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Weak solutions to the Cauchy problem for the diffusive discrete coagulation–fragmentation system

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Abstract

The initial value problem for the discrete coagulation–fragmentation system with diffusion is studied. This is an infinite countable system of reaction–diffusion equations describing coagulation and fragmentation of discrete clusters moving by spatial diffusion in all space $\mathbb{R}^d$. The model considered in this work is a generalization of Smoluchowski's discrete coagulation equations. Existence of global-in-time weak solutions to the Cauchy problem is proved under natural assumptions on initial data for unbounded coagulation and fragmentation coefficients. This work extends existence theory for this system from the case of clusters distribution on bounded domain subject to no-flux boundary condition to the case of all $\mathbb{R}^d$.

1. Introduction

This paper deals with the coagulation–fragmentation equations which are related to a mean-field model describing coalescence and spontaneous fragmentation of clusters moving by diffusion in all space $\mathbb{R}^d$, $d \geqslant 1$. The model describes the space and time evolution of a system of a large number of clusters growing by binary coalescence. The model is a generalization of the classical Smoluchowski coagulation equations which were originally introduced to describe the binary coagulation of colloidal particles moving according to Brownian motions [27,28]. In this approach, the clusters are assumed to be composed of a finite number of identical units (monoclusters), and are fully identified by

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their size, that is, the number of monoclusters they are made of. We refer to [8] and [7] for derivations of the model and physical background. It is worth mentioning here that the coagulation–fragmentation equations appear in many branches of science, e.g., in aerosol science [8,24], polymer science [32], biology [22] and astrophysics [23]. We restrict here to the physical situation in which clusters diffuse according to Fick's law and this is the only process which allows them to approach each other sufficiently close, so that they have a chance to coalesce. The influence of external fields which could induce directional movements and coalescence of clusters (such as temperature or electric field for charged particles) are not taken into account. As in most of papers only the process of spontaneous multiple fragmentation is taken here into account although collisional fragmentation can also be considered together with coagulation.

Since the original work of Smoluchowski, a number of physical and mathematical studies have been devoted to the coagulation–fragmentation equations but most of them are restricted to the case when the spatial fluctuations of clusters are neglected (see, e.g., [1,3,8,20] and references therein). Much less attention has been paid to the spatially inhomogeneous setting, though a reaction–diffusion-type model of diffusive coagulation was derived in [16] and also considered in [7,25,26]. Within the last decade, the diffusive coagulation–fragmentation equations on bounded domain have been further studied from a mathematical point of view in several papers [4,5,15,17,18,30,31] and references therein.

For $i \geqslant 1$, we denote by $c_i = c_i(t,x) \geqslant 0$ the concentration (number density) of i-clusters (that is, clusters composed of i units) at time t and position x. The initial value problem for the diffusive coagulation–fragmentation system (CFD) reads

$$\frac{\partial c_i}{\partial t} - d_i\,\Delta c_i = R_i(c) \quad \text{in } (0,+\infty)\times\mathbb{R}^d, \tag{1.1}$$

$$c_i(0) = c_{0,i} \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

where

$$R_i(c) = K_{1,i}(c) - K_{2,i}(c) - F_{1,i}(c) + F_{2,i}(c), \quad i \geqslant 1,$$

and denoting $c = (c_i)_{i\geqslant 1}$,

$$K_{1,i}(c) = \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}c_{i-j}c_j, \qquad K_{2,i}(c) = c_i\sum_{j=1}^{\infty} a_{i,j}c_j,$$

$$F_{1,i}(c) = B_i c_i, \qquad F_{2,i}(c) = \sum_{j=1}^{\infty} B_{i+j}\beta_{i+j,i}c_{i+j}$$

under convention that $K_{1,i} = F_{1,i} = 0$ for $i = 1$. The real numbers $d_i > 0$, $i \geqslant 1$, denote the diffusion coefficient of an i-clusters. The coagulation coefficients $a_{i,j} = a_{j,i}$ are nonnegative numbers which determine the rate of binary coagulation of i-clusters and j-clusters. The nonnegative real numbers B_i $(B_1 = 0)$ are fragmentation rates and nonnegative real numbers $\beta_{i,j}$, $i, j \geqslant 1$, determine the average number of j-clusters produced during the break-up of an i-cluster. The conservation of mass during a fragmentation event implies

$$\sum_{j=1}^{i-1} j\beta_{i,j} = i, \quad i \geqslant 2.$$

The gain terms $K_{1,i}(c)$, $F_{2,i}(c)$ in the ith equation account for the formation of i-clusters from smaller pieces and appearance of i-clusters resulting from fragmentation of larger clusters, respectively. The loss terms $K_{2,i}(c)$, $F_{1,i}(c)$ describe the depletion of i-clusters due to interactions with other clusters and their break-up.

