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Extraction of nth roots of 2×2 matrices

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Abstract

This paper is concerned with the determination of algebraic formulae giving all the solutions of the matrix equation $X^n = A$ where n is a positive integer greater than 2 and A is a 2×2 matrix with real or complex elements. If A is a 2×2 scalar matrix, the equation $X^n = A$ has infinitely many solutions and we obtain explicit formulae giving all the solutions. If A is a non-scalar 2×2 matrix, the equation $X^n = A$ has a finite number of solutions and we give a formula expressing all solutions in terms of A and the roots of a suitably defined nth degree polynomial in a single variable. This leads to very simple formulae for all the solutions when A is either a singular matrix or a non-singular matrix with two coincident eigenvalues. Similarly when $n = 3$ or 4, we get explicit algebraic formulae for all the solutions. We also determine the precise number of solutions in various cases.

AMS classification: 15A24

Keywords: nth roots of 2×2 matrices; Matrix equation

1. Introduction

Let A be a square matrix whose elements are either real or complex numbers. Any matrix X such that $X^n = A$ is called an nth root of A and the problem of determining all the nth roots X of a given matrix has been dealt with by many mathematicians ([2,3], [4, pp. 120–122], [5], [6, pp. 231–239], [7], [8, pp. 94–97], [9]). However, explicit formulae giving X in terms of A and its elements are not generally known.

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Cayley [1] and Sullivan [10] have obtained algebraic formulae giving the square roots of 2×2 matrices and Damphousse [2] has given formulae expressing the nth roots of non-singular 2×2 matrices in terms of transcendental functions.

This paper is concerned with the determination of algebraic formulae giving the nth roots of 2×2 matrices. As algebraic formulae have already been obtained when $n = 2$, we obtain formulae, not hitherto obtained, whenever $n \geqslant 3$. We first obtain explicit algebraic formulae giving the infinitely many nth roots of scalar 2×2 matrices. Next, we show that a non-scalar 2×2 matrix A has only finitely many nth roots all of which are given by

$$X = \{f_n(t_0)\}^{-1/n}(A + t_0 I),$$

where $f_n(t)$ is a suitably defined polynomial and t_0 is a root of an nth degree polynomial equation in the single variable t such that $f_n(t_0) \neq 0$. Thus the problem of finding all nth roots of a non-scalar matrix is reduced to the simple problem of determining all roots of a polynomial equation. When A is either a non-scalar non-singular matrix with two coincident eigenvalues or a non-scalar singular matrix, we determine all possible values of t_0 and thus get explicit formulae for all the nth roots of A. Similarly when $n = 3$ or 4, the polynomial equation that determines t_0 is solvable by radicals and we obtain explicit algebraic formulae for all the cube roots and fourth roots of a given non-scalar matrix. We give a couple of numerical examples illustrating the application of the formulae obtained.

We also determine the number of distinct nth roots of non-scalar 2×2 matrices. We show that a non-scalar non-singular matrix has precisely n^2 distinct nth roots if the given matrix has two distinct eigenvalues and precisely n distinct nth roots if it has two coincident eigenvalues. Further, a non-scalar singular matrix has precisely n distinct nth roots if its trace is non-zero and no solutions if its trace is zero. These results are different from the conclusion drawn by Damphousse [2, p. 400] that every non-scalar non-singular 2×2 matrix has only n distinct nth roots. In fact, we have given a numerical example explicitly giving 16 distinct fourth roots of a non-scalar non-singular 2×2 matrix.

In Section 2 we prove some preliminary results, Section 3 deals with the roots of scalar matrices while Section 4 deals with the roots of non-scalar matrices. Throughout the paper I denotes the 2×2 identity matrix while $\mathrm{tr}(A)$ and $\det(A)$ denote the trace and determinant respectively of a matrix A.

