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SET EXISTENCE AXIOMS FOR GENERAL (not necessarily countable) STABILITY THEORY

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H. Friedman ([2], [3]) set up the framework for several studies aimed at finding out what are precisely the set existence axioms needed for proving various theorems of 'ordinary mathematical practice' esp. in the fields of analysis and algebra (cf. also [4], [7], [8]). In [5] we considered the similar question for theorems in stability theory.

Friedman's axioms are formulated in the language of second-order artithmetic. This is well suited for classical analysis, the concepts of which are codable as sets of natural numbers, while, in the case of algebra or model theory one has to restrict oneself to *countable* structures and languages to make such a coding possible (this has been done in [4] and [5]). However, the distinction between the countable and the general (i.e., not necessarily countable) theory is not very fashionable nowadays, neither in algebra nor in parts of model theory because, except for a few uses of Zorn's lemma in the latter, the two theories are developed along identical lines.

The aim of this paper is to extend our work [5] by formulating set existence axioms in a general context that can accommodate model theory of arbitrary (not necessarily countable) languages and then examining which of these axioms are needed for several theorems in stability theory. Obviously, the interest of such an enterprise transcends stability theory as the axioms should be suitable, e.g., for general algebra as well. I should emphasize that, while dropping the assumption of countability, we do not assume the language to be uncountable. In fact, we carefully avoid any theorem (such as upward Löwenheim–Skolem) which makes explicit reference to cardinalities. This leaves us with a small portion of model theory which is, nevertheless, large enough to contain the basics of stability theory.

The point of departure for building up the intended universe of our axioms is an abstract infinite set whose elements will serve as symbols of a logical language. We call these elements 'letters' and denote their set by 'Lt'. Next we consider the set Lt* of words over Lt (i.e., finite sequences of elements of Lt) with the operation of concatenation *C*. While Friedman's axioms describe properties of

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natural numbers with the arithmetical operations and of sets of natural numbers, ours refer to words with the operation of concatenation and to sets of words.

Friedman had five axiom systems ordered linearly by deductive strength, the weakest three (denoted 'RCA_0', 'WKL_0' and 'ACA_0') being the relevant ones for stability theory (cf. [5]). We thought to define analogues RCA_0^*, WKL_0^* and ACA_0^* and be content with these, but were soon forced to consider a larger list of axioms no more linearly ordered. The reason for this difference is that the wellordering of the natural numbers implies several forms of the axiom of choice while, in the absence of such an ordering of Lt*, we have to consider these forms as distinct and often incomparable axioms.

Some of our findings are as follows. The theorem on definability of types and the finite equivalence theorem are derivable in RCA_0^* by the same proofs as those for the countable case in [5]. A modification of the proof in [5] shows that the symmetry lemma is also derivable in RCA_0^*. The theorems asserting the existence of defining schemes for types and of nonforking extensions turn out to be equivalent to distinct set existence principles (while both were equivalent to ACA_0 in the countable case); the proofs of these facts are new as the arguments in [5, Theorems 3.7 and 4.8] do not generalize. Although the above list contains the most important of the theorems in stability theory considered in [5], the picture we obtain is far from complete. We leave several questions open, some metamathematical and others concerning the strength of theorems in stability theory (the most tantalizing is the one concerning the statement that if all types are definable, then the theory is stable). These questions make it seem doubtful that we came up with all set existence principles that are relevant to stability theory, not to speak about general algebra.

The paper depends, of course, on [5] especially for the proofs which go over unchanged.

In Section 1 we describe the system RCA_0^* and its consequences. Section 2 describes various distinct principles which in the countable case were all provable in RCA_0. Section 3 is about WKL_0^* and Section 4 about ACA_0^* and related systems. Section 5 contains open problems.

1. The theory RCA_0^*

We use a two-sorted language with variables of the first sort $u, v, w, l, \ldots$ ranging over *words* (i.e., elements of Lt*) and of the second sort $X, Y, \ldots$ ranging over *sets* of words. The language has a constant Λ (for the empty word), a binary function symbol C (for concatenation of words; we will simply write 'uv' for '$C(u, v)$'), a unary predicate Lt, binary predicates $=$ (for identity of words) and $\subset$ ($u \subset v$ meaning that u is a segment of v, cf. axiom (6) below) as well as $\in$ relating words to sets. A formula whose bound variables are all of the first sort will be called *arithmetical*; quantifiers of the form '$\exists u \subset t$' and '$\forall u \subset t$' are called *bounded* and we define Σ^i_j, Π^i_j, Δ^i_j for $i = 0, 1$ and $j \in \omega$ as is customarily done in arithmetic.

The system RCA_0^* has seven axioms and three axiom schemata. The axioms are (the universal closures of) the following:

(1) $uv = \Lambda \leftrightarrow u = \Lambda \wedge v = \Lambda$.