Notice that, in the situation described above there are no sources nor sinks of clusters in the reaction terms. Consequently, the total mass of clusters (the mass of monoclusters being normalized) defined for $t \geqslant 0$ by

$$m(t) = \sum_{i=1}^{\infty} i \int_{\Omega} c_i(t,x)\,dx$$

is expected to be equal to the initial one, provided the latter is finite. It turns out however that it is not true in general for several physically relevant coagulation rates and the breakdown of the mass conservation is then related to the so-called gelation phenomenon (see, e.g., [10,12,14,21] in the spatially homogeneous case and [10,13] for the diffusive case, and references given there). In general, we thus only have that $m(t) \leqslant m(0)$ and this property suggests a natural functional framework to study (1.1)–(1.2). More precisely, we define the Banach space

$$X_1 = \left\{ u = (u_i)_{i\geqslant 1},\ u_i \in L^1(\Omega),\ \sum_{i=1}^{\infty} i|u_i|_{L^1} < \infty \right\},$$

endowed with the norm

$$\|u\|_1 = \sum_{i=1}^{\infty} i|u_i|_{L^1}, \quad u \in X_1.$$

We also denote by X_1^+ the positive cone of X_1, i.e.,

$$X_1^+ = \left\{ u = (u_i)_{i\geqslant 1} \in X_1,\ u_i \geqslant 0 \text{ a.e. in } \Omega \right\}.$$

Thus, within our setting, the total mass of a solution to (1.1)–(1.2) is nothing but its X_1-norm and the above argument suggests that it stays bounded by the initial one throughout time evolution. We assume the same assumptions on the growth of coagulation and fragmentation coefficients as in [17]. Namely,

$$\lim_{j\to\infty} \frac{a_{i,j}}{j} = \lim_{j\to\infty} \frac{B_{i+j}\beta_{i+j,i}}{i+j} = 0, \quad i \geqslant 1. \tag{1.3}$$

Observe that (1.3) excludes the coagulation rates $a_{i,j} = i + j$ and $a_{i,j} = ij$, but includes several cases considered in the literature such as $a_{i,j} = i^\lambda + j^\lambda$ and $a_{i,j} = (ij)^\lambda$, $\lambda \in [0,1)$. We remark also that, the existence of solutions (in the sense of Definition 1.1 below) to (1.1)–(1.2) is still an open question when $a_{i,j} \leqslant A(i+j)$ and only partial results are known [31]. It is also worth pointing out that in the space homogeneous case the existence of solutions to the coagulation–fragmentation equations case was proved in [3] for $a_{i,j} \leqslant i + j$ without any growth assumptions on fragmentation rate—a problem still not solved in the diffusive case.

We use the same notion of solution as in the previous papers [17,19]. In the following, Ω_T denotes the set $(0,T) \times \mathbb{R}^d$ for $T > 0$.

Definition 1.1. Let $T_\star \in (0, +\infty]$. A solution $c = (c_i)_{i\geqslant 1}$ to (CFD) on $[0, T_\star)$ is a mapping from $[0, T_\star)$ in X_1^+ such that, for each $T \in (0, T_\star)$ and $i \geqslant 1$,

(1) $c_i \in \mathcal{C}([0, T]; L^1(\Omega))$,
(2) $K_{1,i}(c), K_{2,i}(c), F_{1,i}(c), F_{2,i}(c) \in L^1(\Omega_T)$,
(3) c_i is a mild solution to the ith equation in (CFD), i.e., for each $t \in [0, T]$,

$$c_i(t) = S_i(t)c_i(0) + \int_0^t S_i(t-s)R_i(c)(s)\,ds,$$

where S_i is the heat semigroup in $L^1(\mathbb{R}^d)$ corresponding to the Laplace operator $d_i\Delta$.

We are now in a position to state our result.

Theorem 1.2. *Assume that there exists $D > 0$ such that*

$$0 < d_i \leqslant D \quad \textit{for } i \geqslant 1 \tag{1.4}$$

and $c_0 = (c_{0,i}) \in X_1^+$. If the kinetic coefficients $(a_{i,j})$, B_i, $\beta_{i,j}$ satisfy (1.3) then there exists at least one solution to (CFD) on $[0, +\infty) \times \mathbb{R}^d$ such that $\|c(t)\|_1 \leqslant \|c_0\|_1$ for $t \geqslant 0$.

