A minimax inequality and its applications
to ordinary differential equations

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Abstract

The aim of this paper is to investigate the minimax inequality which plays a fundamental role in the critical points theorem of B. Ricceri below. Equivalent formulations are shown, and characterization is proved in particular for a special class of functionals. As an application, a multiplicity result for an ordinary Dirichlet problem is emphasized. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

Given two real Gâteaux differentiable functionals \( \Phi \) and \( J \) on a real Banach space \( X \), the minimax inequality

\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda (\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (\rho - J(x))), \quad \rho \in \mathbb{R},
\]

plays a fundamental role for establishing the existence of at least three critical points for the functional \( \Phi(x) - \lambda J(x) \), as the theorem of B. Ricceri below
ensures. In this paper some conditions that imply the minimax inequality (1.1) are pointed out (Proposition 2.1 and Proposition 2.2) and equivalent formulations are proved (Theorem 2.3). Moreover, the main result of this paper (Theorem 3.1) establishes an equivalent statement of minimax inequality for a special class of functionals, while its consequences (Theorem 3.2 and Theorem 3.4) guarantee some conditions so that (1.1) holds; some special cases of Theorem 3.2 which are simpler to apply are pointed out (Proposition 3.6) and an easy assumption such that (1.1) is not true (Proposition 3.5) is observed.

We now recall the three critical points theorem of B. Ricceri.

**Theorem 1.1** [1]. Let $X$ be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda \Psi(x)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, +\infty[ \to \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).$$

Then, there exists an open interval $\Lambda \subseteq ]0, +\infty[$ and a positive real number $q$ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(x) + \lambda \Psi'(x) = 0$$

has at least three solutions in $X$ whose norms are less than $q$.

**Remark 1.1.** An equivalent statement of (1.2) is the following assertion: there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + \rho \lambda) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + \rho \lambda).$$

(1.3)

as it has been proved in [2].

Theorem 2.3 establishes equivalent statements of (1.3) that are even simpler to apply in the field of differential equations than the minimax inequality (1.2); so in Theorem 1.1 we can substitute (1.2) for one of these equivalent statements (see Remark 2.2).

The main result of this paper is Theorem 3.1 that provides an intrinsic characterization of minimax inequality (1.1) for definite functions $\Phi$ and $J$; it is worth pointing out that in its proof the fact that $W_{1,2}^0([0, 1])$ is continuously imbedded in $C^0([0, 1])$ is not used (Remark 3.1).
Finally, we apply Theorem 1.1 to ordinary differential equations, by using an immediate consequence of Theorem 3.1. To be precise, in Section 4, we consider a two point boundary value problem for second order differential equation of the form

\[
\begin{align*}
    u'' + \lambda f(u) &= 0, \\
    u(0) = u(1) &= 0,
\end{align*}
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \lambda \) is a real parameter, and we establish some conditions on \( f \) so that problem (1.4) admits at least three classical solutions.

In recent years, many authors have studied multiple solutions from several points of view and with different approaches (see, for example, \([3–7]\)):

- In their interesting paper \([3]\), R.I. Avery and J. Henderson studied problem (1.4) (independent of \( \lambda \), in that case) by using a multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on \( f \), while in \([4]\), three solutions, under completely different assumptions on \( f \) and by using variational methods, were obtained.

Here, thanks to Theorem 3.2, the recent result of \([4]\) on multiple solutions is reviewed in a more general setting (Theorem 4.1). Moreover, some other special cases of Theorem 4.1 are presented (Remark 4.2), with the following as an example.

**Theorem 1.2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function and let \( g(t) = \int_0^t f(\xi) \, d\xi \) for every \( t \in \mathbb{R} \). Assume that there exist three positive constants \( T, a \) and \( v \), with \( v < 2 \), such that

\[
\begin{align*}
    (i) \quad & \max_{|t| \leq T/2} g(t) < \frac{1}{4} \int_0^T g(t) \, dt, \\
    (ii) \quad & g(t) \leq a \left( 1 + |t|^v \right) \text{ for all } t \in \mathbb{R}.
\end{align*}
\]

Then, there exists an open interval \( \Lambda \subseteq [0, +\infty[ \) and a positive real number \( q \) such that, for each \( \lambda \in \Lambda \), problem (1.4) admits at least three solutions belonging to \( C^2([0, 1]) \) whose norms in \( C^2([0, 1]) \) are less than \( q \).

## 2. Minimax inequality

Given two real functions \( \Phi \) and \( J \) on a non-empty set \( X \), the aim of this section is first to point out some statements that imply the following inequality

\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda (\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (\rho - J(x)))
\] (2.1)
for some \( \rho \in \mathbb{R} \), and next, under further conditions on \( X, \Phi \) and \( J \), to prove that these statements are also necessary so that \( (2.1) \) holds.

The following propositions ensure sufficient conditions for \( (2.1) \).

**Proposition 2.1.** Let \( X \) be a non-empty set, and \( \Phi, J \) two real functions on \( X \). Assume that there are \( \rho \in \mathbb{R}, x_0, x_1 \in X \) such that:

(i) \( J(x_0) < \rho < J(x_1) \),

(ii) \( \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x) > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)} \).