2. Preliminaries

In this section we prove two lemmas. The first lemma gives for an arbitrary 2×2 matrix A and an arbitrary integer $m \geqslant 3$, a formula expressing A^m in terms of A, I and symmetric functions of the eigenvalues of A. The second lemma gives for an arbitrary non-scalar 2×2 matrix A, an arbitrary positive integer n and a variable t, a formula expressing $(A + tI)^n$ in terms of A, I and two functions $f_n(t)$ and $g_n(t)$. We also obtain several formulae concerning the functions $f_n(t)$ and $g_n(t)$.

Lemma 2.1. *If A is any arbitrary 2×2 matrix with eigenvalues λ_1 and λ_2 (not necessarily distinct), then for any positive integer $m \geqslant 3$,*

$$A^m = \phi_{m-1}(\lambda_1, \lambda_2)A - \det(A)\phi_{m-2}(\lambda_1, \lambda_2)I, \tag{2.1}$$

where

$$\phi_m(\lambda_1, \lambda_2) = \lambda_1^m + \lambda_1^{m-1}\lambda_2 + \cdots + \lambda_2^m. \tag{2.2}$$

If A is a singular matrix, then for any positive integer $m \geqslant 2$,

$$A^m = \{\mathrm{tr}(A)\}^{m-1}A. \tag{2.3}$$

Proof. Let p be the trace and q the determinant of the matrix A so that λ_1 and λ_2 satisfy the characteristic equation of A, that is, the equation

$$\lambda^2 - p\lambda + q = 0. \tag{2.4}$$

Thus $\lambda_1 + \lambda_2 = p$ and $\lambda_1\lambda_2 = q$, and it follows from the well-known Cayley–Hamilton theorem that

$$A^2 = pA - qI. \tag{2.5}$$

By multiplying (2.5) by A, and replacing A^2 by $pA - qI$ on the right-hand side of the resulting equation, it is readily verified that (2.1) holds for $m = 3$. We assume that (2.1) holds for any arbitrary integer m, and multiplying (2.1) by A, we get

$$\begin{aligned} A^{m+1} = {} & \left(\lambda_1^{m-1} + \lambda_1^{m-2}\lambda_2 + \cdots + \lambda_2^{m-1}\right)A^2 \\ & - \det(A)\left(\lambda_1^{m-2} + \lambda_1^{m-3}\lambda_2 + \cdots + \lambda_2^{m-2}\right)A, \end{aligned} \tag{2.6}$$

so that on using the relations $\lambda_1 + \lambda_2 = p$, $\lambda_1\lambda_2 = q = \det(A)$ and (2.5), we get on simplification

$$A^{m+1} = \phi_m(\lambda_1, \lambda_2)A - \det(A)\phi_{m-1}(\lambda_1, \lambda_2)I, \tag{2.7}$$

and the result follows by induction.

If A is a singular matrix, $\det(A) = 0$ and the eigenvalues of A are 0 and $\mathrm{tr}(A)$ so that (2.3) follows from (2.1), (2.2) and (2.5).

We note that $\phi_m(\lambda_1, \lambda_2)$, which is a symmetric function of the roots of equation (2.4), can be expressed in terms of p and q for all positive integers m. Table 1 gives the values of $\phi_m(\lambda_1, \lambda_2)$ in terms of p and q for $m = 1, 2, \ldots, 10$. □

Lemma 2.2. *If A is any arbitrary non-scalar 2×2 matrix with eigenvalues λ_1 and λ_2 (not necessarily distinct), trace p, and determinant q, and t is an arbitrary variable, then for any positive integer n,*

$$(A + tI)^n = f_n(t)A + g_n(t)I, \tag{2.8}$$

Table 1
Values of $\phi_m(\lambda_1, \lambda_2)$

m	$\phi_m(\lambda_1, \lambda_2)$
1	p
2	$p^2 - q$
3	$p^3 - 2pq$
4	$p^4 - 3p^2q + q^2$
5	$p^5 - 4p^3q + 3pq^2$
6	$p^6 - 5p^4q + 6p^2q^2 - q^3$
7	$p^7 - 6p^5q + 10p^3q^2 - 4pq^3$
8	$p^8 - 7p^6q + 15p^4q^2 - 10p^2q^3 + q^4$
9	$p^9 - 8p^7q + 21p^5q^2 - 20p^3q^3 + 5pq^4$
10	$p^{10} - 9p^8q + 28p^6q^2 - 35p^4q^3 + 15p^2q^4 - q^5$

