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Best Possible Results in a Class of Inequalities, II

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We give a sufficient condition on a lower triangular infinite matrix A with non-negative entries, and a positive sequence $b = (b_n)$, for an inequality of the form $\|A(b|x|)\|_p \leq K\|x\|_p$, $x \in \ell_p$, to be best possible, in the sense that there is no positive sequence $d = (d_n)$ such that $(d_n b_n^{-1})$ is a monotone unbounded sequence, and an inequality of the form above holds with b replaced by d. This condition permits easy proofs of "best possible" theorems that generalize a previous result concerning Hardy's inequality.

Let ω be the space of scalar (real or complex) sequences. For $b = (b_n)$, $c = (c_n) \in \omega$, we define $b < c \Rightarrow |b_n| \leq M|c_n|$ for all n, for some M. If $b < c$ and $c < b$, we will say that b and c are equivalent.

For $1 \leq p \leq \infty$, let ℓ_p and $\|\ \|_p$, the norm on ℓ_p, be as usual. If A is an infinite matrix with non-negative entries and $b \in \omega$ is a non-negative sequence (A and b will be so from now on), an HPD (for Hardy–Petersen–Davies) inequality for A, b, and ℓ_p is a theorem of the form: for some $K > 0$,

$$\|A(b|x|)\|_p \leq K\|x\|_p \qquad \text{for all } x \in \ell_p. \tag{1}$$

(In this statement, $A(b|x|)$ is the sequence whose nth entry is $\sum_{k=1}^{\infty} a_{nk} b_k |x_k|$.)

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If A and p are given and (1) holds for some K, we will say that b *satisfies* the inequality (1).

Except for the fact that the best possible $K = K(p)$ is known, and the theorem is quantified over all p, $1 < p \leq \infty$, Hardy's inequality for sequences (see [2]) is an HPD inequality, with $b = e = (1, 1, \ldots)$ and $A = (a_{nk})$, the Cesáro matrix, given by

$$a_{nk} = \begin{cases} \dfrac{1}{n}, & k \leq n \\ 0, & k > n. \end{cases}$$

Petersen and Davies [1, 6] generalized Hardy's inequality to a class of HPD inequalities in which A is a lower triangular matrix satisfying, with b, certain requirements.

In [3, 4] we undertook the study of HPD inequalities, with the goal of systematizing and comparing these theorems. To understand the sort of comparison we had in mind, note that if (1) holds and $c < b$, then (1) holds with b replaced by c (and with, possibly, a different K). To put it another way, the "larger" b is with respect to $<$, the better the theorem.

We found (see [4, Theorem 3.5]; the blanket assumption that A be lower triangular is extraneous there) a necessary and sufficient condition for the existence of a best possible HPD inequality. In the current context, assuming A has no zero columns, given A and p there is one and only one candidate (up to equivalence) for a largest possible b satisfying (1); it is the sequence of reciprocals of the ℓ_p norms of the columns of A. (Convention: $(\infty)^{-1} = 0$.) There is a largest possible b satisfying (1) if and only if this particular b satisfies (1).

Using this result, we found [4, Theorem 9.2] that there is no best possible HPD inequality when

$$1 < p \leq \infty, \qquad rp > 1, \qquad \text{and} \qquad a_{nk} = \begin{cases} n^{-r}, & n \geq k \\ 0, & n < k. \end{cases}$$

Taking $r = 1$, we see that Hardy's inequality is not best possible, nor can it be improved to a final, best possible theorem of form (1), with $1 < p \leq \infty$ and A being the Cesáro matrix.

We are fond of negative, killjoy results like this, but we admit that an optimist might find grounds for debate in our choice of the notion of "best possible." For example, to say that Hardy's inequality is not best possible, for some $p \in (1, \infty]$, means that there is a positive sequence b, unbounded and bounded away from zero, such that (1) holds for some K, with A being the Cesáro matrix. But are there any *nice* such b? The only ones we actually found look like the constant sequence e except for large entries every trillion years or so. Does there exist a monotonically increasing such b?

To our surprise, the answer turned out to be no, and, indeed, in the circumstances described in (2), it turns out that (n^{r-1}) is a maximum sequence, with respect to $<$, among those *monotone* sequences b satisfying (1) for some K. (See [5].) In this paper we want to depart in a slightly different direction, to obtain results that are more general, but sometimes weaker, than those of [5].

Suppose (1) holds, for some A, b, p, and K. Let us say that b is *monotonically maximal* (abbreviated m.m.) for the inequality if no such inequality holds, with possibly a different K, with b replaced by $(b_n r_n)$, with (r_n) a monotone positive sequence tending to ∞.