Then, one has

\[
\sup_{\lambda \geq 0} \inf_{x \in X} \left( \Phi(x) + \lambda (\rho - J(x)) \right) < \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda (\rho - J(x)) \right).
\]

**Proof.** From (ii), taking into account that (i) holds, one has

\[
\Phi(x_0) - \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x) > \frac{\Phi(x_1) - \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x)}{J(x_1) - \rho}.
\]

Now, let \( \lambda \in \mathbb{R} \). Taking into account the previous inequality, one has either

\[
\lambda > \frac{\Phi(x_0) - \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x)}{J(x_0) - \rho},
\]

or

\[
\lambda < \frac{\Phi(x_1) - \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x)}{J(x_1) - \rho}.
\]

Therefore, thanks to (i), we obtain

\[
\inf_{x \in X} \left( \Phi(x) + \lambda (\rho - J(x)) \right) < \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x).
\]

Since the function \( \lambda \rightarrow \inf_{x \in X} (\Phi(x) + \lambda (\rho - J(x))) \) is upper semicontinuous in \([0, +\infty[\) and tends to \(-\infty\) as \( \lambda \rightarrow +\infty \), it attains its supremum in \([0, +\infty[\). Hence, one has

\[
\sup_{\lambda \geq 0} \inf_{x \in X} \left( \Phi(x) + \lambda (\rho - J(x)) \right) < \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x).
\]

Therefore, we have the conclusion thanks to the following equality

\[
\inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda (\rho - J(x)) \right) = \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x). \quad \square
\]

**Proposition 2.2.** Let \( X \) be a non-empty set, and \( \Phi, J \) two real functions on \( X \). Assume that there are \( r \in \mathbb{R}, x_0, x_1 \in X \) such that:
(j) \( \Phi(x_0) < r < \Phi(x_1) \),
(jj) \( \sup_{x \in \Phi^{-1}(J^{[-\infty,r]})} J(x) < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)} \).

Then, for each \( \rho \) satisfying
\[
\sup_{x \in \Phi^{-1}(J^{[-\infty,r]})} J(x) < \rho < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)}
\]
one has
\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))).
\]

**Proof.** From \( \sup_{x \in \Phi^{-1}(J^{[-\infty,r]})} J(x) < \rho \) we obtain
\[
r \leq \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x)
\]
and \( J(x_0) < \rho \).

Moreover, from
\[
\rho < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)},
\]
taking into account that \( J(x_0) < \rho \), one has \( \rho < J(x_1) \). Hence \( J(x_0) < \rho < J(x_1) \).

Now, again from
\[
\rho < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)}
\]
we obtain
\[
\frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)} < r.
\]

Hence,
\[
\frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)} < \inf_{x \in J^{-1}([\rho, +\infty[)} \Phi(x).
\]

Now, we can apply Proposition 1 and we obtain
\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))),
\]
and the proof is complete. \( \square \)

**Remark 2.1.** If \( \Phi(x_0) = J(x_0) = 0 \) the assumptions of previous propositions take simpler forms. To be precise:

(i) \( 0 < \rho < J(x_1) \),
(ii) \[ \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x) > \frac{\Phi(x_1)}{J(x_1)} \]

and

(j) \[ 0 < r < \Phi(x_1), \]

(jj) \[ \sup_{x \in \Phi^{-1}([-\infty, r])} J(x) < \frac{J(x_1)}{\Phi(x_1)}. \]

So, we point out that Proposition 2.2 extends Proposition 3.1 of [8].

The following theorem is the main result of this section; it establishes two statements equivalent to minimax inequality (2.1).

**Theorem 2.3.** Let \( X \) be a separable and reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) a sequentially weakly lower semicontinuous functional and \( J : X \to \mathbb{R} \) a sequentially weakly upper semicontinuous functional such that

\[ \lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty \]

for all \( \lambda \geq 0. \)

Then, the following assertions are equivalent:

(a) there are \( \rho \in \mathbb{R} \), \( x_0, x_1 \in X \) such that:

(i) \( J(x_0) < \rho < J(x_1), \)

(ii) \[ \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x) > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)}; \]

(b) there are \( r \in \mathbb{R} \), \( x_0, x_1 \in X \) such that:

(j) \( \Phi(x_0) < r < \Phi(x_1), \)

(jj) \[ \sup_{x \in \Phi^{-1}([-\infty, r])} J(x) < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)}; \]

(c) there exists \( \rho \in \mathbb{R} \), such that:

\[ \sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))). \]

**Proof.** (a) \( \Rightarrow \) (b). Let \( r \in \mathbb{R} \) be such that

\[ \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x) > r > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)}. \]

From

\[ \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x) > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)}. \]
taking into account (i), we obtain that \( \Phi(x_0) < \Phi(x_1) \); hence, from previous inequalities we have \( \Phi(x_0) \leq r < \Phi(x_1) \). From \( \inf_{x \in J^{-1}(|\rho, +\infty|)} \Phi(x) > r \) we obtain that

\[
\sup_{x \in \Phi^{-1}([-\infty, r])} J(x) \leq \rho
\]

and from

\[
r > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)}
\]

we have

\[
\rho < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)};
\]

therefore

\[
\sup_{x \in \Phi^{-1}([-\infty, r])} J(x) < \frac{(\Phi(x_1) - r)J(x_0) + (r - \Phi(x_0))J(x_1)}{\Phi(x_1) - \Phi(x_0)},
\]

that is the conclusion.

(b) \(\Rightarrow\) (c). It follows from Proposition 2.2.