This theorem extends to all $\mathbb{R}^d$ a recent result by Laurençot and Mischler [17] which concerns existence of weak solutions to (CFD) subject to initial data in X_1^+ and no-flux condition imposed on the boundary of a bounded domain $\Omega \subset \mathbb{R}^d$. By now two analytic methods of existence proof for (CFD) appeared in the literature. One of them is based on the contraction argument and can be applied only in the case when the mapping $c \to (R_i(c))_{i\geqslant 1}$ is locally Lipschitz continuous in suitable function spaces. This requirement leads to some restrictions on the growth of coagulation and fragmentation coefficients which exclude many physically relevant cases. In [2] existence of local-in-time mass-conserving solution is proved when $\Omega = \mathbb{R}^d$ for any $d \geqslant 1$. The solution can be prolonged for all $t > 0$ only for one space dimension. We point out that thanks to a very abstract point of view assumed in [2] a continuous model of coagulation–fragmentation with diffusion is treated in a unified way with the discrete one. Results proved there are based on theorems which deal with the generation of semigroups in generalized Slobodeckii space being a subspace of Banach space valued distributions. The contraction mapping method was also used in [31] in a different function setting for both bounded or unbounded domain. In any case additional assumptions on initial data are imposed so this is not enough to assume $c_0 \in X_1$. On the other hand the solution constructed by means of this method is mass-preserving and uniquely determined. However, it can be prolonged for all $t > 0$ only in some particular cases involving additional restrictions on diffusion coefficients or space dimension (see [31]).

In order to take into account unbounded coagulation–fragmentation coefficients and initial conditions in X_1^+ one considers weak solutions in the sense of Definition 1.1. In this case the compactness method has been used. The weak solution is constructed as the limit of solutions to some finite systems of reaction–diffusion equations being defined as a suitable truncation of the original system (1.1). This method ensures neither uniqueness of solution nor mass conservation even if kinetic coefficients warrant both properties in space homogeneous case. Its proof relies on the construction of a sequence of approximating solutions to finite (truncated) systems related to the original one and on the observation that the sequences of reaction terms in the ith equation, $i \geqslant 1$, are weakly compact in $L^1(\Omega_T)$. It then allows passing to the limit in each reaction term and conclude that the limit of the approximating sequence is a solution to (CFD). It is worth noticing that by obvious reasons componentwise compactness in $L^1(0,T;L^1(\mathbb{R}^d))$ of approximating sequence of solutions requires additional arguments with respect to the case of a bounded domain. In the next section we prove a compactness result which provides us with a tool to handle the case of unbounded domain without making use of weighted spaces.

Recently, some results has been obtained on the approximation of solutions to diffusive coagulation–fragmentation equations by means of the stochastic particles approximation [6] (see also [11] and much earlier work [16]). In [6] clusters distribution in all space $\mathbb{R}^d$ has been considered with probability measure on $\mathbb{R}^d$ in the place of Lebesgue measure which is considered in this work.

2. Proof of Theorem 1.2

In this section, we fix $c_0 = (c_{0,i}) \in X_1^+$ and $T > 0$.

We say that a subset A of $L^1(\mathbb{R}^d)$ $(L^1(0,T;L^1(\mathbb{R}^d)))$ has *u-property* if

$$\lim_{R\to+\infty} \sup_{f\in A} \int_{\{|x|>R\}} \left|f(x)\right| dx = 0 \quad \left(\lim_{R\to+\infty} \sup_{f\in A} \int_0^T \int_{\{|x|>R\}} \left|f(t,x)\right| dt\, dx = 0 \right),$$

respectively.

We shall consider the initial value problem

$$v_t - \Delta v = f \quad \text{in } L^1(\mathbb{R}^d),\ v(0) = v_0. \tag{2.1}$$

Given $f \in L^1(0,T;L^1(\mathbb{R}^d))$ and $v_0 \in L^1(\mathbb{R}^d)$ there exists the unique mild solution $v \in C([0,T];L^1(\mathbb{R}^d))$ (see, e.g., [29]). For subsets $I_0 \subset L^1(\mathbb{R}^d)$ and $I_f \subset L^1(0,T;L^1(\mathbb{R}^d))$ let

$$M \subset C\big([0,T];\mathbb{R}^d\big)$$

denote the set of all mild solutions v to (2.1) corresponding to f and v_0 ranging in I_f and I_0, respectively. We are now in a position to state a compactness result which plays a crucial role in the proof of Theorem 1.2. It is based on the following classical result.

