SOME COMPLEX STRUCTURES ON PRODUCTS OF HOMOTOPY SPHERES

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§1. INTRODUCTION

In 1948 H. Hopf ([6]) introduced the complex manifolds, that nowadays are known as Hopf manifolds. Their construction is quite simple: they are nothing but the quotient of \( \mathbb{C}^n - 0 \) by the group of automorphisms, generated by the transformation \( z \rightarrow cz \), where \( c \) is a complex number with \( |c| \neq 1 \). Differentiably, they are the product \( S^1 \times S^{2n-1} \) of a circle and a \((2n - 1)\)-sphere, and complex analytically they are homogeneous manifolds and fibre bundles over the complex projective space \( P_{n-1} \), with fibre an elliptic curve. For \( n \geq 2 \) they do not carry any Kähler structure, hence \( a \text{ fortiori} \) no projective algebraic structure.

Later, E. Calabi and B. Eckmann ([2]) have constructed complex structures on any product of two spheres of odd dimension.

After J. Milnor's discovery ([8]) of exotic spheres (i.e. differentiable manifolds \( \Sigma^n \), which are homeomorphic, but not diffeomorphic to \( S^n \)), the question has been raised by Hopf and others whether a product \( S^1 \times \Sigma^{2n-1} \) (which is never diffeomorphic to \( S^1 \times S^{2n-1} \)) also carries complex structures.

The purpose of this note is to give an affirmative answer to this question in the case where \( \Sigma^{2n-1} \) is any homotopy sphere bounding a parallelisable manifold \( (n \neq 2) \). This is a consequence of recent results of F. Hirzebruch, J. Milnor and one of the present authors, describing exotic spheres as neighbourhood boundaries of isolated singularities of quasi-homogeneous affine varieties. The generalised Hopf manifolds constructed with the help of these results are no longer homogeneous, nor are they holomorphic fibre bundles. But they are holomorphic Seifert fibre spaces over a projective variety \( V \), such that each fibre is an elliptic curve. In certain cases \( V \) is again the complex projective space \( P_{n-1} \), but in general \( V \) is not even nonsingular.†

The generalised Hopf manifolds seem to provide the first examples of compact topological manifolds, carrying complex structures for which the underlying differentiable structures are different.

More generally, it is possible to construct complex structures on any product

† The analogue of the generalised Hopf manifolds for \( n = 2 \) are special elliptic surfaces, as studied by K. Kodaira ([7]).
\[ x^{2k-1} \times S^{21-1} \] of exotic spheres bounding parallelisable manifolds \((k, 1 \neq 2)\). However, it seems to be unknown whether, for \(k, 1 \neq 1\), this product is perhaps always diffeomorphic to \(S^{2k-1} \times S^{21-1}\). It is known that this is the case if one of the spheres is the standard sphere, or if \(k = 1 = 4\). ([11]).

\section{Generalised Hopf Manifolds}

Let \(a = (a_0, \ldots, a_n)\) be an \((n+1)\) tuple of positive integers, let \((z_0, \ldots, z_n)\) be the standard coordinates in \(\mathbb{C}^n\), and let \(X(a) = X(a_0, \ldots, a_n)\) be the affine algebraic variety given by the equation

\[ \sum_{i=0}^{n} a_i z_i^n = 0. \]

If some \(a_i = 1\), the variety \(X(a)\) is non singular. Otherwise, \(X(a)\) has exactly one singular point, namely \(0 = (0, \ldots, 0)\). Let \(S^{2n+1}\) be the sphere \(\sum_{i=0}^{n} z_i \bar{z}_i = 1\). Then the space \(\Sigma(a) = X(a) \cap S^{2n+1}\) carries in a natural way the structure of a \((2n-1)\) dimensional oriented differentiable manifold. These differentiable manifolds have been studied in [1]. Among other things, it has been proved there that all homotopy spheres \(\Sigma^{2n-1}(n \neq 2)\), bounding a parallelisable manifold, appear among the \(\Sigma(a)\). In fact, the following theorem holds

**Theorem 1.** For \(n \neq 2\) and every \((2n-1)\) dimensional homotopy sphere \(\Sigma\) bounding a parallelisable manifold, there are infinitely many \(a = (a_0, \ldots, a_n)\), such that \(\Sigma(a)\) is diffeomorphic to \(\Sigma\).