(2) $uw = vw \rightarrow u = v$.

(3) $wu = wv \rightarrow u = v$.

(4) $u(vw) = (uv)w$.

(5) $\mathrm{Lt}(l) \wedge \mathrm{Lt}(m) \wedge ul = vm \rightarrow l = m$.

(6) $u \subset v \leftrightarrow \exists w, t\, (v = wut)$.

(7) (Induction axiom)

$$[\Lambda \in X \wedge \forall u, l\, (u \in X \wedge \mathrm{Lt}(l) \rightarrow ul \in X)] \rightarrow \forall u\, (u \in X).$$

(Let us remark that if we interpret X in (7) as a truly second-order variable, i.e., ranging over *all* sets of words then, once the set Lt is given, (1)–(7) characterize the system (Lt^*, Λ, Lt, C, $\subset$) up to isomorphism.)

In order to be able to use induction we must stipulate existence of sets. We add:

(8) Δ^0_1-CA: $\forall u\, (\varphi(u) \leftrightarrow \psi(u)) \rightarrow \exists X\, \forall u\, (u \in X \leftrightarrow \varphi(u))$

whenever φ is Σ^0_1 and ψ is Π^0_1.

Remark. In arithmetic (which is Friedman's context) there is a good reason for starting by restricting at first comprehension to Δ^0_1-sets, namely, the fact that these are precisely the recursive (i.e., the 'palpable' kind of) sets. Our situation is more general and we may ask why we should start with the same restriction of the comprehension principle. The formal analogy to RCA_0 is a reason good enough for doing so, especially since, as it turns out, the resulting RCA_0^* is equiconsistent with RCA_0 (hence, with primitive recursive arithmetic). As an additional reason, we can point to the following 'palpability' feature characterizing the Δ^0_1-definable sets. If X is a such, then for each u there is a *concrete direct evidence* that $u \in X$ or $u \notin X$, whichever is the case. Such an evidence is embodied by any $v \in \mathrm{Lt}^*$ which makes either $\varphi_1(u, v)$ or $\neg\psi_1(u, v)$ true, where $\varphi(u) = \exists v\, \varphi_1(u, v)$ and $\psi(u) = \forall v\, \psi_1(u, v)$. In arithmetic, this feature implies the existence of an *algorithm* for finding such an evidence. In our context, however, we cannot conclude the same, as we lack the ability of searching through Lt^* systematically.

As in the case of RCA_0, we need more induction than (7) and (8) are able to provide.

(9) Σ^0_1-induction: $[\varphi(\Lambda) \wedge \forall u, l\, (\varphi(u) \wedge \mathrm{Lt}(l) \rightarrow \varphi(ul))] \rightarrow \forall u\, \varphi(u)$

whenever φ is Σ^0_1.

Finally:

(10) Lt is infinite.

This completes the description of RCA_0^*. Quine [6] remarked that once we have at least two distinct letters, we can encode finite sequences of words as single words using the operation of concatenation alone. Having this in mind, it is easy to see that RCA_0 is interpretable in RCA_0^* by identifying the set ω of natural numbers with the set $\{a\}^*$ of the words based on a given letter $a \in \mathrm{Lt}$; notice that $\{a\}^*$ is definable from a by a Δ_0^0-formula. Conversely (as finite sequences of natural numbers are encodable in arithmetic as single numbers), RCA_0^* is interpretable in RCA_0 by identifying Lt with the set of numbers encoding sequences of length one. Notice that the interpretation of RCA_0 in RCA_0^* is conservative while the converse interpretation is not.

We can define, in an obvious way, the notion of *primitive recursive function* over Lt* the main clause being that whenever g and h are primitive recursive, so is f provided that:

$$f(u_1, \ldots, u_n, \Lambda) = g(u_1, \ldots, u_n)$$

and

$$l \in \mathrm{Lt} \rightarrow f(u_1, \ldots, u_n, vl) = h(u_1, \ldots, u_n, v, l, f(u_1, \ldots, u_n, v)).$$

A useful example of such is the *length* function $\mathrm{lh}(\Lambda) = \Lambda$, $\mathrm{lh}(ul) = \mathrm{lh}(u)a$ from Lt* into ω (i.e., $\{a\}^*$). All primitive recursive functions are representable in RCA_0^*. It follows that if $\mathfrak{A} \models \mathrm{RCA}_0^*$ and $\{a\}_{\mathfrak{A}}^*$ is isomorphic to the standard integers for one $a \in \mathrm{Lt}_{\mathfrak{A}}$, it is so for all $a \in \mathrm{Lt}_{\mathfrak{A}}$; if such is the case, we call $\mathfrak{A}$ an ω-model. RCA_0^* has ω-models of arbitrary cardinality.