Proposition 2.1. *Let A be a bounded subset of $W^{1,1}(\mathbb{R}^d)$ enjoying u-property. Then A is a precompact set in $L^1(\mathbb{R}^d)$.*

Theorem 2.1. *Suppose that $I_0 \subset L^1(\mathbb{R}^d)$ and $I_f \subset L^1(0,T;L^1(\mathbb{R}^d))$ are bounded sets. If I_f and I_0 have u-property then M is precompact in $L^1(0,T;L^1(\mathbb{R}^d))$.*

Proof. The boundedness of I_0 and I_f implies existence of a constant γ such that

$$|f|_{L^1(0,T;L^1(\mathbb{R}^d))} + |v_0|_{L^1(\mathbb{R}^d)} \leqslant \gamma. \tag{2.2}$$

Let us first consider nonnegative data $f \geqslant 0$, $v_0 \geqslant 0$. For any positive function h we put

$$h^r = r \wedge (\chi_{B(0,r)} h),$$

where $B(0,r)$ is a ball centered at 0 of radius r. Then $f^r \in L^\infty(\mathbb{R}^d) \cap L^p(0,T;L^p(\mathbb{R}^d))$, $v_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $f^r \to f$, $v_0^r \to v_0$ as $r \to +\infty$ for each $p \in [1,\infty)$. Let $u_r \in C([0,T]; L^2(\mathbb{R}^d)) \cap W^{1,2}_{\mathrm{loc}}(]0,T]; L^2(\mathbb{R}^d) \cap L^2_{\mathrm{loc}}(]0,T]; H^2(\mathbb{R}^d))$ be the L^2-strong solution to the regularized problem

$$u_{r,t} - D\Delta u_r = f^r, \quad u_r(0) = v_0^r. \tag{2.3}$$

Notice that by the maximum principle $u_r \geqslant 0$. Let us choose a smooth function $\tilde\theta : \mathbb{R} \to [0,1]$ such that

$$\tilde\theta(s) = 0 \quad \text{for } s \in (-\infty, 1] \quad \text{and} \quad \tilde\theta(s) = 1 \quad \text{for } s \geqslant 2.$$

Then there exists a constant $C_0 > 0$ such that $|\tilde\theta'(s)| \leqslant C_0$ for $s \in \mathbb{R}$. Now for $l > k > 0$ and any $x \in \mathbb{R}^d$ we define

$$\theta_{k,l}(x) := \tilde\theta\left(\frac{|x|^2}{k^2}\right) - \tilde\theta\left(\frac{|x|^2}{l^2} - 3\right). \tag{2.4}$$

For convenience, in the sequel, we shall write $\nabla\tilde\theta(|x|^2/k^2)$ and $\nabla\tilde\theta(|x|^2/l^2 - 3)$ to denote the x-derivative of the function $x \mapsto \tilde\theta(|x|^2/k^2)$ and $x \mapsto \tilde\theta(|x|^2/l^2 - 3)$, respectively. Notice that $\theta_{k,l} \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\theta_{k,l} \subset \{x \in \mathbb{R}^d\colon k \leqslant |x| \leqslant \sqrt5\, l\}$ and

$$\operatorname{supp}\nabla\theta_{k,l} \subset \left\{x \in \mathbb{R}^d\colon k \leqslant |x| \leqslant \sqrt2\, k\right\} \cup \left\{x \in \mathbb{R}^d\colon 2l \leqslant |x| \leqslant \sqrt5\, l\right\}. \tag{2.5}$$

Multiplying (2.3) by $\theta_{k,l}$ then integrating over $[0,t) \times \mathbb{R}^d$ and using (2.3)–(2.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} u_r(t,x)\theta_{k,l}(x)\,dx + \int_0^t \int_{\{k \leqslant |x| \leqslant \sqrt2 k\}} \nabla u_r \nabla\tilde\theta\left(\frac{|x|^2}{k^2}\right) dx\,d\tau \\
&\quad + \int_0^t \int_{\{2l \leqslant |x| \leqslant \sqrt5 l\}} \nabla u_r \nabla\tilde\theta\left(\frac{|x|^2}{l^2} - 3\right) dx\,d\tau \\
&\quad = \int_{\mathbb{R}^d} v_0^r(x)\theta_{k,l}\,dx\,d\tau + \int_0^t \int_{\mathbb{R}^d} f(\tau,x)\theta_{k,l}(x)\,dx\,d\tau.
\end{aligned} \tag{2.6}$$

To find L^1-estimate on ∇u_r let us note that u^r being the solution corresponding to $v_0^r \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $f^r \in L^1(0,T;L^1(\mathbb{R}^d)) \cap L^2(0,T;L^2(\mathbb{R}^d))$ coincides with the mild solution

$$u_r(t) = E *_x v_0^r + \int_0^t E(t-s) *_x f^r(s)\,ds, \quad t \geqslant 0, \tag{2.7}$$

where E is the Gauss–Weierstrass kernel corresponding to the Laplace operator. For $t > 0$ we have also

$$\nabla u_r(t) = \nabla E *_x v_0^r + \int_0^t \nabla E(t-s) *_x f^r(s)\,ds. \tag{2.8}$$