where the functions $f_n(t), g_n(t)$ *and their respective derivatives with respect to* t *satisfy the following relations*:

$$f_{n+1}(t) = (2t + p)f_n(t) - (t^2 + pt + q)f_{n-1}(t), \tag{2.9}$$

$$g_{n+1}(t) = (2t + p)g_n(t) - (t^2 + pt + q)g_{n-1}(t), \tag{2.10}$$

$$qf_{n-1}(t) = -g_n(t) + tg_{n-1}(t), \tag{2.11}$$

$$g_{n-1}(t) = f_n(t) - (t + p)f_{n-1}(t), \tag{2.12}$$

$$f'_n(t) = nf_{n-1}(t), \tag{2.13}$$

$$g'_n(t) = ng_{n-1}(t), \tag{2.14}$$

$$\lambda f_n(t) + g_n(t) = (t + \lambda)^n, \tag{2.15}$$

where λ *is any eigenvalue of* A*. Further, when* $n \geqslant 3$,

$$\begin{aligned} f_n(t) &= nt^{n-1} + {}_nC_2pt^{n-2} + \sum_{m=3}^{n} {}_nC_m\phi_{m-1}(\lambda_1, \lambda_2)t^{n-m}, \\ g_n(t) &= t^n - {}_nC_2qt^{n-2} - q\sum_{m=3}^{n} {}_nC_m\phi_{m-2}(\lambda_1, \lambda_2)t^{n-m}, \end{aligned} \tag{2.16}$$

where $\phi_m(\lambda_1, \lambda_2) = \lambda_1^m + \lambda_1^{m-1}\lambda_2 + \cdots + \lambda_2^m$*. Finally if the matrix* A *has two coincident eigenvalues, then for any positive integer* n,

$$\begin{aligned} f_n(t) &= n(2t + p)^{n-1}/2^{n-1}, \\ g_n(t) &= (2t + p)^{n-1}\{2t - (n - 1)p\}/2^n. \end{aligned} \tag{2.17}$$

Proof. Using the Cayley–Hamilton theorem we get

$$A^2 = pA - qI, \tag{2.18}$$

and for $m \geqslant 3$, Lemma 2.1 gives

$$A^m = \phi_{m-1}(\lambda_1, \lambda_2)A - q\phi_{m-2}(\lambda_1, \lambda_2)I, \tag{2.19}$$

where $\phi_m(\lambda_1, \lambda_2) = \lambda_1^m + \lambda_1^{m-1}\lambda_2 + \cdots + \lambda_2^m$. Now using the binomial expansion of $(A + tI)^n$, and substituting the values of various powers of A given by (2.18) and (2.19), we obtain the relation (2.8), and for $n \geqslant 3$, we obtain the relations (2.16). Further, $(A + tI)^n = (A + tI)^{n-1}(A + tI)$ so that

$$f_n(t)A + g_n(t)I = \{f_{n-1}(t)A + g_{n-1}(t)I\}(A + tI) \tag{2.20}$$

which, on using (2.18), reduces to

$$\begin{aligned} &\{f_n(t) - (t + p)f_{n-1}(t) - g_{n-1}(t)\}A \\ &\quad = \{-g_n(t) - qf_{n-1}(t) + tg_{n-1}(t)\}I. \end{aligned} \tag{2.21}$$