Two other primitive recursive functions of interest are defined as follows:

$$\mathrm{pd}(\Lambda) = \Lambda, \quad \mathrm{pd}(ul) = u \quad \text{and} \quad u \dot{-} \Lambda = u, \quad u \dot{-} vl = \mathrm{pd}(u \dot{-} v).$$

Using $\dot{-}$, we can prove as in [4, 1.1]:

Lemma 1.1 (RCA_0^*). *Π_1^0-induction holds as well.*

First-order logic over a (not necessarily countable) language L can be developed in RCA_0^* in a natural way because the formulas of the language are, in fact, words over a sufficiently large set of letters. The notions of formula, logical axiom, proof will have Δ_1^0 descriptions. Theories will be sets of words. An L-structure $M = (|M|, S^M)_{S \in L}$ can be represented as a pair of sets $(|M|, X^M)$ where $X^M = \{(S, u) : u \in S^M\}$. As in [5], we will also (and mostly) speak of the *model* M, i.e., the complete diagram of the structure $(M, a)_{a \in |M|}$; this is a set of $L(|M|)$-sentences where $L(|M|)$ is L augmented by individual constants which name the elements $a \in |M|$. We make no notational distinction between an element and its name. A word of caution: among the symbols of L we have the equality $=$; the corresponding relation $=^M$ is not necessarily the identity relation on $|M|$, but this should not disturb us (by the way, in RCA_0 where every set is well ordered, we could assume w.l.o.g. that the interpretation of the equality symbol was always the identity).

As described in [5], key notions like consistent, complete or stable theory, as well as type, nonforking extension, etc. will all have arithmetical definitions of low complexity. The following basic theorems of stability theory are provable in RCA_0^*: *the local definability theorem* (stating that for a stable T, every type p over a subset of a model of T is definable), *the finite equivalence relation theorem* and the *symmetry lemma*. The proofs of the first two of these theorems are like those in [5] for the countable case. For the symmetry lemma, however, we have to come up with a modified proof:

Theorem 1.2 (Symmetry Lemma) (RCA*). *Let T be stable and $A \cup \{b, c\}$ a subset of one of its models. If $t(b, A \cup \{c\})$ does not fork over A, then $t(c, A \cup \{b\})$ does not fork over A.*

Proof. We are given that $p_1(x) = t(b, A \cup \{c\})$ is a nonforking extension of $p(x) = t(b, A)$ and want to show that $q_1(y) = t(c, A \cup \{b\})$ is a nonforking extension of $q(y) = t(c, A)$ (as in [5], we ask the reader to keep in mind the notational distinction between the type-variables). To this end, assume that $\vDash \varphi(b, c)$, i.e., $\varphi(b, y) \in q_1$; we have to show that q needs the formula $\varphi(x, y)$. Recall that a φ-ladder of length k is a pair of sequences $(a_0, \ldots, a_{k-1})$ and $(b_0, \ldots, b_{k-1})$ such that $\vDash \varphi(a_i, b_j)$ iff $i \leqslant j$. In a stable theory there are no arbitrarily long φ-ladders.

Let $\psi_k(x, x_1, \ldots, x_{k-1}, y, y_1, \ldots, y_{k-1})$ be the formula stating that $(x, \ldots, x_{k-1})$ and $(y, \ldots, y_{k-1})$ are φ-ladders. As b and c are φ-ladders of length 1, the type $p_1(x)$ needs $\psi_1(x, y)$, trivially. By the definition of nonforking, $p(x)$ needs ψ_1 as well. Let k be the largest integer such that $p(x)$ needs ψ_k. Assume this happens via $\delta(\bar{u}, \bar{v})$, i.e. (cf. Definition 4.1 in [5]),

$$\exists \bar{u}\bar{v}\, \delta \wedge \forall \bar{u}\bar{v} \left(\delta \rightarrow \bigvee_{i<l} \psi_k(x, \bar{u}_i, \bar{v}_i) \right) \in p,$$

where $\bar{u}_i$, $\bar{v}_i$ are subsequences of $\bar{u}$, $\bar{v}$ of lengths $k-1$, k respectively. Let $\bar{u}_i = (u_{i1}, \ldots, u_{i,k-1})$ and $\bar{v}_i = (v_{i0}, \ldots, v_{i,k-1})$. Consider the formula

$$\lambda(u_{00}, \ldots, u_{l-1,0}, \bar{u}, \bar{v}) = \delta(\bar{u}, \bar{v}) \wedge \bigwedge_{i<l} (\exists z\, \psi_k(z, \bar{u}_i, \bar{v}_i) \rightarrow \psi_k(u_{i0}, \bar{u}_i, \bar{v}_i)).$$