(c) \(\Rightarrow\) (a). Applying Theorem 1 of [2] we obtain that there exists \( \rho \in \) \(\inf_X J, \sup_X J \) such that

\[
\sup_{x \in J^{-1}([-\infty, \rho])} \frac{\Phi(x) - \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x)}{J(x)} > \inf_{x \in J^{-1}([\rho, \infty])} \frac{\Phi(x_1) - \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x)}{J(x_1)}.
\]

Hence, there exist \( x_0 \) and \( x_1 \in X \) such that \( J(x_0) \leq \rho < J(x_1) \) and

\[
\frac{\Phi(x_0) - \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x)}{J(x_0) - \rho} > \frac{\Phi(x_1) - \inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x)}{J(x_1) - \rho},
\]

that is

\[
\inf_{x \in J^{-1}([\rho, +\infty])} \Phi(x) > \frac{(J(x_1) - \rho)\Phi(x_0) + (\rho - J(x_0))\Phi(x_1)}{J(x_1) - J(x_0)}.
\]

\(\blacksquare\)

**Remark 2.2.** Taking into account Remark 1.1, the assumption (1.2) of Theorem 1.1 can be substituted for (a), (b) or (c) (with \(-J = \Psi\)) of Theorem 2.3.

To be precise, each of the following statements is equivalent to (1.2) of Theorem 1.1:

(a) there are \( \rho \in \mathbb{R}, x_0, x_1 \in X \) such that:

(i) \( \Psi(x_0) < \rho < \Psi(x_1) \),

(ii) \( \inf_{x \in \Psi^{-1}([-\infty, \rho])} \Phi(x) > \frac{(\Psi(x_1) - \rho)\Phi(x_0) + (\rho - \Psi(x_0))\Phi(x_1)}{\Psi(x_1) - \Psi(x_0)}; \)
or
(b) there are \( r \in \mathbb{R}, x_0, x_1 \in X \) such that:

\[
\begin{align*}
(j) & \quad \Phi(x_0) < r < \Phi(x_1), \\
(jj) & \quad \inf_{x \in \Phi^{-1}([-\infty,r])} \Psi(x) > \frac{(\Phi(x_1) - r)\Psi(x_0) + (r - \Phi(x_0))\Psi(x_1)}{\Phi(x_1) - \Phi(x_0)}.
\end{align*}
\]

We explicitly observe that the previous assertions are simpler to apply than the original assumption. The crucial point is to calculate one of the infimum that appear above; a way to estimate this, in a special case, is studied in the next section.

3. Minimax inequality for a special class of functionals

Throughout this section \( X \) is the Sobolev space \( W^{1,2}_0([0,1]) \) endowed with the norm \( \|x\| = (\int_0^1 [x'(t)]^2 \, dt)^{1/2} \), \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( g : \mathbb{R} \to \mathbb{R} \) is the function defined as follows

\[
g(t) = \int_0^t f(\xi) \, d\xi.
\]

We now introduce two special real functions on the Sobolev space \( X \) as follows

\[
\Phi(x) = \frac{1}{2} \|x\|^2
\]

for every \( x \in X \), and

\[
\mathcal{J}(x) = \int_0^1 g(x(t)) \, dt
\]

for every \( x \in X \).

Let \( \rho \in \mathbb{R}, x_0, x_1 \in X \) be such that \( \mathcal{J}(x_0) < \rho < \mathcal{J}(x_1) \), we put

\[
A(\rho, x_0, x_1) = \frac{(\mathcal{J}(x_1) - \rho)\mathcal{J}(x_0) + (\rho - \mathcal{J}(x_0))\mathcal{J}(x_1)}{\mathcal{J}(x_1) - \mathcal{J}(x_0)}
\]

and, taking into account that \( A(\rho, x_0, x_1) > 0 \),

\[
B(\rho, x_0, x_1) = \left( \frac{1}{2} A(\rho, x_0, x_1) \right)^{1/2},
\]

that is

\[
B(\rho, x_0, x_1) = \left( \frac{1}{2} \left( \int_0^1 g(x_1(t)) \, dt - \rho \right) \|x_0\|^2 + (\rho - \int_0^1 g(x_0(t)) \, dt) \|x_1\|^2 \right)^{1/2}.
\]
Moreover, if \( \Phi(x_0) < r < \Phi(x_1) \), we put

\[
C(r, x_0, x_1) = \frac{(\Phi(x_1) - r) \mathcal{J}(x_0) + (r - \Phi(x_0)) \mathcal{J}(x_1)}{\Phi(x_1) - \Phi(x_0)},
\]

that is

\[
C(r, x_0, x_1) = \frac{\|x_1\|^2 - 2r \int_0^1 g(x_0(t)) \, dt + (2r - \|x_0\|^2) \int_0^1 g(x_1(t)) \, dt}{\|x_1\|^2 - \|x_0\|^2}.
\]

Now, we put

\[
K_\rho = \inf \left\{ \frac{1}{2} \|x\| \in \mathbb{R}^+: \mathcal{J}(x) \geq \rho \right\},
\]

\[
D_\rho = \inf \left\{ \frac{1}{2} \|x\| \in \mathbb{R}^+: \max_{|t| \leq \frac{1}{2} \|x\|} g(t) \geq \rho \right\},
\]

\[
\varepsilon_\rho = K_\rho - D_\rho.
\]

(3.3)

Clearly, \( K_\rho \geq D_\rho \). In fact

\[
|x(t)| \leq \frac{1}{2} \|x\|
\]

for every \( t \in [0, 1] \) and for every \( x \in X \), so that

\[
\mathcal{J}(x) = \int_0^1 g(x(t)) \, dt \leq \max_{|t| \leq \frac{1}{2} \|x\|} g(t)
\]

for every \( x \in X \), namely

\[
\mathcal{J}(x) \leq \max_{|t| \leq \frac{1}{2} \|x\|} g(t)
\]

for every \( x \in X \); therefore

\[
\left\{ \frac{1}{2} \|x\| \in \mathbb{R}^+: \mathcal{J}(x) \geq \rho \right\} \subseteq \left\{ \frac{1}{2} \|x\| \in \mathbb{R}^+: \max_{|t| \leq \frac{1}{2} \|x\|} g(t) \geq \rho \right\}.
\]

Hence \( \varepsilon_\rho \geq 0 \) and, if \( \rho < \mathcal{J}(x_1) \), \( \varepsilon_\rho \leq \frac{1}{2} \|x_1\| - D_\rho \).

