lambda = J_\lambda$. The point of Lemma 3.2 is that the classical notion of a critical point of $V_\lambda$ (Definition 3.1) coincides with the notation of a critical point of $J_\lambda$ introduced in Section 4. Henceforth we shall only use $V_\lambda$ for the functional dual to $J_\lambda$.

**Proof of Theorem 5.1.** In Section 4 of [4] the following result is established. An element $v \in L^2[0,1]$ is a critical point of $V_\lambda$ if and only if $v \in \mathcal{E}$, $\lambda v = u'$ a.e. and $u(s) = -sv'(s)/(1 - v'(s)^2)^{1/2}$ a.e. where $u \in V$ is a critical point of $J_\lambda$. But $u$ is a critical point of $J_\lambda$ if and only if $u$ satisfies (1.5). The rest of the proof is a consequence of Kolodner's Theorem 1 and Lemma 2.

With the existence theorem for (2.3) (a), (b) such a simple consequence of Kolodner's paper we are now in a position to apply the duality theory of Section 4 to prove the following stability theorem.

**Theorem 5.2.** (a) If $\lambda \in [0, \lambda_1]$ then the zero function is the unique minimiser of $V_\lambda$;

(b) if $\lambda_n < \lambda \leq \lambda_{n+1}$, $n \geq 1$, then $v_\lambda(\lambda)$ is the unique minimiser of $V_\lambda$, $v_\lambda(\lambda)$, $\nu = 2, \ldots, n$, and 0 are all critical points of $V_\lambda$ but none of them is a local minimiser.

**Proof.** It is clear that $\partial G_\lambda$ and $\partial G_\lambda^*$ are both set-valued u.s.c. and that $G_\lambda^*$ is everywhere subdifferentiable on $V$ (see [4] for details). Since $F^*$ is clearly strictly convex and $\partial F$ is clearly everywhere non-empty it suffices in order to verify that all the hypotheses of Theorem 4.4 hold to show that $\partial F$ is set-valued u.s.c. For each $u \in V$ it has been shown [4] that $A^*v \in \partial F(u)$ if and only if $v \in \mathcal{E}$ and
\[
u(s) = -u(s)/(u(s)^2 + s^2)^{1/2} \quad \text{a.e.}
\]

In other words $v(s) = -u(s)/(u(s)^2 + s^2)^{1/2}$ or
\[
u(s) = \int_s^1 \frac{u(t)}{(u(t)^2 + t^2)^{1/2}} \, dt.
\]

Now it is clear that the mapping $\partial F$ which sends $u$ to the unique element $A^*v$ in $\partial F(u)$ is continuous from $V$ into $V$. 

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**Notes:**
- The above text is a continuation from page 25.
- The page contains a proof of theorems and lemmas related to functional analysis, specifically concerning the dual space and critical points of certain functionals.
- The text is dense with mathematical notation and requires a good understanding of functional analysis concepts.
- The proof involves establishing conditions under which certain functionals have unique minimisers and identifying critical points.
- The text cites [4] for further details on the topics discussed.

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**References:**

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**Mathematical Symbols:**
- $A^*$: Dual operator
- $F^*$: Dual functional
- $\lambda$: Parameter
- $V_\lambda$: Functional
- $\lambda_1$: Specific parameter value
- $\mathcal{E}$: Subset of $L^2[0,1]$
Therefore, as a consequence of Theorem 4.4 it is sufficient to prove the present theorem, word for word, with $J_\lambda$ replacing $V_\lambda$ and $u_\lambda(\lambda)$ instead of $v_\rho(\lambda)$. Here $u_\lambda(\lambda)$ denotes the critical point of $J_\lambda$ which is in duality with the critical point $v_\rho(\lambda)$ of $V_\lambda$.

This result is implicit in Lemma 2 of Kolodner’s paper [3] provided that the Jacobi theory of conjugate points is taken into account.