Taking into account that the function $t \to |\nabla E(t,\cdot)|_{L^1(\mathbb{R}^d)}$ belongs to $L^p(0,T)$ for $p \in [1,2)$ and using Young's inequality we conclude that

$$|\nabla u_r|_{L^1(0,T;L^1(\mathbb{R}^d))} \leqslant C_1\big(|f^r|_{L^1(0,T;L^1(\mathbb{R}^d))} + |v_0^r|_{L^1(\mathbb{R}^d)}\big) \leqslant C_1\gamma, \tag{2.9}$$

where $C_1 = C_1(T,d)$. Letting $f^r \to f$, $v_0^r \to v_0$ as $r \to +\infty$ and making use of (2.6) and (2.7) we obtain

$$u_r \to v \quad \text{in } C\big([0,T];\mathbb{R}^d\big),$$
$$\nabla u_r \to \nabla v \quad \text{in } L^1\big(0,T;L^1(\mathbb{R}^d)\big).$$

Moreover, by (2.9),

$$|\nabla v| \leqslant C_1\gamma. \tag{2.10}$$

We may now replace u_r in (2.6) by v. Then fixing k we shall let $l \to +\infty$ in (2.6). To this end we notice that for each $b \in \mathbb{R}$ we have

$$\left|\nabla\tilde{\theta}\left(\frac{|x|^2}{b^2}\right)\right| \leqslant |x|\,\frac{2C_0}{b^2}, \quad x \in \mathbb{R}^d,$$

and hence

$$\int_0^t \int_{\{2l \leqslant |x| \leqslant \sqrt{5}\,l\}} \nabla v \nabla\tilde{\theta}\left(\frac{|x|^2}{l^2} - 3\right) dx\,d\tau \leqslant 2\sqrt{5}\,C_0 C_1 \frac{\gamma}{l}. \tag{2.11}$$

Using Lebesgue's dominated convergence theorem we arrive from (2.6) (with u_r replaced by v) at

$$\begin{aligned}
&\int_{\mathbb{R}^d} v(t,x)\tilde{\theta}\left(\frac{|x|^2}{k^2}\right) dx \\
&\quad = \int_{\mathbb{R}^d} v_0\tilde{\theta}\left(\frac{|x|^2}{k^2}\right) dx + \int_0^t \int_{\mathbb{R}^d} f(\tau,x)\tilde{\theta}\left(\frac{|x|^2}{k^2}\right) dx\,d\tau \\
&\qquad + \int_0^T \int_{\{k \leqslant |x| \leqslant \sqrt{2}\,k\}} \nabla v \nabla\tilde{\theta}\left(\frac{|x|^2}{k^2}\right) dx\,d\tau.
\end{aligned}$$

Since the last term can be estimated in a similar way as that in (2.11) we obtain for $k > 0$,

$$\int_{\{|x|>\sqrt{2}k\}} v(t,x)\,dx \leqslant \int_{\{|x|>k\}} v_0(x)\,dx + \int_0^T \int_{\{|x|>k\}} \bigl|f(\tau,x)\bigr|\,dx\,d\tau + \frac{2\sqrt{2}\,C_0C_1\gamma}{k}. \tag{2.12}$$

Notice that we have used in the first term the nonnegativity of v. In order to consider the general case we consider data (f_+, v_{0+}) and (f_-, v_{0-}) separately, where $f_+ = f \vee 0$ and $f_- = -(f \wedge 0)$. Consequently, (2.12) holds for both v_+ and v_-. As I_0 and I_f have *u-property* we conclude that given $\delta > 0$ there exists R_δ such that for all $v \in M$,

$$\sup_{t\in[0,T]} \int_{|x|>R_\delta} \bigl|v(t,x)\bigr|\,dx < \delta. \tag{2.13}$$

To show that M is a precompact set in $L^1(0,T;L^1(\mathbb{R}^d))$ we shall apply Theorem 2.6.1 from [29]. Taking into account accretivity and maximality properties of the Laplace operator in $L^1(\mathbb{R}^d)$ we only have to check that given $\varepsilon > 0$ there exists $Q_\varepsilon \subset L^1(\mathbb{R}^d)$ such that for each $v \in M$ there exists a measurable subset $E_{v,\varepsilon}$ in $[0,T]$ such that $|E_{v,\varepsilon}| < \varepsilon$ and $v(t) \in Q_\varepsilon$ for each $v \in M$ and $t \in [0,T]\backslash E_{v,\varepsilon}$. From (2.2), (2.8) and (2.10) it follows that there is a constant C_γ such that

$$|v|_{L^1(0,T;W^{1,1}(\mathbb{R}^d))} < C_\gamma. \tag{2.14}$$

Let us define

$$\tilde{M} = \bigl\{v(t\cdot)\colon\ v \in M,\ t \in [0,T]\bigr\}$$

and $\psi(t) = |v(t)|_{W^{1,1}(\mathbb{R}^d)}$ for almost all $t \in [0,T]$. Now let $\varepsilon > 0$ be fixed and for $v \in M$ we set

$$E_{v,\varepsilon} = \bigl\{t \in [0,T]\colon\ \psi(t) > C_\gamma \varepsilon^{-1}\bigr\}.$$