Moreover, we put

\[
\max g(t) = \begin{cases} 
\max_{t \in [-v, v]} g(t) & \text{if } v > 0, \\
g(0) & \text{if } v = 0, \\
-\infty & \text{if } v < 0.
\end{cases}
\]

The main result of this paper is the following theorem, which ensures a form equivalent to inequality (2.1) for the functionals \( \Phi \) and \( \mathcal{J} \).

**Theorem 3.1.** The following assertions are equivalent:
(a) there exist \( \rho \in \mathbb{R}, x_0, x_1 \in X \) such that

\[
(i) \quad \mathcal{J}(x_0) < \rho < \mathcal{J}(x_1), \\
(ii) \quad \max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon_\rho} g(t) < \rho,
\]

where \( B(\rho, x_0, x_1) \) is given by (3.1) and \( \varepsilon_\rho \) by (3.3);

(b) there exist \( \rho \in \mathbb{R} \) such that

\[
\sup_{\lambda \geq 0} \inf_{x \in X} \left( \Phi(x) + \lambda (\rho - \mathcal{J}(x)) \right) < \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda (\rho - \mathcal{J}(x)) \right).
\]

Proof. (a) \( \Rightarrow \) (b). First, we assume

\[
B(\rho, x_0, x_1) - \varepsilon_\rho \geq 0.
\]

From (ii) we obtain

\[
\inf_{\lambda \geq 0} \{ v \in \mathbb{R}^+ : \max_{|t| \leq v} g(t) \geq \rho \} = B(\rho, x_0, x_1) - \varepsilon_\rho.
\]

Moreover,

\[
\inf \left\{ v \in \mathbb{R}^+ : \max_{|t| \leq v} g(t) \geq \rho \right\} > \inf_{\lambda \geq 0} \left( \Phi(x) + \lambda (\rho - \mathcal{J}(x)) \right).
\]

in fact, arguing by contradiction, we assume that there is \( w \in \mathbb{R}^+ \) such that

\[
\max_{|t| \leq w} g(t) \geq \rho \quad \text{and} \quad w < B(\rho, x_0, x_1) - \varepsilon_\rho,
\]

so

\[
\max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon_\rho} g(t) \geq \max_{|t| \leq w} g(t) \geq \rho
\]

and this is a contradiction.

Further, taking into account that the function \( v \mapsto \max_{|t| \leq v} g(t) \) is continuous in \([0, +\infty[\), one has

\[
\inf \left\{ v \in \mathbb{R}^+ : \max_{|t| \leq v} g(t) \geq \rho \right\} > B(\rho, x_0, x_1) - \varepsilon_\rho.
\]

Therefore,

\[
B(\rho, x_0, x_1) - \varepsilon_\rho < \inf \left\{ \frac{1}{2} \| x \| \in \mathbb{R}^+ : \max_{|t| \leq \frac{1}{2} \| x \|} g(t) \geq \rho \right\};
\]

namely \( B(\rho, x_0, x_1) < K_\rho \). So, we obtain

\[
\inf \left\{ \frac{1}{2} \| x \|^2 \in \mathbb{R}^+ : \mathcal{J}(x) \geq \rho \right\} > A(\rho, x_0, x_1),
\]

namely

\[
\inf_{x \in \mathcal{J}^{-1}(0, +\infty)} \Phi(x) > \frac{(\mathcal{J}(x_1) - \rho) \Phi(x_0) + (\rho - \mathcal{J}(x_0)) \Phi(x_1)}{\mathcal{J}(x_1) - \mathcal{J}(x_0)},
\]

and, taking into account Theorem 2.3, we obtain (b).
Now, we assume $B(\rho, x_0, x_1) - \varepsilon \rho < 0$; therefore, one has $B(\rho, x_0, x_1) < K_\rho - D_\rho \leq K_\rho$. Arguing as before, we obtain the conclusion.

(b) \implies (a). Applying Theorem 2.3 we obtain that

\[ J(x_0) < \rho < J(x_1) \]

and

\[ \inf \left\{ \frac{1}{2} \| x \|^2 \in \mathbb{R}^+ : J(x) \geq \rho \right\} > A(\rho, x_0, x_1); \]

hence, $K_\rho > B(\rho, x_0, x_1)$, and so $B(\rho, x_0, x_1) - \varepsilon \rho < D_\rho$. It follows that

\[ \max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon \rho} g(t) < \rho, \]

that is the conclusion. \Box

Now, we examine some consequences of the main result. The most important is the following.

**Theorem 3.2.** Assume that there exist $\rho \in \mathbb{R}, x_0, x_1 \in X$ such that

(i) \hspace{5mm} $J(x_0) < \rho < J(x_1)$,

(ii) \hspace{5mm} $\max_{|t| \leq B(\rho, x_0, x_1)} g(t) < \rho$,

where $B(\rho, x_0, x_1)$ is given by (3.1).