Obviously, λ is consistent, i.e. $\vDash \exists u_{00} \cdots \bar{u}\bar{v}\, \lambda$. We *claim* that:

$$\vDash \forall u_{00} \cdots u_{l-1,0} \bar{u}\bar{v} \left(\lambda \rightarrow \bigvee_{i<l} \bigvee_{j<k} \varphi(u_{ij}, c) \right),$$

showing that $q(y)$ needs $\varphi(x, y)$ via λ. Proof of the claim: Otherwise,

$$\vDash \exists u_{00} \cdots \bar{u}\bar{v} \left(\lambda \wedge \bigwedge_{i<l} \bigwedge_{j<k} \neg \varphi(u_{ij}, c) \right)$$

and as we know also that

$$\vDash \forall \bar{u}\bar{v} \left(\delta(\bar{u}, \bar{v}) \rightarrow \bigvee_{i<l} \psi_k(b, \bar{u}_i, \bar{v}_i) \right)$$

we infer, with the definition of λ in mind, that

$$\vDash \forall u_{00} \cdots \bar{u}\bar{v}\Big(\lambda \wedge \bigwedge_{i<l} \bigwedge_{j<k} \neg\varphi(u_{ij}, c) \to \bigvee_{i<l} \psi_{k+1}(b, u_{i0}, \bar{u}_i, c, \bar{v}_i)\Big).$$

This means that p_1 needs ψ_{k+1}. But then, p needs ψ_{k+1} as well, contradicting the maximality of k.

Let us remark that this proof is completely elementary in the sense that it does not involve at all constructions of infinite objects, not even recursive ones. This stands in contrast with the proof of the symmetry lemma in [5]. That proof is not formalizable in RCA_0^* because it uses a theorem, 4.5 of [5], which is derivable in RCA_0 but not in RCA_0^* (we discuss the derivability of this theorem in our general context, in the second problem of Section 5). This is by no means an isolated phenomenon. A much more important example is the following. In RCA_0 we could prove that every complete theory has a model but we cannot do the same in RCA_0^*. the reason for this difference is explained next.

2. VWKL_0^* and related systems

Consider the following set existence principles:

VWKL (*Very Weak König Lemma*): "Every infinite tree $\mathcal{T} \subset {}^{<\omega}2$ with no terminal nodes has a branch".

OP (*Ordering Principle*): "Every set can be linearly ordered".

AC (*Axiom of Choice*): "If I and R are sets such that for all $i \in I$, $R_i = \{u : (i, u) \in R\} \neq \emptyset$, then there is a function $F \subset R$ with $\operatorname{dom} F = I$".

WOP (*Wellordering Principle*): "Every set can be wellordered".

All of these principles are provable in RCA_0 (due to the fact that every set is wellordered with order type ω). VWKL is the principle needed for constructing a model of a countable complete theory.

Neither of the last three principles mentioned above is provable in RCA_0^*. VWKL is still provable (recall that in this system, $\omega = \{a\}^*$) but is insufficient for building a model of an arbitrary complete theory. What we need is the following:

Ext VWKL (*Extended Very Weak König Lemma*): "Let X be a set and $\mathcal{F}$ a family of functions from finite subsets of X into $\{0, 1\}$ such that: (a) the restriction of any function in $\mathcal{F}$ to a subset of its domain belongs to $\mathcal{F}$; (b) if $f \in \mathcal{F}$ and $u \in X$, then f has an extension $g \in \mathcal{F}$ with $u \in \operatorname{dom} g$. Under these conditions, there is a function $F : X \to \{0, 1\}$ such that $F \restriction X_0 \in \mathcal{F}$ for all finite $X_0 \subset X$".

This principle is not provable in RCA_0^*. As a matter of fact we have:

Theorem 2.1. Ext VWKL *is not provable in* RCA_0^* *together with the full (second-order) comprehension axiom* CA.

Proof (a rudimentary Fraenkel-type argument). Let X be an infinite set and E an equivalence relation on X all of whose classes have precisely two elements. Let $\mathscr{P}$ be the family of subsets of X^*, second-order definable with (first-order) parameters in the structure $(X^*; C, E)$ (where C is the concatenation on X^*). Then $\mathfrak{A} = (X^*; \mathscr{P}; \Lambda, C, \subset; \in)$ is a model of RCA_0^* with full comprehension but not of Ext VWKL. A counterexample to the latter is provided by the set $\mathscr{F}$ of those functions f from finite subsets of X to $\{0, 1\}$ satisfying that $f(a) \neq f(b)$ whenever $a, b \in \operatorname{dom} f$ and $a\, E\, b$.