We will begin by proving the instability results. In Lemma 2 (p. 406 of [3]) Kolodner proves the following result about the Jacobi equation corresponding to $J_\lambda$:

$$h''(s) + \frac{\lambda s^2 h(s)}{(u_\lambda(\lambda)(s)^2 + s^2)^{1/2}} = 0, \quad h(0) = 0, \quad h'(0) = 1.$$  \hfill (5.1)$_{\lambda \rho}$

If $n \geq 1$ and $\lambda_n < \lambda < \lambda_{n+1}$ the solution $h$ of (5.1)$_{\lambda \rho}$, $\nu \geq 2$ has exactly $(\nu - 1)$ zeros in $[0, 1]$. This means that Jacobi’s necessary condition [2] for $u_\lambda(\lambda)$, $\nu \geq 2$, $\lambda > \lambda_1$ to be a minimiser of $J_\lambda$ for the fixed end-point problem is violated, and so, a fortiori $u_\lambda(\lambda)$ cannot be a local minimiser for $J_\lambda$ on $V$, when $\lambda > \lambda_1$, $\nu \geq 2$.

Now to establish the instability of the zero solution when $\lambda > \lambda_1$ we can not rely on the violation of Jacobi’s necessary condition for the fixed end-point problem, because it is not violated. In this case the Jacobi equation corresponding to the zero critical point of $J_\lambda$ is:

$$h''(s) + \frac{\lambda h(s)}{s} = 0, \quad h(0) = 0, \quad h'(0) = 1.$$  \hfill (5.2)

The second variation of $J_\lambda$ at the zero solution is given by

$$\int_0^1 \left[ h'(s)^2 - \frac{\lambda h(s)^2}{s} \right] ds$$

and it will suffice to show that for $\lambda > \lambda_1$ this functional is not positive definite. But the operator $A: \mathcal{D}(A) \subset L^2 \to L^2$ given by

$$Ah = -h'' - \frac{\lambda h}{s}$$

$$\mathcal{D}(A) = \{ h \in C^\infty[0, 1] : h(0) = h'(1) = 0 \}$$

is essentially self-adjoint.

Now, by definition, corresponding to $\lambda_1$ there exists a non-zero element $h_1 \in C^2[0, 1]$ such that

$$-h_1''(s) - \frac{\lambda_1 h_1(s)}{s} = 0, \quad h_1(0) = h_1'(1) = 0,$$
and so
\[-h''_1(s) - \lambda h_1(s) = (\lambda_1 - \lambda) \frac{h_1(s)}{s}\]

So multiplying by $h_1$ and integrating gives that, for $\lambda > \lambda_1,$
\[
\int_0^1 \left[ h'_1(s)^2 - \lambda h_1(s)^2 \right] ds = -\left(\lambda - \lambda_1\right) \int_0^1 \frac{h_1(s)^2}{s} ds = -\left(\frac{\lambda - \lambda_1}{\lambda_1}\right) \| h_1 \|^2.
\]

The second variation is not positive on $V$ and so 0 is not a minimiser of $J_\lambda$ if $\lambda > \lambda_1.$

Now $G_\lambda \circ \Lambda : V \to \mathbb{R}$ is convex and continuous in the norm-topology on $V$ and so it is convex and lower semi-continuous on $V$ with respect to the weak topology. Since convergence in the weak topology on $V$ implies the $L^2$-convergence of a subsequence we can assert that any minimising sequence of $J_\lambda$ has a subsequence which converges weakly to a minimiser of $J_\lambda.$ Since we have established already that $F$ is everywhere subdifferentiable it is subdifferentiable at this minimiser. So the minimiser is a critical point of $I_\lambda$ and if $\lambda > \lambda_1,$ it can only be $u_1(\lambda).$ If $\lambda < \lambda_1$ it can only be zero. This completes the proof of the theorem.

**Theorem 5.3.** If $\lambda > \lambda_1$ there exists $\delta(\lambda) > 0$ and an increasing function $d_\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that, if $| h | < \delta(\lambda)$ and $v_\lambda(\lambda) + h \in \delta,$ then
\[
V_\lambda(v_\lambda(\lambda) + h) - V_\lambda(v_\lambda(\lambda)) \geq d_\lambda(| h |).
\]
(Here $| \cdot |$ is the $L^2$ norm of $k$).

