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Separators of fat points in $\mathbb{P}^n$

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ABSTRACT

In this paper we extend the definition of a separator of a point P in $\mathbb{P}^n$ to a fat point P of multiplicity m. The key idea in our definition is to compare the fat point schemes $Z = m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s \subseteq \mathbb{P}^n$ and $Z' = m_1P_1 + \cdots + (m_i - 1)P_i + \cdots + m_sP_s$. We associate to P_i a tuple of positive integers of length $\nu = \deg Z - \deg Z'$. We call this tuple the degree of the minimal separators of P_i of multiplicity m_i, and we denote it by $\deg_Z(P_i) = (d_1, \ldots, d_\nu)$. We show that if one knows $\deg_Z(P_i)$ and the Hilbert function of Z, one will also know the Hilbert function of Z'. We also show that the entries of $\deg_Z(P_i)$ are related to the shifts in the last syzygy module of I_Z. Both results generalize well-known results about reduced sets of points and their separators.

1. Introduction

Given a finite set of reduced points X in $\mathbb{P}^n$, it is a classical idea to derive either algebraic or geometric information about X by using the notion of a separator. Our goal in this paper is to extend the definition of a separator so that it also includes the class of non-reduced sets points which are usually called fat points.

Let k be an algebraically closed field of characteristic zero. A hypersurface defined by the homogeneous form $F \in R = k[x_0, \ldots, x_n] = k[\mathbb{P}^n_k]$ is said to be a *separator* of $P \in X$ if $F(P) \neq 0$, but $F(Q) = 0$ for all $Q \in X \setminus \{P\}$, i.e., the hypersurface defined by F passes through all the points of X but P. The *degree of a point P in X* is then defined to be

$$\deg_X(P) := \min\{\deg F \mid F \text{ is a separator of } P\}.$$

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Separators first appeared in Orecchia's work [20] on the conductor of a set of points, although the term separator does not appear until the paper of Geramita, Kreuzer, and Robbiano [9]. Orecchia showed that the conductor of the coordinate ring A of a finite set of reduced points $X = \{P_1, \ldots, P_s\}$, that is, the largest ideal of A that coincides with its extension in the integral closure of A, is generated by forms whose degrees are in the set $\{\deg_X(P_1), \ldots, \deg_X(P_s)\}$. For this reason, the degree of a point P in X is sometimes called the *conductor degree*. Geramita, Kreuzer, and Robbiano [9] introduced separators to study sets of points with the Cayley–Bacharach property. Later investigations of the properties of separators have included the work of Bazzotti [3], Beccari and Massaza [5], Sabourin [21], and Sodhi [22]. The definition of separators has also been generalized to different contexts. For example, Bazzotti and Casanellas defined a separator for reduced points on a surface [4], while the authors have studied separators of reduced sets of points in a multiprojective space (see [15,16,19]). The paper of Abbott, Bigatti, Kreuzer, and Robbiano [1] contains a discussion on how to compute the separators of a set of points.

Of particular importance to this paper are the results of Geramita, Maroscia, and Roberts [10] and Abrescia, Bazzotti, and the second author [2]. If X is a reduced set of points in $\mathbb{P}^n$, and $d = \deg_X(P)$, then Geramita et al. showed that the Hilbert function of $X \setminus \{P\}$ can be determined by knowing the Hilbert function of X and the value of d. This result nicely illustrates the idea that a separator gives information about passing from X to a subset of the type $X \setminus \{P\}$. Abrescia et al. then found a relationship between the shifts in the last syzygy module of I_X and the degree of a point. This result, originally only proved for points in $\mathbb{P}^2$, was independently extended to $\mathbb{P}^n$ by the second author [19] and Bazzotti [3].

In the above cited work, the sets of points being considered are almost always a reduced set of points. The work of Geramita, Kreuzer, and Robbiano [9], Kreuzer [18], and Kreuzer and Kreuzer [17] relaxed this condition and studied zero-dimensional subschemes Z, and considered subschemes of colength 1, i.e., zero-dimensional subschemes $Y \subseteq Z$ such that $\deg Y = \deg Z - 1$. In this paper, however, we are interested in the case that both Y and Z are sets of fat points ($\deg Y = \deg Z - 1$ is rarely true in this case), and to define separators of fat points in this context. If $X = \{P_1, \ldots, P_s\}$ is a set of reduced points in $\mathbb{P}^n$, and $m_1, \ldots, m_s$ are positive integers, then let Z be the scheme defined by $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ where each I_{P_i} is the defining ideal of the point P_i. The scheme Z, which we shall denote by $Z = m_1 P_1 + \cdots + m_s P_s$, is usually called a *set of fat points* of $\mathbb{P}^n$.