Then, there exists $\rho \in \mathbb{R}$ such that

\[ \sup_{\lambda \geq 0} \inf_{x \in X} \left( \Phi(x) + \lambda \left( \rho - J(x) \right) \right) < \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda \left( \rho - J(x) \right) \right). \]

**Proof.** Taking into account that $\varepsilon \rho \geq 0$ one has

\[ \max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon \rho} g(t) \leq \max_{|t| \leq B(\rho, x_0, x_1)} g(t) \]

and the conclusion follows from Theorem 3.1. \Box

**Remark 3.1.** The proof of Theorem 3.1 makes no use of the fact that $W^{1,2}_0([0, 1])$ is continuously embedded in $C^0([0, 1])$; on the contrary, (3.4) is fundamental in the proof of Theorem 3.2.

Theorem 3.4 below is another consequence of Theorem 3.1. First, we point out the following proposition.

**Proposition 3.3.** The following assertions are equivalent:

(a) \hspace{5mm} there are $\rho \in \mathbb{R}, x_0, x_1 \in X$ such that:
we assume that

On the other hand, from (j), we have $\Phi(x_0) > 1$, and this is a contradiction.

where $B(\rho, x_0, x_1)$ is given by (3.1);

(b) there are $r \in \mathbb{R}$, $x_0, x_1 \in X$ such that:

- (j) $\Phi(x_0) < r < \Phi(x_1)$,
- (jj) $\max_{|t| \leq \sqrt{r/2}} g(t) < C(r, x_0, x_1),

where $C(r, x_0, x_1)$ is given by (3.2).

**Proof.** (a) $\Rightarrow$ (b). First we note that $\Phi(x_0) < \Phi(x_1)$. In fact, arguing by contradiction, we assume that $\Phi(x_0) \geq \Phi(x_1)$. It follows that

$$A(\rho, x_0, x_1) = \frac{(J(x_1) - \rho) \Phi(x_0) + (\rho - J(x_0)) \Phi(x_1)}{J(x_1) - J(x_0)} \geq \Phi(x_1),$$

namely $\frac{1}{2} \|x_1\| \leq B(\rho, x_0, x_1)$. Hence, taking into account (ii), one has

$$J(x_1) \leq \max_{|t| \leq \frac{1}{2} \|x_1\|} g(t) \leq \max_{|t| \leq B(\rho, x_0, x_1)} g(t) < \rho,$$

that is in contradiction to (i).

We now put $r = A(\rho, x_0, x_1)$. We obtain $\rho = C(r, x_0, x_1)$ and $B(\rho, x_0, x_1) = \sqrt{r/2}$. Therefore, from (ii) we have the conclusion.

(b) $\Rightarrow$ (a). First we note that $\Phi(x_0) < \Phi(x_1)$. In fact, arguing by contradiction, we assume that $\Phi(x_0) \geq \Phi(x_1)$. It follows that

$$C(r, x_0, x_1) = \frac{(\Phi(x_1) - r) \Phi(x_0) + (r - \Phi(x_0)) \Phi(x_1)}{\Phi(x_1) - \Phi(x_0)} \leq J(x_0);$$

hence, from (jj) we obtain

$$\max_{|t| \leq \sqrt{r/2}} g(t) < J(x_0).$$

On the other hand, from (j), we have $\frac{1}{2} \|x_0\| < \sqrt{r/2}$; therefore,

$$J(x_0) \leq \max_{|t| \leq \frac{1}{2} \|x_0\|} g(t) \leq \max_{|t| \leq \sqrt{r/2}} g(t) < J(x_0),$$

and this is a contradiction.

Now, we put

$$\rho = C(r, x_0, x_1) = \frac{(\Phi(x_1) - r) \Phi(x_0) + (r - \Phi(x_0)) \Phi(x_1)}{\Phi(x_1) - \Phi(x_0)}.$$

We have $r = A(\rho, x_0, x_1)$; hence $\sqrt{r/2} = B(\rho, x_0, x_1)$. Therefore, from (jj), one has

$$\max_{|t| \leq B(\rho, x_0, x_1)} g(t) < \rho. \quad \square$$
Theorem 3.4. Assume that there exist \( r \in \mathbb{R}, \ x_0, x_1 \in X \) such that

\begin{align*}
(\text{j}) & \quad \Phi(x_0) < r < \Phi(x_1), \\
(\text{jj}) & \quad \max_{|t| \leq \sqrt{\frac{r}{2}}} g(t) < C(r, x_0, x_1),
\end{align*}

where \( C(r, x_0, x_1) \) is given by (3.2).

Then, there exists \( \rho \in \mathbb{R} \) such that

\[
\sup_{\lambda \geq 0} \inf_{x_0 \in X} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right) < \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right).
\]

Proof. It follows from Theorem 3.2 and Proposition 3.3. \( \square \)

Remark 3.2. Applying Proposition 2.2 it is possible to verify that the conclusion of Theorem 3.4 also holds for every \( \rho \) satisfying

\[
\max_{|t| \leq \sqrt{\frac{r}{2}}} g(t) < \rho < C(r, x_0, x_1).
\]

Theorem 3.1 can also be used to establish a minimax equality. The following proposition is a further consequence in this direction.

Proposition 3.5. Assume that 0 is a global maximum for \( g \) in \( \mathbb{R} \). Then, one has

\[
\sup_{\lambda \geq 0} \inf_{x_0 \in X} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right) = \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right)
\]

for every \( \rho \in \mathbb{R} \).