Let us remark that a similar argument shows that Ext VWKL does not follow even from $RCA_0^* + CA + OP$. On the other hand, RCA_0^* + Ext VWKL implies OP. Also, the independence of AC from the Prime Ideal Theorem shows that AC does not follow from $RCA_0^* + CA +$ Ext VWKL. We *conjecture* that $RCA_0^* +$ WOP (or even $ACA^* +$ WOP, with ACA_0^* defined as in Section 4) does not imply Ext VWKL.

We now set $VWKL_0^* = RCA_0^* +$ Ext VWKL. Let MECT be the statement that every complete theory has a model.

Theorem 2.2 (RCA_0^*). MECT *is equivalent to* $VWKL_0^*$.

Proof (Sketch). To show that $VWKL_0^*$ implies MECT we proceed as in the countable case with a slight complication due to the fact that we cannot use "the first constant not mentioned so far" (see 5.1 below for similar complication). For the reverse implication let P_u, $u \in X$ be unary predicates; denote

$$P_f(x) = \bigwedge \{P_u(x) : u \in \operatorname{dom} f, f(u) = 0\} \cup \{\neg P_u(x) : u \in \operatorname{dom} f, f(u) = 1\}$$

and let T be the theory whose axioms insure that $\{x : P_f(x)\}$ is non-empty iff $f \in \mathscr{F}$. T is complete, and if $M \vDash T$, then any $a \in |M|$ picks up an F as desired.

3. WKL_0^*

The extended weak König lemma is:

Ext WKL: "If X and $\mathscr{F}$ satisfy condition (a) of Ext VWKL as well as: (c) for every finite $X_0 \subset X$ there is an $f \in \mathscr{F}$ with $\operatorname{dom} f = X_0$, then there is an $F : X \to \{0, 1\}$ such that $F \restriction X_0 \in \mathscr{F}$ for all finite $X_0 \subset X$".

This principle was formulated by Rado and later by Engeler and Robinson (cf. [1]). We let WKL_0^* be RCA_0^* + Ext WKL.

Theorem 3.1 (RCA_0^*). (a) WKL_0^* *is equivalent to the completeness theorem.*
(b) WKL_0^* *is equivalent to the compactness theorem.*

The *proof* is similar to its unstarred counterpart and to 2.1.

4. ACA_0^*

We now consider the arithmetic comprehension axiom-scheme:

Σ_∞^0-CA: $\exists X \forall u\, (u \in X \leftrightarrow \varphi(u))$

whenever φ is an arithmetic formula.

Proposition 4.1 (RCA_0^*). (a) Σ_∞^0-CA *is equivalent to* Σ_1^0-CA (*i.e., like* Σ_∞^0-CA *but* φ *restricted to* Σ_1^0*-formulas*).
(b) Σ_∞^0-CA *is equivalent to the following axiom*: *if* $F: I \to \mathrm{Lt}^*$ *is a function, then there is an* X *s.t.* $\forall u\, (u \in X \leftrightarrow \exists i \in I\, (F(i) = u))$ (*i.e., "if* F *is a function, then its range exists"*).

Proof. (a) Like 2.7 in [7].
(b) Let $\varphi(u) = \exists v\, \psi(u, v)$ with ψ Δ_0^0. Then $I = \{(u, v) : \psi(u, v)\}$ exists by Δ_1^0-CA and so does $F = \{((u, v), u) : (u, v) \in I\}$. F is a function whose range is $X = \{u : \varphi(u)\}$.

We let ACA_0^* be $\mathrm{VWKL}_0^* + \Sigma_\infty^0$-CA. With this definition we have:

Proposition 4.2. ACA_0^* *implies* WKL_0^*.

Proof. If $\mathscr{F}$ satisfies the hypothesis (a) and (c) of Ext WKL, then its subset $\mathscr{F}' = \{f : \text{for all finite } X_0 \subset X\, \exists g \in \mathscr{F}\, (f \subset g \wedge X_0 \subset \operatorname{dom} g)\}$ exists by Σ_∞^0-CA, is nonempty and satisfies conditions (a) and (b) of Ext VWKL.

We now turn to the global definability and extension theorems. If $p \in S(A)$ and $\varphi(x, \bar{y})$ is an L-formula, then we say that $(\alpha(x, \bar{z}), \beta(\bar{y}, \bar{z}))$ is a quasi φ-definition for p if $\alpha(x, \bar{b}) \in p$ for some $\bar{b} \in A$ and for any such $\bar{b}$, $\beta(\bar{y}, \bar{b})$ is a φ-definition for p (i.e., for all $\bar{a} \in A$, $\varphi(x, \bar{a}) \in p$ iff $\vDash \beta[\bar{a}, \bar{b}]$). The sharp version of the local definability theorem which is provable in RCA_0^*, says that for a stable T, if $p \in S(A)$, then p has a quasi φ-definition for all φ. A function d is called a (quasi) defining scheme for $p \in S(A)$ if for all $\varphi(x, \bar{y})$, $d(\varphi)$ is a (quasi) φ-definition for p. Consider the following:

GDT (*Global Definability Theorem*). "If T is stable, $p \in S(A)$, then p has a defining scheme".