Since A is a non-scalar matrix, it follows that

$$\begin{aligned} f_n(t) &= (t + p)f_{n-1}(t) + g_{n-1}(t), \\ g_n(t) &= -qf_{n-1}(t) + tg_{n-1}(t), \end{aligned} \tag{2.22}$$

and the relations (2.11) and (2.12) follow readily. If E is the translation operator defined for any function $\psi_n(t)$ by $E\psi_n(t) = \psi_{n+1}(t)$, we may write the relations (2.22) as follows:

$$\begin{aligned} &(E - t - p)f_{n-1}(t) - g_{n-1}(t) = 0, \\ &qf_{n-1}(t) + (E - t)g_{n-1}(t) = 0, \end{aligned} \tag{2.23}$$

and eliminating $g_{n-1}(t)$ from the equations (2.23), we get

$$\{E^2 - (2t + p)E + t^2 + pt + q\}f_{n-1}(t) = 0, \tag{2.24}$$

and the relation (2.9) follows. The relation (2.10) is obtained similarly. Further, on differentiating both sides of (2.8) with respect to t, we get

$$n(A + tI)^{n-1} = f_n'(t)A + g_n'(t)I \tag{2.25}$$

or,

$$n\{f_{n-1}(t)A + g_{n-1}(t)I\} = f_n'(t)A + g_n'(t)I \tag{2.26}$$

and, since A is a non-scalar matrix, the relations (2.13) and (2.14) now follow readily.

If λ is any eigenvalue of A so that $\lambda^2 - p\lambda + q = 0$, we obtain from (2.22) the recurrence relation

$$\lambda f_n(t) + g_n(t) = (t + \lambda)(\lambda f_{n-1}(t) + g_{n-1}(t)). \tag{2.27}$$

When $n = 1$, we get from (2.8), $A + tI = f_1(t)A + g_1(t)I$ so that $f_1(t) = 1$ and $g_1(t) = t$, and now the recurrence relation (2.27) leads to

$$\lambda f_n(t) + g_n(t) = (t + \lambda)^{n-1}(\lambda f_1(t) + g_1(t)), \tag{2.28}$$

and hence we get the relation (2.15).

Finally if the matrix A has two coincident eigenvalues, we must have $q = p^2/4$, and the recurrence relations (2.9) and (2.10) may be written as

$$\begin{aligned} &\{E - (t + p/2)\}^2 f_{n-1}(t) = 0, \\ &\{E - (t + p/2)\}^2 g_{n-1}(t) = 0, \end{aligned} \tag{2.29}$$

and, using the standard methods of solving recurrence relations, we readily obtain (2.17). □

3. Roots of scalar matrices

Theorem 3.1. *All solutions of the matrix equation*

$$X^n = kI, \tag{3.1}$$

where k is an arbitrary non-zero real or complex number, are given by

$$X = k^{1/n} I, \tag{3.2}$$

and

$$X = \begin{bmatrix} \{k^{1/n}(1+\rho) + 2u\}/2 & v\{k^{1/n}(1-\rho) + 2u\}/2 \\ \{k^{1/n}(1-\rho) - 2u\}/(2v) & \{k^{1/n}(1+\rho) - 2u\}/2 \end{bmatrix} \tag{3.3}$$

where ρ is any nth root of unity other than 1, *and u, v are arbitrary parameters such that $v \neq 0$. Further, all solutions of the matrix equation*

$$X^n = O, \tag{3.4}$$

apart from the trivial solution $X = O$, are given by

$$X = \begin{bmatrix} a & b \\ -a^2/b & -a \end{bmatrix} \tag{3.5}$$

where a and b are arbitrary parameters such that $b \neq 0$.