The sequence b can be m.m. for an inequality without being monotone itself. Also, if b is m.m. for the inequality (1), and $c = (c_n)$ is any positive sequence, then (b_n/c_n) is m.m. for the same inequality, rewritten with AD_c playing the role of A and (b_n/c_n) playing the role of b. ($D_c = (c_n\delta_{nk})$ is the diagonal matrix with main diagonal c.) In particular, taking $c = b$, if b is strictly positive, we see that b is m.m. for (1) if and only if e is m.m for the same inequality with A replaced by AD_b. Clearly e is m.m. for an inequality if and only if e is maximal (equivalently, maximum; see [5]) among the monotone sequences satisfying the inequality. Thus the question of whether or not a given b satisfying (1) is m.m. for the inequality is always equivalent to the question of whether or not e is maximum among the monotone sequences satisfying a reconstructed version of (1).

But beware! If b satisfies (1), is monotone, and is m.m. for (1), it does not follow that b is maximal among the monotone sequences satisfying (1). One reason for this is that for any monotone positive b converging to 0 or to ∞, there is a monotone sequence d such that $b < d$, $d \not< b$, and $(r_n) = (b_n^{-1}d_n)$ is *not* monotone, nor is it equivalent to any monotone sequence. For such a d, take $A = D_{d^{-1}}$, where $d^{-1} = (d_n^{-1})$. By Theorem 3.5 of [4], $\|A(d|x|)\|_p = \|x\|_p$ is a best possible HPD inequality of the form (1), for each p, so since $b < d$ and d $\not< b$, b is not maximal among the monotone sequences satisfying (1). However, it is easy to arrange for b to be m.m. for the inequality (1); just arrange for $b_n = d_n$ for infinitely many n, in the choice of d. We omit the details.

The grim moral is that being monotonic, and monotonically maximal for an inequality, does not ensure that a sequence is maximal among the monotonic sequences satisfying the inequality.

In the other direction, if a sequence b is monotone non-decreasing to ∞, and maximal (and therefore maximum; see [5]) among the monotonic sequences satisfying (1), take any sequence $n_1 = 1 < n_2 < n_3 < \ldots$ of integers such that $(b_{(n_{k+1}-1)}b_{n_k}^{-1})$ is unbounded and set $c_n = b_{n_k}$, $n_k \leq n <$

n_{k+1}, $k = 1, 2, \ldots$. Then $c = (c_n)$ is non-decreasing to ∞, and $c < b$,so c satisfies (1); $b \not< c$ so c is not maximal among the monotone sequences satisfying (1); but c is m.m. for the inequality (1), it is easy to see, because of the maximality of b, the monotonicity of c, and the fact that $c_j = b_j$ for infinitely many j. In fact, by taking different sequences (n_k), we can produce uncountably many mutually non-comparable such c's. The point is that the maximality enjoyed by b in these circumstances is unique, up to equivalence, while monotone maximality is not—far from it.

When b is non-increasing to 0 and maximal among the monotone sequences satisfying (1), we do not know whether or not b is necessarily monotonically maximal for the inequality, even in the special cases when A is lower triangular. We leave this question open.

We have been making a fuss about the tangled relations between "monotonically maximal" and "maximal among the monotonics" in order to sort out the relations between the results of this paper and the results in [5]. We shall have some more to say about those relations after presenting our current results.

THEOREM 1. *Suppose that A is lower triangular, b satisfies the inequality* (1), *and for all positive, non-decreasing sequences $r = (r_k)$ tending to ∞, $(\sum_{k=1}^{n} a_{nk}b_k r_k)_n$ tends to ∞. Then b is monotonically maximal for the inequality* (1).

Proof. Suppose that $r = (r_k)$ is a non-decreasing sequence tending to ∞. With A and b satisfying the hypothesis of the theorem, we wish to show that $br = (b_k r_k)$ does not satisfy the inequality (1).

In the case $p = \infty$, observe that $A(bre) = (\sum_{k=1}^{n} a_{nk}b_k r_k)_n$, which tends to ∞, by hypothesis, so no inequality of form (1), with b replaced by br, is possible.

Suppose that $p < \infty$. For a positive integer m, let $x^{(m)} = \sum_{k=1}^{n} e_k = (1, 1, \ldots, 1, 0, 0, \ldots)$, the sequence with m ones in the first m places and zeroes elsewhere. We have

$$\|x^{(m)}\|_p^{-1}\|A(brx^{(m)})\|_p \geq \left[m^{-1}\left(\sum_{n=1}^{m}\left(\sum_{k=1}^{n} a_{nk}b_k r_k\right)^p\right)\right]^{1/p}$$

$$\to \infty \qquad \text{as } m \to \infty,$$

because $\gamma_n = (\sum_{k=1}^{n} a_{nk}b_k r_k)^p \to \infty$, by hypothesis, as $n \to \infty$, so $m^{-1}\sum_{n=1}^{m} \gamma_n \to \infty$, as $m \to \infty$.