We want to define a separator of a fat point so that we recover fat point analogs of the results of Geramita et al. and Abrescia et al., that were mentioned above. The key insight that we need to carry out this program is to view passing from X to $X \setminus \{P\}$ as "reducing" the multiplicity of P by one, as opposed to "removing" the point P from X. This point-of-view appears to be the correct perspective in order to get the desired generalizations.

Once we dispense with the preliminaries in Section 2, in Section 3 we introduce our definition of a separator for fat points. In keeping with our idea of dropping the multiplicity of a point by one, let $Z = m_1 P_1 + \cdots + m_i P_i + \cdots + m_s P_s$ and $Z' = m_1 P_1 + \cdots + (m_i - 1) P_i + \cdots + m_s P_s$. A *separator* of the point P_i of Z of multiplicity m_i is any form F such that $F \in I_{Z'} \setminus I_Z$. In Theorem 3.3 we show that there exists a set of $\nu = \deg Z - \deg Z'$ separators of the point P_i of multiplicity m_i, say $\{F_1, \ldots, F_\nu\}$, such that the ideal $I_{Z'}/I_Z$ is minimally generated by $(\overline{F}_1, \ldots, \overline{F}_\nu)$ in the ring R/I_Z. The *degree of the minimal separators of the fat point* P_i *of multiplicity* m_i, which is denoted $\deg_Z(P_i)$, is the ν-tuple $(\deg F_1, \ldots, \deg F_\nu)$.

In Section 4 and Section 5 we use our new definition to prove fat point analogs of the results mentioned above. In particular, we prove the following results:

Theorem 1.1. *Let Z and Z' be the fat point schemes in $\mathbb{P}^n$ defined as above, and suppose* $\deg_Z(P_i) = (d_1, \ldots, d_\nu)$ *where* $\nu = \deg Z - \deg Z'$.

(a) (*Theorem* 4.1) *For all* $t \in \mathbb{N}$,

$$\Delta H_{Z'}(t) = \Delta H_Z(t) - \left|\left\{d_j \in (d_1, \ldots, d_\nu) \;\middle|\; d_j = t\right\}\right|$$

where ΔH_Y denotes the first difference Hilbert function of Y.

(b) (*Theorem* 5.4) *If*

$$0 \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_0 \to I_Z \to 0$$

is a minimal graded free resolution of I_Z, then the last syzygy module has the form

$$\mathbb{F}_{n-1} = \mathbb{F}'_{n-1} \oplus R(-d_1 - n) \oplus R(-d_2 - n) \oplus \cdots \oplus R(-d_\nu - n).$$

As an interesting corollary of Theorem 1.1(b), we note that if $m = \max\{m_1, \dots, m_s\}$ is the maximum of the multiplicities of a set of fat points in $\mathbb{P}^n$, then $\operatorname{rk}\mathbb{F}_{n-1} \geqslant \binom{m+n-2}{n-1}$. See Corollary 5.9 for more details.

We end our paper in Section 6 by calculating $\deg_Z(P)$ when Z is a special class of fat points. We show that if Z is a *homogeneous* set of fat points, i.e., $m_1 = \cdots = m_s$, whose support is a complete intersection, then for every point P in the support of Z, the degree of the minimal separators of the fat point P of multiplicity m is the same (see Theorem 6.4). This result can be viewed as a Cayley–Bacharach type of result since a set of reduced points has the *Cayley–Bacharach property* if and only if the degree of every point in X is the same. The results of this section extend our understanding of fat points in special position (see, for example, [13,14] and references there within).

2. Preliminaries and notation

In this section we collect together some well-known results which we shall need; we continue to use the notation and definitions from the introduction.

Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$. The positive integers $m_1, \dots, m_s$ are called the *multiplicities*. If $m_1 = \cdots = m_s = m$, then we refer to Z as a *homogeneous scheme of fat points*, otherwise Z is *non-homogeneous*. The set of reduced points $X = \{P_1, \dots, P_s\}$ is called the *support* of Z, and is denoted by $\operatorname{Supp}(Z)$. The *degree* of the fat point scheme $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$ is given by the formula $\deg Z := \sum_{i=1}^{s} \binom{m_i+n-1}{n}$.

The defining ideal of Z, denoted I_Z, is a homogeneous ideal in the ring $R = k[x_0, \dots, x_n]$. The *Hilbert function of Z*, denoted H_Z, is the numerical function $H_Z : \mathbb{N} \to \mathbb{N}$ defined by

$$H_Z(t) := \dim_k(R/I_Z)_t = \dim_k R_t - \dim_k(I_Z)_t \quad \text{for } t \in \mathbb{N}.$$

The first difference function of Z, denoted ΔH_Z, is defined by

$$\Delta H_Z(t) := H_Z(t) - H_Z(t-1) \quad \text{where } H_Z(t) = 0 \text{ if } t < 0.$$

The eventual value of H_Z is given by the degree of Z:

Lemma 2.1. *Let $Z \subseteq \mathbb{P}^n$ be a fat point scheme. Then $H_Z(t) = \deg Z$ for all $t \gg 0$.*

We also require information about the ideal of a single (fat) point in $\mathbb{P}^n$.