GQDT (*Global Quasi Definability Theorem*). "If T is stable, $p \in S(A)$, then p has a quasi defining scheme".

ET (*Extension theorem*). "If T is stable, $p \in S(A)$ and $A \subset B$, then p has a nonforking extension $q \in S(B)$".

In the countable case these theorems were equivalent. In general, the situation is as follows:

Proposition 4.3. (a) GDT *is provable in* $\mathrm{RCA}_0^* + \Sigma_\infty^0\text{-CA} + \mathrm{AC}$.
(b) GQDT *is provable in* $\mathrm{RCA}_0^* + \Sigma_\infty^0\text{-CA}$.
(c) ET *is provable in* ACA_0^*.

Proof. As in [5, Theorems 3.6 and 4.7] (for the proof of (b) we should keep in mind that for any φ and p, p has a quasi φ-definition (α, β) such that all extralogical symbols occurring in α and β occur in φ as well; this fact is provable in RCA_0^*).

Proposition 4.4 (RCA_0^*). *Each of* GDT, GQDT *and* ET *implies* Σ_∞^0-CA.

Proof. Let F be a function with domain I. We want to prove that the range of F exists. The argument in [5, Theorems 3.7 and 4.8] relied on countability in an essential way, so we have to come up with a new proof.

We define a theory T in the language L containing a binary predicate E_u for each $u \in \mathrm{Lt}^*$ and a unary predicate P_i for each $i \in I$. The axioms of T insure that:

(1) Each E_u is an equivalence relation with infinitely many classes, all infinite.

(2) The relations E_u are independent in the sense that whenever $u_1, \ldots, u_k$ are distinct and $Y_1, \ldots, Y_k$ are equivalence classes of $E_{u_1}, \ldots, E_{u_k}$, then the intersection $Y_1 \cap \cdots \cap Y_k$ is infinite.

(3) If $F(i) = u$, then P_i is an equivalence class of E_u.

(4) If $F(i) = F(j)$, then $P_i = P_j$.

It is easy to verify that T has quantifier elimination and is complete.

Next, we build a model M of T. The elements of $|M|$ will be pairs (J, f) where J is a finite subsets of I, f a finite function with range $f \subset \omega - \{0, 1\}$ such that $\operatorname{dom} f \cap F(J) = \emptyset$ (where $F(J) = \{F(i) : i \in J\}$). Intuitively, $c = (J, f)$ is meant to belong to those and only those P_i for which $F(i) \in F(J)$ while f indicates to which class of E_u c belongs for $u \in \operatorname{dom} f$. Formally, for $c = (J, f)$ define a function $g_c : \mathrm{Lt}^* \to \omega$ by:

$$g_c(u) = \begin{cases} 0 & \text{if } u \in F(J), \\ f(u) & \text{if } u \in \operatorname{dom} f, \\ 1 & \text{otherwise.} \end{cases}$$

Let $c \in P_i^M$ iff $g_c(F(i)) = 0$, $M \models c = c'$ iff $g_c = g_{c'}$ (i.e., $F(J) = F(J')$ and $f = f'$) and $c\, E_u^M\, c'$ iff $g_c(u) = g_{c'}(u)$.

Keeping in mind that T has quantifier elimination, one sees that we have, indeed, constructed a model $M \models T$. Define $p \in S(M)$ as the unique type containing

$$\{P_i(x) : i \in I\} \cup \{\neg x\, E_u\, c : c \in |M| \text{ and } g_c(u) \neq 0\}.$$

p can be shown to exist, in RCA_0^*. It is easily seen that for all $c \in |M|$, $x \neq c \in p$ and $x\, E_u\, c \in p$ iff $g_c(u) = 0$. The important property of p follows:

$$u \in \operatorname{range} F \quad \text{iff} \quad x\, E_u\, c \in p \text{ for some } c \in |M|.$$

We can now conclude the proof:

To see that GDT implies Σ^0_∞-CA notice that if d is a defining scheme, then, letting $\varphi_u(x, y) = x\, E_u\, y$, we have:

$$u \in \operatorname{range} F \quad \text{iff} \quad M \models \exists y\, d(\varphi_u)(y).$$