Proof. If μ_1 and μ_2 are the eigenvalues of any solution X of Eq. (3.1), from Lemma 2.1 we get

$$X^n = \phi_{n-1}(\mu_1, \mu_2)X - \det(X)\phi_{n-2}(\mu_1, \mu_2)I \tag{3.6}$$

and using (3.1) we get

$$\phi_{n-1}(\mu_1, \mu_2)X - \det(X)\phi_{n-2}(\mu_1, \mu_2)I = kI. \tag{3.7}$$

If X is a scalar matrix, $X = sI$ so that (3.1) gives $s^n I = kI$. If $k \neq 0$, we must have $s^n = k$ and we thus get the solutions (3.2) of (3.1), while if $k = 0$ we get the trivial solution of (3.4). If X is not a scalar matrix, it follows from (3.7) that

$$\phi_{n-1}(\mu_1, \mu_2) = \mu_1^{n-1} + \mu_1^{n-2}\mu_2 + \cdots + \mu_2^{n-1} = 0, \tag{3.8}$$

so that $\mu_2 = \mu_1\rho$ where ρ is any nth root of unity other than 1, $\det(X) = \mu_1\mu_2 = \mu_1^2\rho$ and (3.7) reduces to $\mu_1^n I = kI$ so that $\mu_1^n = k$. There are now two possibilities:

(i) If $k \neq 0$, we get $\mu_1 = k^{1/n}$ and hence $\mu_2 = k^{1/n}\rho$. It follows from (3.7) that any matrix X with eigenvalues $\mu_1 = k^{1/n}$ and $\mu_2 = k^{1/n}\rho$ is a solution of (3.1). If we now write

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.9}$$

we get

$$\begin{aligned} a + d &= \mu_1 + \mu_2 = k^{1/n}(1+\rho), \\ ad - bc &= \mu_1\mu_2 = k^{2/n}\rho. \end{aligned} \tag{3.10}$$

Eqs. (3.10) are easily solved for $a,\ b,\ c,\ d$ to get the solution

$$\begin{aligned} a &= \{k^{1/n}(1+\rho) + 2u\}/2, \\ b &= v\{k^{1/n}(1-\rho) + 2u\}/2, \\ c &= \{k^{1/n}(1-\rho) - 2u\}/(2v), \\ d &= \{k^{1/n}(1+\rho) - 2u\}/2, \end{aligned} \tag{3.11}$$

where u and v are arbitrary parameters such that $v \neq 0$ and we thus get the solutions (3.3) of Eq. (3.1).

(ii) If $k = 0$, we get $\mu_1 = 0$ and hence also $\mu_2 = 0$. Thus both eigenvalues of X are zero and if we take X as in (3.9), we get $a + d = 0$ and $ad - bc = 0$ and these two equations are readily solved for c and d to obtain the solutions (3.5) of Eq. (3.4). □

4. Roots of non-scalar matrices

Theorem 4.1. *If A is a non-scalar matrix with trace p, determinant q and eigenvalues λ_1 and λ_2 (not necessarily distinct), and the functions $\phi_m(\lambda_1, \lambda_2),\ f_n(t),\ g_n(t)$ are defined by* (2.2) *and* (2.16), *all solutions of the matrix equation*

$$X^n = A, \quad n \geqslant 3, \tag{4.1}$$

are given by

$$X = \{f_n(t_0)\}^{-1/n}(A + t_0 I), \tag{4.2}$$

where t_0 is any root of the polynomial equation $g_n(t) = 0$ such that $f_n(t_0) \neq 0$. Further, if the matrix A is non-singular and has two distinct eigenvalues, Eq. (4.1) *has n^2 distinct solutions while if A has two coincident eigenvalues, it has n distinct solutions which are given by*

$$X = n^{-1}(2p^{n-1})^{-1/n}\{2A + (n-1)pI\}. \tag{4.3}$$

Finally, if the matrix A is singular and $p = 0$, *Eq.* (4.1) *has no solutions while if* $p \neq 0$ *it has n distinct solutions which are given by*

$$X = p^{-(n-1)/n} A. \tag{4.4}$$

Proof. Let X be an nth root of the matrix A and let μ_1 and μ_2 be the eigenvalues of X. Using Lemma 2.1 we get

$$X^n = \phi_{n-1}(\mu_1,\ \mu_2)X - \det(X)\phi_{n-2}(\mu_1,\ \mu_2)I \tag{4.5}$$

and hence it follows from (4.1) that

$$\phi_{n-1}(\mu_1,\ \mu_2)X - \det(X)\phi_{n-2}(\mu_1,\ \mu_2)I = A. \tag{4.6}$$