**Proof.** Because of Theorem 4.5 it will suffice to prove the following result.

**Lemma 5.4.** If $\lambda > \lambda_1$ there exists $\epsilon(\lambda) > 0$ and a number $c_\lambda > 0$ such that, if $h \in V$ and $\| h \| < \epsilon(\lambda)$ then
\[
J_\lambda(u_1(\lambda) + h) - J_\lambda(u_1(\lambda)) \geq c_\lambda \| h \|^3.
\]

**Proof.** For convenience we shall write $u$ in place of $u_1(\lambda)$ throughout.
\[
J_\lambda(u + h) - J_\lambda(u) = \int_0^1 \left[ \frac{(u'(s) + h'(s))^2}{2\lambda} - \frac{u'(s)^2}{2\lambda} \right] ds - \left\{ ((u(s) + h(s))^2 + s^3)^{1/2} - (u(s)^2 + s^3)^{1/2} \right\} ds
\]
\[
= \int_0^1 \left[ \left\{ \frac{u'(s) h'(s)}{\lambda} + \frac{h'(s)^2}{2\lambda} \right\} - \left\{ \frac{u(s) h(s)}{\lambda} \right\} \left( \frac{u(s)^2 + s^3}{2(u(s)^2 + s^3)^{1/2}} + \frac{s^2 h(s)^2}{2(u(s)^2 + s^3)^{3/2}} \right) + \frac{s^2(u(s) + h(s))^2}{2((u(s) + h(s))^2 + s^3)^{3/2}} \right\} ds
\]
\[
- \frac{s^2(u(s) + h(s)) h(s)^2}{2((u(s) + h(s))^2 + s^3)^{3/2}} \right\} ds
\]
where \( 0 \leq |h(s)| \leq |h(s)| \), by Taylor’s theorem,

\[
= \int_0^1 \left[ \frac{h'(s)^2}{2\lambda} - \frac{s^2 h(s)^2}{2(u(s)^2 + s^2)^{3/2}} \right] ds + \int_0^1 \frac{1}{2} \frac{s^2(u(s) + \dot{h}(s)) h(s)^2}{(u(s) + \dot{h}(s))^2 + s^2} ds
\]

since \( u \) is a solution of (1.5),

But

\[
\left| \int_0^1 \frac{1}{2} \frac{s^2(u(s) + \dot{h}(s)) h(s)^2}{((u(s) + \dot{h}(s))^2 + s^2)^{3/2}} ds \right| \leq \left| \int_0^1 \frac{sh(s)^2}{4((u(s) + \dot{h}(s))^2 + s^2)^{3/2}} ds \right| \leq \left| \int_0^1 \frac{h(s)^2}{4s^2} ds \right|
\]

Now \( h \in W^{1,2}[0, 1] \) which is continuously embedded in the Hölder class \( C_{0,1/2}[0, 1] \). So for some \( M > 0 \),

\[
\left| \int_0^1 \frac{h(s)^2}{4s^2} ds \right| \leq \frac{M^3}{4} \| h \|_3^3 \int_0^1 \frac{1}{s^{1/2}} ds = \frac{M^3}{2} \| h \|_3^3.
\]

If suffices then to show that for \( \| h \| \) sufficiently small

\[
\int_0^1 \left[ \frac{h'(s)^2}{\lambda} - \frac{s^2 h(s)^2}{(u(s)^2 + s^2)^{3/2}} \right] ds \geq c_1 \| h \|_3^3
\]

for some \( c_1 > 0 \).

Now the operator \( B_\lambda : \mathcal{D}(B_\lambda) \subset L^2 \rightarrow L^2 \) defined by

\[
B_\lambda h = \frac{\lambda s^2 h(s)}{(u(s)^2 + s^2)^{3/2}}
\]

for all \( h \in \mathcal{D}(B_\lambda) = \{ h \in C^\infty[0, 1] : h(0) - h'(1) = 0 \} \) is essentially self-adjoint.

We will prove first of all that all the eigenvalues of \( B_\lambda \) are strictly positive when \( \lambda > \lambda_1 \).