Now we define a holomorphic operation of \(\mathbb{C}\) on \(X(a) - 0\) by

\[ \iota(z_0, \ldots, z_n) = (e^{i/a_0}z_0, \ldots, e^{i/a_n}z_n) \]

\(\mathbb{Z}\) and \(\mathbb{R}\) operate on \(X(a) - 0\) as subgroups of \(\mathbb{C}\). Since the action of \(\mathbb{Z}\) on the complex manifold \(X(a) - 0\) is free, the quotient

\[ H(a) = X(a) - 0/\mathbb{Z} \]

carries in a canonical way the structure of a \(\mathbb{C}\) manifold \(H(a) = H(a_0, \ldots, a_n)\). \(H(1, \ldots, 1)\) is a Hopf manifold, as defined in the introduction. There is a diffeomorphism of \(\Sigma(a) \times \mathbb{R}\) onto \(X(a) - 0\) defined by \((z, t) \rightarrow \iota(z, t)\), inducing a diffeomorphism from \(H(a)\) onto \(\Sigma(a) \times S^1\). Combining this observation with Theorem 1, we find

**Theorem 2.** For \(n \neq 2\) and every homotopy sphere \(\Sigma\) of dimension \(2n - 1\), bounding a parallelisable manifold, there exist complex structures on \(S^1 \times \Sigma\).

These complex structures \(H(a)\) on \(S^1 \times \Sigma(a)\) for an exotic sphere \(\Sigma(a)\) are different from previously known structures. More interesting is the fact that the underlying differentiable structures of \(S^1 \times \Sigma(a)\) are exotic. In fact, we have

**Proposition 3.** Let \(\Sigma, \Sigma'\) be homotopy spheres of dimension \(n \geq 5\). Then \(S^1 \times \Sigma\) is diffeomorphic to \(S^1 \times \Sigma'\), if and only if \(\Sigma\) is diffeomorphic to \(\Sigma'\).
We learned the following proof for this fact in a discussion with several specialists. If \( S^1 \times \Sigma' \) is diffeomorphic to \( S^1 \times \Sigma \), there exists a diffeomorphism of the universal coverings, \( f: \mathbb{R} \times \Sigma' \to \mathbb{R} \times \Sigma \).

Identify \( \Sigma' \) with the submanifold \( f(\{0\} \times \Sigma') \) of \( \mathbb{R} \times \Sigma \), and \( \Sigma \) with \( \{t\} \times \Sigma \), where \( t \) is chosen large enough so that \( \Sigma \) and \( \Sigma' \) do not intersect. Let \( W \) be the closure of the connectedness component of \( \mathbb{R} \times \Sigma - (\Sigma \cup \Sigma') \) which has boundary \( \Sigma \cup \Sigma' \). Then \( W \) is a compact manifold of dimension \( \geq 6 \), with boundary \( \Sigma \cup \Sigma' \). \( W \) is simply connected, because it is a deformation retract of the closure of one component of \( \mathbb{R} \times \Sigma - \{0\} \times \Sigma' \), which in turn can be identified with the closure of one component of \( \mathbb{R} \times \Sigma' - \{0\} \times \Sigma' \), i.e., with \( \mathbb{R}^+ \times \Sigma' \). The relative homology group \( H_p(W, \Sigma') \) vanishes, as can be seen from the exact homology sequence of the pair \( (W, \Sigma') \) and the fact that \( H_p(\Sigma') \to H_p(W) \) is an isomorphism, because its composition with the isomorphism \( H_p(W) \to H_p(\mathbb{R}^+ \times \Sigma') \) is an isomorphism. Hence the triad \( (W, \Sigma, \Sigma') \) satisfies the conditions of the h-cobordism theorem (\([9] \), 9.1), and \( \Sigma \) is diffeomorphic to \( \Sigma' \).

Remark. As N. Kuiper pointed out to us, the existence of an exotic 8-sphere \( \Sigma^8 \) implies the existence of 30 different differentiable structures on \( S^1 \times S^7 \), namely those obtained as \( S^1 \times \Sigma' \) and as \( (S^1 \times \Sigma') \# S^8 \), where \( \Sigma' \) is any one of the 28 homotopy 7-spheres. It would be interesting to know whether the \( (S^1 \times \Sigma') \# S^8 \) also admit complex structures.