Owing to (2.14) we obtain

$$|E_{v,\varepsilon}| = \int_0^T \chi_{\{t:\ \psi(t)>C_\gamma\varepsilon^{-1}\}}\,dt \leqslant C_\gamma^{-1}\varepsilon \int_0^T \psi(t)\,dt < \varepsilon.$$

Next we put

$$Q_\varepsilon = \bigl\{w \in W^{1,1}(\mathbb{R}^d)\colon\ |w|_{W^{1,1}(\mathbb{R}^d)} \leqslant C_\gamma \varepsilon^{-1}\bigr\} \cap \tilde{M}.$$

Thanks to (2.13) the set $\tilde{M}$ has u-property and consequently by Proposition 2.1 Q_ε is a precompact set in $\mathbb{R}^d$. Finally we notice that $v(t) \in Q_\varepsilon$ for all $v \in M$ and $t \in [0,T]\backslash E_{v,\varepsilon}$ which completes the proof. □

We shall need the following auxiliary fact.

Lemma 2.3. *Let $\{u^n\colon\ n \geqslant 1\}$ be a sequence of L^2-strong solutions to the initial value problem*

$$u_t - \Delta u + gu = f, \quad u(0) = u_0, \tag{2.15}$$

such that f, g *and* u_0 *are a.e. nonnegative functions. Moreover let* $f \in I_f$, $g \in I_g$, $u_0 \in I_0$, *where* I_f, I_g, (I_0) *are bounded subsets of* $L^1(0,T;L^1(\mathbb{R}^d))$ $(L^1(\mathbb{R}^d))$, *respectively. If* I_f *and* I_0 *have u-property then the set* $\tilde{I}_g = \{gu\colon\ u$ *is a solution to* (2.15) *and* $g \in I_g,\ f \in I_f,$ $u_0 \in I_0\}$ *too.*

Proof. By the maximum principle u is nonnegative. Multiplying (2.15) by (2.4) and proceeding as in (2.6), (2.11) and (2.12) we arrive at the following inequality:

$$\int\limits_{\{|x|>\sqrt{2}k\}} u(t,x)\,dx + \int\limits_0^T \int\limits_{\{|x|>\sqrt{2}k\}} g(t,x)u(t,x)\,dx\,dt \leqslant \int\limits_{\{|x|>k\}} u_0(x)\,dx + \int\limits_0^T \int\limits_{\{|x|>k\}} \bigl|f(\tau,x)\bigr|\,dx\,d\tau + \frac{\mathrm{const}}{k},$$

whence we deduce that $\tilde{I}_g$ has u-property which completes the proof. □

Now we proceed to the proof of Theorem 1.2. We want to underline at this point that we adopt here the method used in [17] in the case of (CFD) on bounded domain with no-flux boundary condition. Therefore we pay more attention on steps of proof in which the lack of boundedness of the domain plays a role.

Proof of Theorem 1. The proof consists of 4 steps.

Step 1. Approximation (truncated system).

We first define a sequence of solutions to finite reaction–diffusion system obtained from (1.1)–(1.2) by a suitable truncation (see [17]). For $N \geqslant 3$ we put

$$a_{i,j}^N = \begin{cases} a_{i,j} + \frac{1}{N} & \text{if } (i \vee j) \leqslant N, \\ 0 & \text{if } (i \vee j) > N, \end{cases} \tag{2.16}$$

$$B_i^N = \begin{cases} B_i & \text{if } i \leqslant N, \\ 0 & \text{if } i \geqslant N, \end{cases} \tag{2.17}$$

and for $i \geqslant 1$,

$$c_{0,i}^N = \begin{cases} c_i \wedge N & \text{if } i \leqslant N, \\ 0 & \text{if } i > N. \end{cases} \tag{2.18}$$

Next we consider the system of $2N$ equations

$$c_{i,t}^N - d_i \Delta c_i^N = R_i^N(c^N) \quad \text{in } (0,+\infty) \times \mathbb{R}^d, \tag{2.19}$$

$$c_i^N(0) = c_{0,i}^N \quad \text{in } \mathbb{R}^d, \tag{2.20}$$

for $1 \leqslant i \leqslant 2N$, where $c^N = (c_i^N)_{i\geqslant 1}$, $c_i^N = 0$ for $i > 2N$ and R_i^N is equal to R_i with $(a_{i,j})$ and (B_i) replaced by $(a_{i,j}^N)$ and (B_i^N). After suitable modifications related to the fact that the domain of c is unbounded we may use now Lemma 2.2 from [30] to conclude that there exists a unique global-in-time solution to (2.19)–(2.20) such that each component of c^N is bounded in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. It is worth noticing that the L^∞-bound of each component (which is needed for the global in time existence of solution) depends here in general on N in contrast to [30] where the existence of a solution with L^∞-bounded components was studied. To show the L^∞-bound by means of Lemma 2.2 in [30] one has to check the following technical condition: for each $i \geqslant 1$ there exists $\gamma_i > 0$ such that