Proof. Arguing by contradiction we assume that there exists \( \rho \in \mathbb{R} \) such that

\[
\sup_{\lambda \geq 0} \inf_{x_0 \in X} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right) < \inf_{x \in X} \sup_{\lambda \geq 0} \left( \Phi(x) + \lambda \left( \rho - \overline{J}(x) \right) \right).
\]

From Theorem 3.1 we obtain that there exist \( \rho \in \mathbb{R} \) and \( x_0, x_1 \in X \) such that

\begin{align*}
(\text{i}) & \quad \overline{J}(x_0) < \rho < \overline{J}(x_1), \\
(\text{ii}) & \quad \max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon_\rho} g(t) < \rho.
\end{align*}

Therefore, taking into account our assumption, one has

\[0 = \max_{|t| \leq B(\rho, x_0, x_1) - \varepsilon_\rho} g(t) < \rho < \overline{J}(x_1) \leq \max_{|t| \leq \frac{1}{2} \|x_1\|} g(t) = 0,\]

and this is a contradiction. \( \square \)

Theorem 3.2 may assume simpler forms if we a priori choose \( x_0, x_1 \in X \). As an example of this claim, we now give the following proposition.

Proposition 3.6. Assume one of the following assertions:
(a) there exist \( s \in ]0, 1[ \), \( k_0, k_1 \in \mathbb{R} - \{0\} \) such that

\[
\begin{align*}
(i) & \quad \frac{1}{k_0} \int_0^{k_0/2} g(t) \, dt \neq \frac{1}{k_1} \int_0^{k_1/2} g(t) \, dt, \\
(ii) & \quad \max_{|t| \leq (sk_0^2 + (1-s)k_1^2)^{1/2}} g(t) < \frac{2s}{k_0} \int_0^{k_0/2} g(t) \, dt + \frac{2(1-s)}{k_1} \int_0^{k_1/2} g(t) \, dt,
\end{align*}
\]

(b) there exist \( s \in ]0, 1[ \), \( k \in \mathbb{R} - \{0\} \) such that

\[
\max_{|t| \leq (s)^{1/2}|k|/2} g(t) < \frac{2s}{k} \int_0^{k/2} g(t) \, dt,
\]

(c) there exist \( s \in ]0, 1[ \), \( k_0, k_1 \in \mathbb{R} \) such that

\[
\begin{align*}
(j) & \quad k_0 \neq k_1; \int_0^{k_0} g(t) \, dt \geq 0; \int_0^{k_1} g(t) \, dt \geq 0; \\
(jj) & \quad \max_{|t| \leq (2sk_0^2 + 2(1-s)k_1^2)^{1/2}} g(t) < \frac{sg(k_0) + (1-s)g(k_1)}{2}.
\end{align*}
\]

Then, there exist \( \rho \in \mathbb{R} \), \( x_0, x_1 \in X \) such that

\[
\begin{align*}
(i) & \quad \mathcal{J}(x_0) < \rho < \mathcal{J}(x_1), \\
(ii) & \quad \max_{|t| \leq B(\rho, x_0, x_1)} g(t) < \rho,
\end{align*}
\]

where \( B(\rho, x_0, x_1) \) is given by \( (3.1) \).

**Proof.** (a) We put

\[
x(t) = \begin{cases} 
  t & \text{if } 0 \leq t \leq \frac{1}{2}, \\
  1 - t & \text{if } \frac{1}{2} < t \leq 1,
\end{cases}
\]

and \( x_0(t) = k_0x(t) \), \( x_1(t) = k_1x(t) \) for every \( t \in [0, 1] \). We obtain

\[
\begin{align*}
\mathcal{F}(x_0) &= \frac{k_0^2}{2}, \quad \mathcal{F}(x_1) = \frac{k_1^2}{2}, \\
\mathcal{J}(x_0) &= \frac{2}{k_0} \int_0^{k_0/2} g(t) \, dt, \quad \mathcal{J}(x_1) = \frac{2}{k_1} \int_0^{k_1/2} g(t) \, dt.
\end{align*}
\]
Thanks to (i), one has \( \bar{J}(x_0) \neq \bar{J}(x_1) \). Now we assume \( \bar{J}(x_0) < \bar{J}(x_1) \) (it is the same if we assume \( \bar{J}(x_1) < \bar{J}(x_0) \)) and we put

\[
\rho = s \bar{J}(x_0) + (1 - s) \bar{J}(x_1) = \frac{2s}{k_0} \int_0^{k_0/2} g(t) \, dt + \frac{2(1 - s)}{k_1} \int_0^{k_1/2} g(t) \, dt.
\]

We obtain

\[
A(\rho, x_0, x_1) = \frac{(\bar{J}(x_1) - \rho)\bar{\Phi}(x_0) + (\rho - \bar{J}(x_0))\bar{\Phi}(x_1)}{\bar{J}(x_1) - \bar{J}(x_0)}
\]

\[
= s\bar{\Phi}(x_0) + (1 - s)\bar{\Phi}(x_1) = \frac{s k_0^2 + (1 - s) k_1^2}{2},
\]

therefore

\[
B(\rho, x_0, x_1) = \frac{(s k_0^2 + (1 - s) k_1^2)^{1/2}}{2}.
\]

Hence, the conclusion.