If d is a quasi defining scheme, then we have:

$$\begin{aligned} u \in \operatorname{range} F \quad &\text{iff} \quad \exists \bar{c} \in |M|\, \exists \alpha, \beta\, [d(\varphi_u) = (\alpha, \beta) \wedge \alpha(x, \bar{c}) \in p \wedge M \models \exists y\, \beta(y, \bar{c})] \\ &\text{iff} \quad \forall \bar{c} \in |M|\, \forall \alpha, \beta\, [d(\varphi_u) = (\alpha, \beta) \wedge \alpha(x, \bar{c}) \in p \rightarrow M \models \exists y\, \beta(y, \bar{c})]; \end{aligned}$$

thus, range F exists by Δ^0_1-CA, showing that GQDT implies Σ^0_∞-CA. Finally, to show that ET implies Σ^0_∞-CA, let c be an element realizing p and $q \in S(M \cup \{c\})$ a nonforking extension of p. Then we have: $u \in \operatorname{range} F$ iff $x\, E_u\, c \in q$.

The proof is complete.

Proposition 4.5 (RCA_0^*). GDT implies AC.

Proof. Assume that $R_i = \{u : (i, u) \in R\} \neq \emptyset$ for all $i \in I$. We want to show that there is a function $F \subset R$ with $\operatorname{dom} F = I$. Let T be the complete stable theory of independent equivalence relations E_i, $i \in I$, each having infinitely many equivalence classes. T has elimination of quantifiers. Let $M \models T$ be defined as follows.

The elements of M are all finite functions f with $\operatorname{dom} f \subset I$ and such that for all $i \in \operatorname{dom} f$, $f(i) \in R_i \cup (\omega - \{0\})$ (we assume, w.l.o.g., that $R_i \cap \omega = \emptyset$ for all $i \in I$). For such f, define $f' \supset f$ with $\operatorname{dom} f' = I$ by letting $f'(i) = 0$ for all $i \notin \operatorname{dom} f$. Let $f E_i^M g$ iff $f'(i)$, $g'(i) \in R_i$ or else $f'(i) = g'(i)$. Let $p \in S(M)$ be the unique type containing $\{x\, E_i\, f : f'(i) \in R_i\} \cup \{\neg x\, E_i\, f : f'(i) \notin R_i\}$. Assuming GDT, p has a defining scheme d. If $\varphi_i(x, y) = x\, E_i\, y$, then $d(\varphi_i)$ must contain at least one f with $f(i) \in R_i$ (this is easily seen if we keep in mind that, due to the quantifier elimination, $d(\varphi_i)$ is equivalent to a quantifier free formula). Define $F(i) = f(i)$ where f is the leftmost such f occurring in $d(\varphi_i)$. F is the desired function.

Summing up, we have the following.

Theorem 4.6. (a) (RCA_0^*) GDT *is equivalent to* Σ^0_∞-CA + AC.

(b) (RCA_0^*) GQDT *is equivalent to* Σ^0_∞-CA.

(c) (VWKL_0^*) ET *is equivalent to* ACA_0^*.

5. Open problems

First problem. Can we prove in RCA_0^*, that ET implies Ext VWKL? In other words, is ET equivalent to ACA_0^* over RCA_0^*? A positive answer to this question amounts to constructing in RCA_0^*, for any $\mathscr{F}$ satisfying (a) and (b), a stable theory T, a model $M \models T$ and a type $p \in S(\emptyset)$ such that any nonforking extension of p to $S(M)$ would pickup a 'branch' F through $\mathscr{F}$.

Second problem. In RCA_0 we could prove the following weak extension theorem:

WET: "If T is stable, $p \in S(\emptyset)$, then there is an $M \models T$ and a $q \in S(M)$ which is a nonforking extension of p".

(Cf. [5, Theorem 4.5].) It seems that WET is not provable in VWKL_0^*; the reason is as follows.

Let VWKL^+ be VWKL stated for a $\mathscr{T}$ which is a Σ_0^1-definable class rather than a set. VWKL^+ is provable in RCA_0. The extension Ext VWKL^+ which is Ext VWKL formulated for any Σ_0^1-definable class $\mathscr{F}$ is certainly not provable in RCA_0^*. We *conjecture* that Ext VWKL^+ is not provable in VWKL_0^*. Using the method of proof of 4.5 in [5] we can show that WET is provable in RCA_0^* + Ext VWKL^+. Our second problem is the following:

Is WET equivalent to Ext VWKL^+ over RCA_0^*?

A positive answer amounts to constructing, for any Σ_1^0-class $\mathscr{F}$ satisfying (a) and (b), a stable theory T and a type $p \in S(\emptyset)$ such that for any $M \models T$ and $q \in S(M)$, if q is a nonforking extension of p, then it picks up a 'branch' F through $\mathscr{F}$.

Third problem. Our last problem concerns the relationship between definability of types and stability. Generalizing 3.3 of [5], we can show:

Proposition 5.1 (WKL_0^*). *If for all $M \models T$ and $p \in S(M)$, p is definable, then T is stable.*

Sketch of the *Proof.* We indicate how to handle the part of the argument in [5] which used countability. The method is routine, but somewhat tedious.