Suppose then that there exists \( h \in C^2[0, 1] \), \( \mu \geq 0 \) such that

\[
-\ddot{h}(s) - \left( \frac{\lambda s^2}{(u(s)^2 + s^2)^{3/2}} - \mu \right) h(s) = 0 \tag{5.3}
\]

\[
h(0) = h'(1) = 0, \quad h'(0) \neq 0
\]

where

\[
-\ddot{u}(s) - \frac{\lambda u(s)}{(u(s)^2 + s^2)^{1/2}} = 0 \tag{5.4}
\]

\[
u(0) = u'(1) = 0
\]
and neither \( u \) nor \( u' \) has a zero in \([0, 1]\). Let \( k \) denote a solution on \([0, \infty)\) of the equations

\[
-k''(s) - \frac{\lambda s^2 k(s)}{(u(s)^2 + s^3)^{3/2}} = 0
\]

\[k(0) = 0, \quad k'(0) = 1 \tag{5.5}\]

That such a solution exists is proved in [3]. Suppose that \( k'(0) > 0 \) and multiply (5.3) by \( k \), (5.5) by \( h \). Then

\[
\{-h'(s) k(s) + h(s) k'(s)\} - \mu \int_0^1 h(s) k(s) \, ds = 0
\]

and hence

\[
\hat{h}(1) k'(1) + \mu \int_0^1 h(s) k(s) \, ds = 0.
\]

Either \( \mu \int_0^1 h(s) k(s) \, ds < 0 \), and hence, by Sturm's comparison theorem, \( k \) has a turning-point in \([0, 1] \), or else \( \mu \int_0^1 h(s) k(s) \, ds \geq 0 \) in which case \( h(1)k'(1) \leq 0 \).

In any case we have shown that \( k' \) has a zero in \([0, 1] \).

Now put

\[
\bar{u}(t) = \lambda u(t/\lambda) \quad \text{for all} \quad t \in [0, \lambda],
\]

\[
\bar{k}(t) = \lambda k(t/\lambda) \quad \text{for all} \quad t \in [0, \lambda].
\]

Then

\[
-k''(t) - \frac{\bar{u}(t)}{(ar{u}(t)^2 + t^2)^{1/2}} = 0
\]

\[\bar{u}(0) = 0, \quad \bar{u}'(\lambda) = 0,
\]

and

\[
-k''(t) - \frac{t^2 \bar{k}(t)}{(ar{u}(t)^2 + t^2)^{3/2}} = 0
\]

\[\bar{k}(0) = 0, \quad \bar{k}'(0) = 1, \quad \bar{k}'(\alpha) = 0
\]

for some \( \alpha \in (0, \lambda) \).

This contradicts Lemma 2 of [3] and so all the eigenvalues of \( B_\lambda \) are strictly positive when \( \lambda > \lambda_1 \).

We have shown that for each \( \lambda > \lambda_1 \) if \( u = u_\lambda(\lambda) \) then there exists \( c_\lambda > 0 \) such that

\[
\int_0^1 \left\{ h'(s)^2 - \frac{\lambda s^2 h(s)^2}{(u(s)^2 + s^3)^{3/2}} \right\} \, ds \geq c_\lambda \| h \|^2.
\]

Now suppose that there is a sequence \( \{h_n\} \subset V \) such that \( \| h_n \| = 1 \) but

\[
\int_0^1 \left\{ h_n'(s)^2 - \frac{\lambda s^2 h_n(s)^2}{(u(s)^2 + s^3)^{3/2}} \right\} \, ds \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence $|h_n| \to 0$ and we may suppose without loss of generality that $h_n(s) \to 0$ almost everywhere as $n \to \infty$.

But

$$\left| \frac{\lambda s^2 h_n(s)^2}{(u(s)^2 + s^2)^{3/2}} \right| \leq \left| \frac{\lambda h_n(s)^2}{s} \right| \leq \lambda M^2 \|h_n\| = \lambda M^2,$$

for some $M > 0$ (by the continuous embedding of $V$ in $C_{0,1/2}[0, 1]$).

The Lebesgue Dominated Convergence theorem then implies that

$$\int_0^1 \left\{ \frac{s^2 h_n(s)^2}{(u(s)^2 + s^2)^{3/2}} \right\} ds \to 0 \quad \text{as} \quad n \to \infty.$$

This is a contradiction. So the second variation of $J_\lambda$ is positive definite on $V$ (with respect to the $W^{1,2}$ norm). We have already verified that this is enough to ensure that the solution $u_1(\lambda)$ of $\lambda > \lambda_1$, lies at the bottom of a potential well in $V$. Thus the proof of the lemma is complete. Theorem 5.3 now follows as a consequence of Theorem 4.5.

REFERENCES