Since A is a non-scalar matrix, it follows from (4.6) that $\phi_{n-1}(\mu_1,\ \mu_2)$ must necessarily be non-zero, and hence there exists a non-zero number s and a number t such that

$$X = s(A + tI). \tag{4.7}$$

Substituting this value of X in (4.1) and using the relation (2.8), we get

$$s^n\{f_n(t)A + g_n(t)I\} = A. \tag{4.8}$$

Since A is non-scalar, it follows that $g_n(t) = 0$ and $s^n f_n(t) = 1$. Thus Eq. (4.1) has finitely many roots all of which are given by (4.2) where t_0 is any root of the polynomial equation $g_n(t) = 0$ such that $f_n(t_0) \neq 0$.

We will now determine the number of distinct solutions of (4.1). If A is a non-scalar non-singular matrix and $\lambda_1 \neq \lambda_2$, using (2.15) we get for any n, the two relations

$$\begin{aligned} \lambda_1 f_n(t) + g_n(t) &= (t + \lambda_1)^n, \\ \lambda_2 f_n(t) + g_n(t) &= (t + \lambda_2)^n, \end{aligned} \tag{4.9}$$

and it now follows that $f_n(t)$ and $g_n(t)$ cannot have a common root. Thus if $g_n(t_0) = 0$, we will necessarily have $f_n(t_0) \neq 0$. Next we note that if $g_n(t) = 0$ has a repeated root t_0, then $g_n(t_0) = 0$ and $g'_n(t_0) = 0$, and from (2.14) we get $g_{n-1}(t_0) = 0$ so that it follows from (2.11) that $f_{n-1}(t_0) = 0$ and hence from (2.12) we get $f_n(t_0) = 0$ which is a contradiction. Thus $g_n(t) = 0$ cannot have a repeated root and since $g_n(t)$ is a polynomial of degree n in t, the equation $g_n(t) = 0$ has n distinct roots. Moreover, for each such root t_0, $\{f_n(t_0)\}^{-1/n}$ takes n distinct values, and thus each root t_0 of $g_n(t) = 0$ leads to n distinct solutions of (4.1). Any two solutions of Eq. (4.1) are of the type $s_1(A + t_1I)$ and $s_2(A + t_2I)$ and they can be equal if and only if $s_1 = s_2$ and $t_1 = t_2$, but these conditions do not hold. Thus a non-scalar non-singular matrix A with two distinct eigenvalues has exactly n^2 distinct nth roots.

If A is a non-scalar non-singular matrix with two coincident eigenvalues, the functions $f_n(t)$ and $g_n(t)$ are given by (2.17). Thus there is only one admissible root of the equation $g_n(t) = 0$, namely $t_0 = (n-1)p/2$, for which $f_n(t_0) \neq 0$ and with this value of t_0, (4.2) yields exactly n distinct nth roots of A which are given by (4.3).

If A is a non-scalar singular matrix, $q = 0$ and from (2.16) we get $g_n(t) = t^n$ so that $t_0 = 0$. Since $q = 0$, the eigenvalues of A are 0 and p, and hence it follows from (2.16) that $f_n(t_0) = p^{n-1}$. If $p = 0$ then $f_n(t_0) = 0$ and hence we get no solutions while if $p \neq 0$ then $f_n(t_0) \neq 0$ and substituting $t_0 = 0$ and $f_n(t_0) = p^{n-1}$ in (4.2), we get n distinct solutions which are given by (4.4). □