(b) We put \( x(t) \) as in the proof of (a) and \( x_0(t) = 0, x_1(t) = k x(t) \) for every \( t \in [0, 1] \). We obtain

\[
\bar{\Phi}(x_0) = \bar{J}(x_0) = 0, \quad \bar{\Phi}(x_1) = \frac{k^2}{2}, \quad \bar{J}(x_1) = \frac{2}{k} \int_0^{k_1/2} g(t) \, dt.
\]

Thanks to our assumption, one has

\[
0 \leq \max_{|t| \leq (s)^{1/2}|k|/2} g(t) < \frac{2s}{k} \int_0^{k/2} g(t) \, dt \leq \bar{J}(x_1).
\]

Now, putting

\[
\rho = \frac{2s}{k} \int_0^{k/2} g(t) \, dt
\]

we obtain

\[
A(\rho, x_0, x_1) = \frac{\rho}{\bar{J}(x_1)} \bar{\Phi}(x_1) = s \frac{k^2}{2},
\]

therefore

\[
B(\rho, x_0, x_1) = \frac{(s)^{1/2} |k|}{2}.
\]

Hence, the conclusion.
(c) We put
\[ x(t) = \begin{cases} 
4t & \text{if } t \in [0, \frac{1}{4}], \\
1 & \text{if } t \in [\frac{1}{4}, \frac{3}{4}], \\
4(1-t) & \text{if } t \in [\frac{3}{4}, 1] 
\end{cases} \]
and \( x_0(t) = k_0 x(t), \) \( x_1(t) = k_1 x(t) \) for every \( t \in [0, 1] \). We obtain \( \Phi(x_0) = 4k_0^2, \)
\( \Phi(x_1) = 4k_1^2 \), and, since (j) holds,
\[ \mathcal{J}(x_0) \geq \frac{1}{2} g(k_0), \quad \mathcal{J}(x_1) \geq \frac{1}{2} g(k_1). \]

Now, taking into account that \( k_0 \neq k_1 \), one has \( \Phi(x_0) \neq \Phi(x_1) \). Moreover, putting \( r = s \Phi(x_0) + (1-s) \Phi(x_1) \) one has
\[ C(r, x_0, x_1) = \frac{(\Phi(x_1) - r) \mathcal{J}(x_0) + (r - \Phi(x_0)) \mathcal{J}(x_1)}{\Phi(x_1) - \Phi(x_0)} = s \mathcal{J}(x_0) + (1-s) \mathcal{J}(x_1) \geq \frac{sg(k_0) + (1-s)g(k_1)}{2} \]
and
\[ \left( \frac{r}{2} \right)^{\frac{1}{2}} = \left( 2sk_0^2 + 2(1-s)k_1^2 \right)^{\frac{1}{2}}. \]
So, from our assumptions it follows that \( \Phi(x_0) < r < \Phi(x_1) \) and
\[ \max_{|t| \leq \sqrt{r/2}} g(t) < C(r, x_0, x_1). \]

Hence, from Proposition 3.3 the conclusion. \( \square \)

**Remark 3.3.** The assumptions of Proposition 1 of [4] (see also Remark 2 of that paper),

assume that there exist two positive constants \( c, d \), with \( c < \sqrt{2d} \), such that:

\[ (j) \quad \int_0^d g(t) \, dt \geq 0; \]

\[ (ji) \quad \frac{\max_{|t| \leq c} g(t)}{c^2} < \frac{1}{4} \frac{g(d)}{d^2}, \]

follow from assumption (c) of Proposition 3.6 by choosing \( k_0 = d, k_1 = 0, s = c^2/(2d^2) \).

**Remark 3.4.** If we a priori fix the constants in the assumptions of Proposition 3.6, it takes an even simpler form. As an example we point out the following assumptions:
(a) there exists a constant $T > 0$ such that

\[
\int_0^{(1/8)T} g(t) \, dt \neq \int_0^{(3/8)T} g(t) \, dt,
\]

(ii) $\max_{|t| \leq \sqrt{\frac{3}{8} T}} g(t) < \frac{3}{4} \int_0^{(1/8)T} g(t) \, dt + \frac{1}{4} \int_0^{(3/8)T} g(t) \, dt$

obtained by choosing $k_0 = \frac{1}{4} T$; $k_1 = \frac{3}{4} T$; $s = \frac{3}{4}$;

(b) there exists a constant $T > 0$ such that

\[
\max_{|t| \leq \frac{T}{2}} g(t) < \frac{1}{4} \int_0^{T} g(t) \, dt
\]

obtained by choosing $s = \frac{1}{4}$ and $k = 2T$.

Remark 3.5. In the present section function $g$ is the antiderivative of $f$ such that $g(0) = 0$. We explicitly observe that even if $g$ is an antiderivative of $f$ such that $g(0) \neq 0$, arguing in a similar way, we obtain the same results. In the latter case we point out that the functional $\overline{J}(x) = \int_0^1 g(x(t)) \, dt$ might not have zeros.

4. Applications to ordinary differential equations

In this section we study an ordinary Dirichlet problem

\[
\begin{cases}
    u'' + \lambda f(u) = 0, \\
    u(0) = u(1) = 0,
\end{cases}
\]

and we establish some conditions such that (4.1) admits at least three classical solutions; we will use Theorem 1.1 and, to prove the minimax inequality (1.2) which plays a fundamental role in that theorem, we will use Theorem 3.2.

The main result of this section is the following theorem.

**Theorem 4.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g(t) = \int_0^t f(\xi) \, d\xi$. Assume that there exist $\rho \in \mathbb{R}$, $x_0, x_1 \in W^{1,2}_0$ be such that

(i) $\int_0^1 g(x_0(t)) \, dt < \rho < \int_0^1 g(x_1(t)) \, dt,$

(ii) $\max_{|t| \leq B(\rho, x_0, x_1)} g(t) < \rho,$
where
\[
B(\rho, x_0, x_1) = \frac{1}{2} \left( \left( \int_0^1 g(x_1(t)) \, dt - \rho \right) \|x_0\|^2 + \left( \rho - \int_0^1 g(x_0(t)) \, dt \right) \|x_1\|^2 \right)^{1/2}.
\]

Further, assume that

(iii) there exist two positive constants \(a\) and \(v\), with \(v < 2\), such that
\[
g(t) \leq a \left( 1 + |t|^v \right)
\]
for all \(t \in \mathbb{R}\).