Therefore, no inequality of form (1), with b replaced by br, is possible. ▌

COROLLARY 1. *Suppose that A is lower triangular, b satisfies the inequality* (1), *$e < (\sum_{k=1}^{n} a_{nk}b_k)_n$, and for each $n > 1$, $a_{nk}b_k \leq a_{n,k+1}b_{k+1}$,*

$k = 1, \ldots, n-1$. *(That is, the rows of AD_b are non-decreasing, up to and including the main diagonal). Then b is monotonically maximal for the inequality* (1).

Proof. The hypothesis that $e < (\sum_{k=1}^{n} a_{nk}b_k)_n$ means that the sums $\sum_{k=1}^{n} a_{nk}b_k$ are bounded away from zero. Let us say that $(\sum_{k=1}^{n} a_{nk}b_k \geq \rho > 0$ for all $n = 1, 2, \ldots$.

Suppose that $r = (r_k)$ is a non-decreasing positive sequence tending to ∞. We want to show that $\sum_{k=1}^{n} a_{nk}b_k r_k \to \infty$ as $n \to \infty$.

Suppose that $1 \leq m < n$. Observe that

$$\sum_{k=1}^{n} a_{nk}b_k = \sum_{k=1}^{m} a_{nk}b_k + \sum_{k=m+1}^{n} a_{nk}b_k \leq m a_{nm}b_m + \sum_{k=m+1}^{n} a_{nk}b_k$$
$$\leq \left(\frac{m}{n-m} + 1\right) \sum_{k=m+1}^{n} a_{nk}b_k,$$

since $a_{nk}b_k$ is non-decreasing with k, $k = 1, \ldots, n$. Thus

$$\sum_{k=m+1}^{n} a_{nk}b_k \geq \frac{n-m}{n} \sum_{k=1}^{n} a_{nk}b_k.$$

Therefore,

$$\sum_{k=1}^{n} a_{nk}b_k r_k \geq \sum_{k=m+1}^{n} a_{nk}b_k r_k \geq r_m \sum_{k=m+1}^{n} a_{nk}b_k$$
$$\geq r_m \frac{n-m}{n} \sum_{k=1}^{n} a_{nk}b_k \geq r_m \frac{n-m}{n} \rho.$$

Since $r_m \to \infty$ as $m \to \infty$, it is now clear that $\sum_{k=1}^{n} a_{nk}b_k r_k \to \infty$ as $n \to \infty$. ∎

COROLLARY 2. *Suppose that A, r, and p are as given in* (2). *Then (n^{r-1}) is monotonically maximal for the inequality* (1), *for such A and p.*

Proof. That $b = (n^{r-1})$ satisfies the inequality (1) in this case is shown in [4] (see the remarks at the end of Section 8 in [4]; the case $p = \infty$ is omitted there, but it is easy to see that (1) holds in that case).

When $r \geq 1$, the claim of this corollary follows straightforwardly from Corollary 1. Suppose that $r < 1$. Suppose that (r_k) is a positive monotone sequence tending to ∞. Then

$$\sum_{k=1}^{n} a_{nk}b_k r_k = n^{-r}\sum_{k=1}^{n} k^{r-1}r_k$$

$$\geq n^{-r}n^{r-1}\sum_{k=1}^{n} r_k = n^{-1}\sum_{k=1}^{n} r_k \to \infty$$

as $n \to \infty$, so the results follows from the theorem.

By the remarks preceding the theorem, the result of Corollary 2 is weaker than the result in [5] when $r > 1$, is the same as the result in [5] when $r = 1$ (Hardy's inequality!), and when $r < 1$, we do not know how to compare the results; they might be non-comparable.

Here is an application of the theorem in happier, more certain circumstances.

COROLLARY 3. *Let*

$$a_{nk} = \begin{cases} \dfrac{1}{\sqrt{n}}\dfrac{1}{\sqrt{n-k+1}}, & 1 \leq k \leq n \\ 0, & n < k, \end{cases}$$

and $1 < p \leq \infty$. *Then* e *is monotonically maximal for (and, therefore, maximum among the monotonic sequences satisfying)* $\|A|x|\|_p = \|A(e|x|)\|_p \leq K\|x\|_p$, *for all* $x \in \ell_p$, *for some* K.

Proof. That the inequality (1) holds with this A and $b = e$ is Proposition 7.11 of [4], for $1 < p < \infty$. For $p = \infty$, it is easy to see that the inequality holds, because the sequence of row sums of A is equivalent to e.

The maximality of e for this inequality follows easily from Corollary 1. ∎

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