$$B_j \beta_{j,i} \leqslant \gamma_i a_{i,j} \quad \text{for } j \geqslant i+1,$$

which extends hypothesis (H1) from [30] on the case of multiple fragmentation. It is easily seen that the condition is satisfied by (2.17) and (2.18). Using equality (2.2) from [30] and (2.20), we also obtain

$$\sup_{t \in [0,+\infty)} \|c^N\|_1 \leqslant \|c_0\|_1, \tag{2.21}$$

uniformly with respect to N.

Step 2. L^1-bound and u-property of reaction terms.

In what follows we denote by C_i, $i \geqslant 1$, a sequence of generic constants such that for fixed i, C_i does not depend on N. It follows from (1.3) that for each $i \geqslant 1$ there is a constant C_i such that for $j \geqslant 1$,

$$\frac{a_{i,j}^N}{j} + B_i^N + \frac{B_{i+j}^N \beta_{i+j,i}}{i+j} \leqslant C_i. \tag{2.22}$$

Owing to (2.21) and (2.22) we obtain for $i \geqslant 1$,

$$\left|F_1^N(c^N)\right|_{L^1(0,T;L^1(\mathbb{R}^d))} + \left|F_2^N\right|_{L^1(0,T;L^1(\mathbb{R}^d))} \leqslant C_i, \tag{2.23}$$

$$\left|\sum_{j=1}^N a_{i,j}^N c_j^N\right|_{L^1(0,T;L^1(\mathbb{R}^d))} \leqslant C_i. \tag{2.24}$$

Now we may proceed in the same way as in [17] using induction argument starting from $i = 1$. The induction step is based on the following observation:

$$\text{for } i \geqslant 1, \quad K_{1,i+1}^N(c^N) \leqslant \sum_{j=1}^i K_{2,j}^N(c^N). \tag{2.25}$$

Consequently we obtain that for each $i \geqslant 1$,

$$\left|R_i^N\right|_{L^1(0,T;L^1(\mathbb{R}^d)} \leqslant C_i. \tag{2.26}$$

We next claim that for each $i \geqslant 1$ the set

$$\left\{R_i^N(c^N):\ N \geqslant 3\right\} \quad \text{has u-property.} \tag{2.27}$$

One proceeds by induction using similar arguments as before. We consider first the case $i = 1$. From (2.22) it is easily seen that for each $i \geqslant 1$,

$$F_{1,i}^N + F_{2,i}^N \leqslant C_i \sum_{j=1}^{2N} j c_j^N \quad \text{in } (0,T) \times \mathbb{R}^d. \tag{2.28}$$

Thus, in order to show that each fragmentation term has u-property it remains to prove that the set

$$\left\{\sum_{j=1}^{2N} j c_j^N:\ N \geqslant 3\right\} \quad \text{has u-property.} \tag{2.29}$$

Indeed multiplying ith equation in (2.19) by $i\theta_{k,l}$ (see (2.4)) and then using equality (2.2) from [30] we find

$$\int_{\mathbb{R}^d} \sum_{j=1}^{2N} j c_j^N(t,x)\theta_{k,l}(x)\,dx = \int_{\mathbb{R}^d} \sum_{j=1}^{2N} j c_j^N(t,x)\theta_{k,l}(x)\,dx + \int_0^t \int_{\mathbb{R}^d} i d_i \Delta c_i^N(\tau,x)\theta_{k,l}\,dx\,d\tau. \tag{2.30}$$

We next integrate by part in the last term and let $l \to \infty$ using similar arguments as in (2.12). It finally yields

$$\int_{\mathbb{R}^d} \sum_{j=1}^{2N} j c_j^N(t,x)\tilde{\theta}\left(\frac{|x|^2}{k}\right) dx = \int_{\mathbb{R}^d} \sum_{j=1}^{2N} j c_{0,j}^N(x)\tilde{\theta}\left(\frac{|x|^2}{k}\right) dx + \int_0^t \int_{\mathbb{R}^d} i d_i c_i^N(\tau,x)\Delta\tilde{\theta}\left(\frac{|x|^2}{k}\right) dx\,d\tau. \tag{2.31}$$