As mentioned in the introduction, the generalised Hopf manifolds \( H(a) \) can be given the structure of a Seifert fibre space in the sense of Holmann (\([4]\)). The fibration map is simply the canonical projection
\[
X(a) - 0/\mathbb{Z} \to X(a) - 0/C.
\]
It is convenient to describe this map in a slightly different way. Let \( \Gamma_a \) be the discrete subgroup of \( \mathbb{C} \) generated by 1 and \( 2\pi i[a] \), where \( [a] = [a_0, \ldots, a_n] \) is the least common multiple of \( a_0, \ldots, a_n \). Then the action of \( \mathbb{C} \) on \( X(a) - 0 \) induces a proper holomorphic action of the complex 1-torus \( T_a = \mathbb{C}/\Gamma_a \) on \( H(a) \). By theorems of H. Holmann (\([4]\), Satz 8) and (\([5]\), Satz 3) the quotient \( H(a)/T_a \) is in a natural way a normal complex space \( V(a) \), the canonical projection
\[
\pi_a: H(a) \to H(a)/T_a
\]
is holomorphic, and \( (H(a), \pi_a, V(a)) \) is a holomorphic Seifert principal fibre space with \( T_a \) as fibre and structure group. \( T_a \) acts on \( H(a) \) fixed point free, therefore each fibre of \( \pi_a \) is a quotient of \( T_a \) by a finite subgroup, hence an elliptic curve. However, the action of \( T_a \) is in general not free, consequently \( H(a) \) is in general not a fibre bundle over \( V(a) \), and \( V(a) \) is in general not free from singular points.

We want to determine those \( H(a) \) for which \( \Sigma(a) \) is a homotopy sphere and \( V(a) \) a complex manifold. According to general results of Holmann (\([4]\), p. 389, 397) the quotient space \( V(a) \) can be described locally in the following way. For any point \( p \in H(a) \) there exists a 1-codimensional nonsingular analytic subset \( Y \) of a neighborhood of \( p \), transversal to the fibre \( T_a \cdot p \), and invariant under the isotropy group \( I_p \) of \( p \), such that a neighbourhood of \( \pi_a(p) \) in \( H(a)/T_a \) is isomorphic to \( Y/I_p \). Now, by a theorem of D. Prill (\([10]\), p. 382), \( Y/I_p \) will be nonsingular at \( p \) if and only if \( I_p \), considered as a transformation group of \( Y \), is equal to
its subgroup \( I_p \), generated by elements with \( 1 \)-codimensional fixed point sets passing through \( p \). This condition is obviously equivalent to the corresponding condition for \( I_p \) as a transformation group of \( H(a) \). Let \( p = (z_0, \ldots, z_k) \) be such that exactly \( k \) coordinates \( z_i, \ldots, z_k \) are different from zero, \( k \geq 2 \). Then
\[
I_p = \ker(T_{(a_0, \ldots, a_n)} \to T_{(a_1, \ldots, a_k)}),
\]
\[
J_p = \prod_{j \neq i} \ker(T_{(a_0, \ldots, a_n)} \to T_{(a_0, \ldots, a_j, \ldots, a_n)}).
\]
The groups \( I_p \) and \( J_p \) are equal if and only if they have the same order, i.e.
\[
\frac{[a_0, \ldots, a_n]}{[a_{i_1}, \ldots, a_{i_k}]} = \prod_{j \neq i} \frac{[a_0, \ldots, a_n]}{[a_0, \ldots, a_j, \ldots, a_n]}.
\]

**Proposition 4.** Let \( \Sigma(a_0, \ldots, a_n) \) be a homotopy sphere, with all \( a_i \neq 1 \). Then the generalised Hopf manifold \( H(a) \) is a Seifert fibre space \( (H(a), \pi_a, V(a)) \) with elliptic curves as fibres.

(i) \( V(a) \) is nonsingular, if and only if one of the following two conditions is satisfied:

1. For all \( i, j \) with \( i \neq j \), g.c.d. \( (a_i, a_j) = 1 \);
2. \( n \) is odd, and up to a permutation of indices, \( a = (a_0, 2b_1, \ldots, 2b_n) \), with \( (a_0, 2) = 1 \), \( (a_0, b_i) = 1 \), and \( (b_i, b_j) = 1 \) for all \( i, j \) with \( i \neq j \).

(ii) If \( V(a) \) is nonsingular, then \( V(a) \) is biregularly equivalent to the complex projective space \( P_n \).