Assume that $\varphi(\bar{x}, \bar{y})$ is an unstable formula with $\bar{x}, \bar{y}$ of lengths k, m. We want to conclude that there is an $M \models T$ and a $p \in S_k(m)$ with p not definable. We start by defining a set of *new* constants C as follows ($a, b_0, \ldots, b_{m-1}$ being fixed distinct words):

$$C_0 = \emptyset,$$
$$C_{n+1} = C_n \cup \{\langle a, \exists x\, \psi(x)\rangle : \exists x\, \psi(x) \text{ a sentence of } L(C_n)\}$$
$$\cup \{\langle b_i, \theta(\bar{y})\rangle : \theta(\bar{y}) \text{ an } L(C_n) \text{ formula}, i < m\},$$
$$C = \bigcup \{C_n : n < \omega\}.$$

The intended meaning of these constants will be made clear in a moment. Letting $\langle \bar{b}, \theta \rangle$ denote the sequence $\langle \langle b_0, \theta(\bar{y}) \rangle, \ldots, \langle b_{m-1}, \theta(\bar{y}) \rangle \rangle$, consider the following set of sentences: $\Gamma = T \cup \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{\exists x\, \psi(x) \to \psi(\langle a, \exists x\, \psi \rangle) : \exists x\, \psi \text{ an } L(C)\text{-sentence}\} \quad \text{and}$$

$$\Gamma_2 = \left\{\exists \bar{x} \bigwedge_{j<l} (\varphi(\bar{x}, \langle \bar{b}, \theta_j \rangle) \not\leftrightarrow \theta_j(\langle \bar{b}, \theta_j \rangle) : l \in \omega_j,\ \theta_j(\bar{y}) \text{ an } L(C)\text{-formula for } j < l\right\}.$$

We *claim* that Γ is finitely consistent and we postpone for the moment the proof. Using compactness, hence WKL_0^*, we conclude that there is a model $N \models \Gamma$. The truth of Γ_1 insures that if M is the set of elements of N denoted by C-constants, then $M \prec N$, hence $M \models T$ and $M \models \Gamma_2$; this second fact insures that the set of formulas

$$\{\varphi(\bar{x}, \langle \bar{b}, \theta \rangle) \not\leftrightarrow \theta(\langle \bar{b}, \theta \rangle) : \theta(\bar{y}) \text{ an } L(C)\text{-formula}\}$$

is finitely satisfiable in M, hence, using compactness again, it can be extended to a type $p \in S_k(M)$. p has no φ-definition (as for every $\theta(\bar{y})$, $\langle \bar{b}, \theta \rangle$ witnesses that θ is not a φ-definition of p). From this point on, one proceeds like in [5].

We still have to prove the claim that Γ is finitely consistent. Notice that the set of constants C is primitive recursive and so is the partial order $\leqslant$ over it defined by $c \leqslant c'$ iff $c = c'$ or c occurs in c' (in the obvious sense). Let $\Gamma' \subset \Gamma$ be finite and let $C' = C_0' \cup C_1' \cup \cdots \cup C_{n-1}'$ be the set of constants in Γ' such that each C_i' has either the form $\{\langle a, \exists x\, \psi \rangle\}$ or $\{\langle b_0, \theta \rangle, \ldots, \langle b_{m-1}, \theta \rangle\}$. We may further assume that C' is closed downwards with respect to $\leqslant$ and that $d' \leqslant d''$ whenever $d' \in C_i$, $d'' \in C_j$ and $i < j$. Now use the instability of φ and take a model $M \models T$ and a φ-tree $\{\varphi(\bar{x}, \bar{d}_s) : s \in {}^{<n}2\}$ with $\bar{d}_s \in M$. One can show by Σ_1^0-induction on $i \leqslant n$ that there is a function $f_i : C_0' \cup \cdots \cup C_{i-1}' \to M$ that provides an interpretation under which $\Gamma' \cap L(C_0' \cup \cdots \cup C_{i-1}')$ is made true. To do this, we use a stronger induction assumption, which implies, among others, that f_i maps the C_j''s that are of the form $\langle \bar{b}, \theta \rangle$ to sequences $\bar{d}_s$ which are parameters of an initial segment of a branch in our φ-tree. We skip any further details.

The third problem we leave open is the following:

Is the statement of 5.1 equivalent to WKL_0^*?

A positive answer amount to producing, for any $\mathcal{F}$ satisfying conditions (a) of Ext VWKL and (c) of Ext WKL, an unstable complete theory T such that for all $M \models T$ and nondefinable $p \in S(M)$, M and p pick up a 'branch' F of $\mathcal{F}$.

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