Since

$$\Delta\tilde{\theta}\left(\frac{|x|^2}{k}\right) = \frac{2d}{k^2}\tilde{\theta}'\left(\frac{|x|^2}{k}\right) + \frac{4|x|^2}{k^4}\tilde{\theta}''\left(\frac{|x|^2}{k}\right),$$

using nonnegativity of c_i^N, (1.4), (2.21) and (2.5) we deduce that for $t \in (0,T]$,

$$\int_{\{|x|>\sqrt{2}k\}} \sum_{j=1}^{2N} j c_j^N(t,x)\,dx = \int_{\{|x|>k\}} \sum_{j=1}^{2N} j c_{0,j}^N(x)\,dx + T\frac{(2d+8)C_0'D}{k^2}\|c_0\|_1, \tag{2.32}$$

where C_0' is a constant such that $|\tilde{\theta}'| + |\tilde{\theta}''| \leqslant C_0'$. Now it is enough to observe that

$$\left\{\sum_{j=1}^{2N} j c_{0,j}^N:\ N \geqslant 3\right\} \tag{2.33}$$

has u-property and consequently (2.29) follows from (2.32) and (2.33). In view of (2.28) we deduce that each fragmentation term has u-property. Taking into account (2.24) we may apply Lemma 2.3 to conclude that

$$\left\{K_{2,1}^N(c^N):\ N \geqslant 3\right\} \quad \text{has u-property.} \tag{2.34}$$

It then follows from (2.25) that $\{K^N_{1,2}(c^N)\colon N \geqslant 3\}$ has u-property as well. Now using again Lemma 2.3 and (2.24) we infer that $\{K^N_{2,2}(c^N)\colon N \geqslant 3\}$ enjoys u-property and further we proceed by induction for $i \geqslant 3$ making use of (2.25) in the induction step. It proves claim (2.27).

Step 3. L^1-strong compactness of each component of $(c^N_i)_{i\geqslant 1}$.

Taking into account (2.26), (2.27) and (2.33) we may apply Theorem 2.2 to each equation in (2.19). Consequently for each $i \geqslant 1$, $\{c^N_i\colon N \geqslant 3\}$ is a precompact set in $L^1(0, T; L^1(\mathbb{R}^d))$. Using the diagonal process we deduce that there is a subsequence of (c^N) (not relabeled) and $c \in X^+_1$ such that for each $i \geqslant 1$,

$$c^N_i \to c_i \quad \text{in } L^1\big(0, T; L^1(\mathbb{R}^d)\big) \text{ and a.e. in } (0, T) \times \mathbb{R}^d. \tag{2.35}$$

From (2.35), (2.16), (2.17) and (1.3) we next deduce that for each $i \geqslant 1$,

$$F^N_{1,i}(c^N) \to F_{1,i}(c) \quad \text{in } L^1\big(0, T; L^1(\mathbb{R}^d)\big), \tag{2.36}$$

$$F^N_{2,i}(c^N) \to F_{2,i}(c) \quad \text{in } L^1\big(0, T; L^1(\mathbb{R}^d)\big), \tag{2.37}$$

$$\sum_{j=1}^{N} a^N_{i,j} c^N_j \to \sum_{j=1}^{\infty} a_{i,j} c_j \quad \text{in } L^1\big(0, T; L^1(\mathbb{R}^d)\big). \tag{2.38}$$

Step 4. Weak-compactness in L^1 of reaction terms and passing to the limit.

It remains passing to the limit in coagulation terms. To this end we first show that all nonlinear terms are weakly precompact in the space $L^1(0, T; L^1(\mathbb{R}^d))$. We begin with the first equation. In order to use the Dunford–Pettis theorem (see, e.g., [9]) we have to show firstly that

$$\begin{cases} \text{given } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } \iint_E K^N_{2,1}\big(c^N(t,x)\big)\,dx\,dt \leqslant \varepsilon, \\ \text{provided } E \text{ is a measurable subset of } (0, T) \times \mathbb{R}^d \text{ and } |E| < \delta, \end{cases} \tag{2.39}$$

and secondly that $\{K^N_{2,1}(c^N)\colon N \geqslant 3\}$ has u-property. The latter requirement has been just shown in (2.27) and (2.39) was originally proved in [17]. Note that for the last step one needs (2.38). Next by induction we show that all coagulation terms are weakly precompact in $L^1(0, T; L^1(\mathbb{R}^d))$. Taking now into account almost everywhere convergence of reaction terms resulting from (2.35) and Vitali's theorem we conclude that for a subsequence (not relabeled) and each $i \geqslant 1$,

$$K^N_{1,i}(c^N) \to K_{1,i}(c), \qquad K^N_{2,i}(c^N) \to K_{2,i}(c),$$

strongly in $L^1(0, T; L^1(\mathbb{R}^d))$. Using also (2.36), (2.37) and (2.35) we may pass to the limit in each equation thanks to continuous dependence of mild solutions on data. At last notice that time $T > 0$ was taken arbitrary so, the solution may be prolonged for all $t > 0$. It completes the proof. □

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