**Proof.** (i) follows immediately from the formula (*), combined with Theorem 1 of [1].

For the proof of (ii), observe that the map from \( C_n \) onto itself, given by
\[
(z_0, \ldots, z_n) \to (z_0^{a_0}, \ldots, z_n^{a_n})
\]
duces a ramified covering map
\[
f_a: V(a_0, \ldots, a_n) \to V(1, \ldots, 1),
\]
where \( V(1, \ldots, 1) \) is of course nothing but \( P_{n-1} \). If condition (i), (1) is satisfied, then \( f_a \) is bijective, and hence a biholomorphic map from \( V(a_0, \ldots, a_n) \) onto \( P_{n-1} \). If condition (i), (2) is satisfied, there is a factorisation \( f_a = h \cdot g_a \), where \( g_a: V(a_0, 2b_1, \ldots, 2b_n) \to V(1, 2, \ldots, 2) \) is the map induced by
\[
(z_0, \ldots, z_n) \to (z_0^{a_0}, z_1^{b_1}, \ldots, z_n^{b_n})
\]
and \( h: V(1, 2, \ldots, 2) \to V(1, \ldots, 1) \) is induced by \( (y_0, \ldots, y_n) \to (y_0, y_1^2, \ldots, y_n^2) \). The map \( h \) is the Segre covering (of degree \( 2^{n-1} \)) of \( P_{n-1} \) by \( P_{n-1} \), and \( g_a \) is bijective. Therefore, \( V(a) \) is again a \( P_{n-1} \).

**Remark.** Since \( V(a) \) is a normal complex space, which is a ramified covering of \( P_{n-1} \), it follows from a theorem of H. Grauert and R. Remmert ([3], Satz 2), that \( V(a) \) is projective, as claimed in the introduction.

### §3. GENERALISED CALABI–ECKMANN MANIFOLDS

In this section we construct complex structures on \( \Sigma(a) \times \Sigma(b) \). The complex manifolds, constructed by E. Calabi and B. Eckmann ([2]) as well as the generalised Hopf manifolds \( H(a) \) of §2 are special cases of these complex structures.
Let \( a = (a_0, \ldots, a_m) \) and \( b = (b_0, \ldots, b_n) \) be tuples of natural numbers, and let \( \tau \) be any complex number with \( \text{Im}(\tau) \neq 0 \). Define an action \( f_{\tau} \) of \( C \) on \( (X(a) - 0) \times (X(b) - 0) \) by

\[
f_{\tau}(t; x_0, \ldots, x_m; y_0, \ldots, y_n) = (e^{t/a_0}x_0, \ldots, e^{t/a_m}x_m; e^{t/b_0}y_0, \ldots, e^{t/b_n}y_n)
\]

\( f_{\tau} \) is a free, holomorphic, locally proper action, therefore, by [4], Satz 10 the quotient space is a complex manifold

\[
H(a, b)_\tau = H(a_0, \ldots, a_m, b_0, \ldots, b_n)_\tau = (X(a) - 0) \times (X(b) - 0)/C.
\]

It is easy to see that one has a diffeomorphism of \( \mathbb{C} \times \Sigma(a) \times \Sigma(b) \) onto \( (X(a) - 0) \times (X(b) - 0) \), defined by \( (t, (x, y)) \to f_{\tau}(t, x, y) \), hence a diffeomorphism from \( \Sigma(a) \times \Sigma(b) \) onto \( H(a, b)_\tau \).

One readily verifies that \( H(a; 1, 1)_{2\pi i} \) is biholomorphic to \( H(a) \), and that the complex manifolds \( M_{p,q;\tau} \), constructed in [2] occur among the \( H(a_0, \ldots, a_m; b_0, \ldots, b_n)_\tau \), in fact \( M_{n-1,m-1;-\tau} \) is just the manifold \( H(1, \ldots, 1; 1, \ldots, 1) \).

**ADDED IN PROOF.** It has just been proved by R. de Sapio (Notices of the American Mathematical Society, 15 (1968), p. 628) that the product of two homotopy spheres \( \Sigma \) and \( \Sigma' \), each bounding a parallelisable manifold, is always diffeomorphic to \( S^{2k-1} \times S^{2l-1} \), provided of course that \( k, l \geq 2 \). In fact, de Sapio determines all differentiable structures on a product of two spheres of dimension \( \geq 2 \). Part of these results has been obtained independently by both D. Burghelea and N. Kuiper (unpublished), who both also settled the question of all differentiable structures on a product \( S' \times S^k \). (D. Burghelea: Some Applications of the Browder-Levine-Theorem, Forschungsinstitut für Mathematik, ETH, Zürich).

**REFERENCES**


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