Then, there exists an open interval \(\Lambda \subseteq ]0, +\infty[\) and a positive real number \(q\) such that, for each \(\lambda \in \Lambda\), problem (4.1) admits at least three solutions belonging to \(C^2([0, 1])\) whose norms in \(W_0^{1,2}([0, 1])\) are less than \(q\).

**Proof.** We put \(X = W_0^{1,2}([0, 1])\) and, for each \(u \in X\)
\[
\Phi(u) = \frac{1}{2} \|u\|^2 = \frac{1}{2} \int_0^1 \left[ u'(t) \right]^2 \, dt, \quad \overline{\Phi}(u) = - \int_0^1 \left( \int_0^t f(s) \, ds \right) \, dx,
\]
\[
R(u) = \Phi(u) + \lambda \overline{\Phi}(u).
\]

It is well known that the critical points in \(X\) of the functional \(R\) are precisely the weak solutions of problem (4.1). So, our aim is to apply Theorem 1.1 to \(\Phi\) and \(\overline{\Phi}\). Clearly, \(\overline{\Phi}\) is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \(X^*\), and \(\overline{\Phi}\) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Moreover, thanks to (iii) and to the Hölder inequality, one has
\[
\lim_{\|u\| \to +\infty} \left( \Phi(u) + \lambda \overline{\Phi}(u) \right) = +\infty
\]
for all \(\lambda \in [0, +\infty[\).

Furthermore, thanks to Theorem 3.2, from (i) and (ii) it follows that the assumption (1.2) is satisfied.

Therefore, we can apply Theorem 1.1. It follows that there exists an open interval \(\Lambda \subseteq ]0, +\infty[\) and a positive real number \(q\) such that, for every \(\lambda \in \Lambda\), the functional \(\overline{R} = \overline{\Phi} + \lambda \overline{\Phi}\) has three critical points that are three weak solutions of problem (1) whose norms in \(W_0^{1,2}([0, 1])\), are less than \(q\). By using classical methods it is easy to verify that the weak solutions belong to \(C^2([0, 1])\) and that they are classical solutions; hence, the conclusion is obtained. \(\square\)
Remark 4.1. Clearly, assumption (ii) of Theorem 4.1 can be improved with assumption (ii) of Theorem 3.1, but the latter is not easy to apply.

Remark 4.2. We can rephrase conclusion of Theorem 4.1 as follows: there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number $\bar{q}$ such that, for each $\lambda \in \Lambda$, problem (4.1) admits at least three solutions belonging to $C^2([0, 1])$ whose norms in $C^2([0, 1])$ are less than $\bar{q}$.

In fact, it is enough to take $\Lambda \subseteq \Lambda$ such that $\sup \Lambda < +\infty$ and

$$\bar{q} > \max \left\{ \frac{q}{2}, (\sup \Lambda) \sup_{|t| \leq q/2} |f(t)| \right\}.$$ 

Remark 4.3. Theorem 4.1 extends Theorem 2 of [4] (see Remark 3.2). Moreover, thanks to Proposition 3.6 we can give several special cases of Theorem 4.1 which are easier to apply. As an example we point out the following special case of Theorem 4.1.

**Theorem 4.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g(t) = \int_0^t f(\xi) d\xi$ for every $t \in \mathbb{R}$. Assume that there exist four positive constants $c$, $d$, $a$ and $v$, with $c < d$ and $v < 2$, such that

(i) $\max_{|t| \leq c} g(t) < \left( \frac{c}{d} \right)^2 \left( \frac{1}{d} \int_0^d g(t) dt \right)$,

(ii) $g(t) \leq a \left( 1 + |t|^v \right)$ for all $t \in \mathbb{R}$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number $q$ such that, for each $\lambda \in \Lambda$, problem (4.1) admits at least three solutions belonging to $C^2([0, 1])$ whose norms in $W^{1,2}_0([0, 1])$ are less than $q$.

**Proof.** It follows from (b) of Proposition 3.6 by choosing $s = \frac{c}{T}$ and $k = 2d$. □

Remark 4.4. Theorem 1.2 in the introduction is obtained from Theorem 4.2 by choosing $c = T/2$ and $T = d$.

Remark 4.5. Assumption (ii) of Theorem 1.2 cannot be dropped as example $f(u) = e^u$ shows: only assumption (i) of Theorem 1.2 is satisfied (by choosing, for instance, $T = 8$), and the problem

$$\begin{cases}
    u'' + \lambda e^u = 0, \\
    u(0) = u(1) = 0,
\end{cases}$$

for every $\lambda \geq 0$, has, at the most, two solutions (see [9]).
Example 4.1. Function \( f(t) = e^{-t}t^6(7 - t) \) satisfies all the assumptions of Theorem 1.2. Indeed, by choosing \( T = 2 \), one has

\[
\frac{1}{4} \int_0^T g(t) \, dt = \frac{1}{4} \int_0^2 e^{-t} t^7 \, dt > \frac{1}{e} = \max_{|t| \leq \frac{T}{2}} g(t).
\]

Remark 4.6. Similar arguments would allow one to study the nonautonomous quasilinear problem

\[
\begin{cases}
\frac{d^2 u}{dx^2} + \lambda f(x, u) h(u') = 0, \\
u(a) = u(b) = 0
\end{cases}
\]

so that the recent results of [10] and [11] may be reviewed in a more general setting.

References