Corollary 4.2. *If a non-scalar matrix A has distinct eigenvalues and ω denotes a primitive cube root of unity, the* 9 *cube roots of A are given by*

$$X = (3t_0^2 + 3pt_0 + p^2 - q)^{-1/3}(A + t_0 I), \tag{4.10}$$

where t_0 takes any of the three values

$$\{(p + (p^2 - 4q)^{1/2})q/2\}^{1/3}\omega^k + \{(p - (p^2 - 4q)^{1/2})q/2\}^{1/3}\omega^{2k}, \tag{4.11}$$

where $k = 0, 1, 2,$ *while the* 16 *fourth roots of A are given by*

$$X = \left\{4t_0^3 + 6pt_0^2 + 4(p^2 - q)t_0 + p^3 - 2pq\right\}^{-1/4}(A + t_0 I), \tag{4.12}$$

where t_0 takes any of the four values

$$\begin{array}{ll} -q^{1/2} + (2q - pq^{1/2})^{1/2}, & -q^{1/2} - (2q - pq^{1/2})^{1/2}, \\ q^{1/2} + (2q + pq^{1/2})^{1/2}, & q^{1/2} - (2q + pq^{1/2})^{1/2}. \end{array} \tag{4.13}$$

Proof. When $n = 3$ or 4, the equation $g_n(t) = 0$ is a cubic or a quartic equation which is solvable by radicals using standard methods. A direct application of solution (4.2) of Theorem 4.1 yields the solutions of the equations $X^3 = A$ and $X^4 = A$ as stated above.

Finally we give a couple of numerical examples to illustrate the use of the formulae obtained above.

The non-singular matrix $\begin{bmatrix} 25 & 7 \\ -7 & 39 \end{bmatrix}$ has two coincident eigenvalues and, using (4.3), its only real fifth root is readily found to be $\begin{bmatrix} 153/80 & 7/80 \\ -7/80 & 167/80 \end{bmatrix}$.

The 16 fourth roots of the matrix $\begin{bmatrix} -179 & 390 \\ -130 & 276 \end{bmatrix}$ are found using (4.12) to be the following matrices:

$$\pm\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}, \quad \pm\begin{bmatrix} -i & 6i \\ -2i & 6i \end{bmatrix}, \quad \pm\begin{bmatrix} -17 & 30 \\ -10 & 18 \end{bmatrix}, \quad \pm\begin{bmatrix} -17i & 30i \\ -10i & 18i \end{bmatrix},$$

$$\pm\begin{bmatrix} -9+8i & 18-12i \\ -6+4i & 12-6i \end{bmatrix}, \quad \pm\begin{bmatrix} 8+9i & -12-18i \\ 4+6i & -6-12i \end{bmatrix},$$

$$\pm\begin{bmatrix} 9+8i & -18-12i \\ 6+4i & -12-6i \end{bmatrix}, \quad \pm\begin{bmatrix} 8-9i & -12+18i \\ 4-6i & -6+12i \end{bmatrix}. \qquad \square$$

References

[1] A. Cayley, A memoir on the theory of matrices, Philos. Trans. Roy. Soc. London 148 (1858) 17–37.
[2] P. Damphousse, The arithmetic of powers and roots in $Gl_2(C)$ and $Sl_2(C)$, Fibonacci Quart. 27 (1989) 386–401.
[3] E.D. Denman, Roots of real matrices, Linear Algebra Appl. 36 (1981) 133–139.
[4] L.E. Dickson, Modern Algebraic Theories, B.H. Sanborn, Chicago, 1926.
[5] J.-C. Evard, F. Uhlig, On the matrix equation $f(X) = A$, Linear Algebra Appl. 162–164 (1992) 447–519.
[6] F.R. Gantmacher, The Theory of Matrices, Chelsea, New York, 1960.
[7] C.K. Koc, M. Inceoglu, A parallel algorithm for principal nth roots of matrices, Automatica 33 (1997) 1735–1738.
[8] C.C. MacDuffee, Theory of Matrices, Chelsea, New York, 1946.
[9] D.E. Otero, Extraction of mth roots in matrix rings over fields, Linear Algebra Appl. 128 (1990) 1–26.
[10] D. Sullivan, The square roots of 2 X 2 matrices, Math. Mag. 66 (1993) 314–316.