Lemma 2.2. *Let I_P be the prime ideal associated to a point $P \in \mathbb{P}^n$.*

(a) *The ideal I_P^m is I_P-primary.*
(b) *The minimal free graded resolution of I_P has the form*

$$0 \to R(-n) \to R^{\binom{n}{n-1}}(-n+1) \to \cdots \to R^{\binom{n}{1}}(-1) \to I_P \to 0.$$

Proof. (a) Since I_P is a complete intersection, $I_P^m = I_P^{(m)}$, the m-th symbolic power of I_P. This fact follows from a classical result of Zariski and Samuel [23, Lemma 5, Appendix 6]. But $I_P^{(m)}$ is the I_P-primary part of I_P^m, so the conclusion follows.

For (b), one appeals to the Koszul resolution. □

3. Defining separators of fat points

In this section we extend the definitions of a separator of a reduced point P in $\mathbb{P}^n$ and the degree of P in a set of points to the case of fat points. At the heart of our definition is the point-of-view that the comparison of the reduced sets of points X and $X \setminus \{P\}$ used to define separators should be seen as "reducing" the multiplicity of the point P by one, as opposed to "removing" the point P from X. It is this feature, i.e., the idea of reducing the multiplicity of the fat point by one, that we will generalize when defining a separator for a fat point.

The following convention shall be useful throughout this paper.

Convention 3.1. Consider the fat point scheme

$$Z := m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s \subseteq \mathbb{P}^n,$$

and fix a point $P_i \in \operatorname{Supp}(Z)$. We then let

$$Z' := m_1P_1 + \cdots + (m_i - 1)P_i + \cdots + m_sP_s,$$

denote the fat point scheme obtained by reducing the multiplicity of P_i by one. If $m_i - 1 = 0$, then the point P_i does not appear in the support of Z'.

Note that when $m_j = 1$ for $j = 1, \ldots, s$, then Z is simply the reduced set of points $X = \operatorname{Supp}(Z)$, and Z' is $X \setminus \{P_i\}$, i.e., we revert to the original context in which separators were defined. A separator will now be defined in terms of forms that pass through Z' but not Z.

Definition 3.2. Let $Z = m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$. We say that F is a *separator of the point P_i of multiplicity m_i* if $F \in I_{P_i}^{m_i-1} \setminus I_{P_i}^{m_i}$ and $F \in I_{P_j}^{m_j}$ for all $j \neq i$.

If F is a separator of the point P_i of multiplicity m_i, then $F \in I_{Z'} \setminus I_Z$. Thus, to compare Z and Z', we need to compare the ideals I_Z and $I_{Z'}$. We can do this algebraically by investigating the ideal $I_{Z'}/I_Z$ in the ring R/I_Z.

Theorem 3.3. *Let Z and Z' be the fat point schemes of Convention* 3.1. *Then there exists $\nu = \deg Z - \deg Z'$ homogeneous polynomials $\{F_1, \ldots, F_\nu\}$ such that*

(a) *each F_i is a separator of P_i of multiplicity m_i, and*
(b) *in the ring R/I_Z, the ideal*

$$I_{Z'}/I_Z = (\overline{F}_1, \ldots, \overline{F}_\nu) \quad \text{where } \overline{F}_i \text{ denotes the class of } F_i.$$

Furthermore, these polynomials form a minimal set of generators, where by minimal we mean that no set of cardinality less than ν generates $I_{Z'}/I_Z$.

Proof. Because $I_{Z'}/I_Z$ is an ideal in the ring R/I_Z, there exists $F_1, \ldots, F_s \in R$ such that $I_{Z'}/I_Z = (\overline{F}_1, \ldots, \overline{F}_s)$. Moreover, because R/I_Z is a Noetherian ring, we can assume that this s is minimal, that is, for any set $\{G_1, \ldots, G_t\}$ with $t < s$, then $I_{Z'}/I_Z \neq (\overline{G}_1, \ldots, \overline{G}_t)$. Because each $\overline{F}_j \neq 0$, this means that $F_j \notin I_Z$. However, $F_j \in I_{Z'}$. So, this implies that each F_j is a separator of P_i of multiplicity m_i. To complete the proof, it suffices to show that $s = \deg Z - \deg Z'$.

Let $P = P_i$ and $m = m_i$. After a linear change of variables, we can assume that $P = [1:0:\cdots:0]$, and hence $I_P = (x_1, \ldots, x_n)$. We can also assume that the hyperplane defined by $L = x_0$ does not pass through any of the points of $\mathrm{Supp}(Z)$.

We first show that $s \leqslant \deg Z - \deg Z'$. For all non-negative integers t we have the following short exact sequence of vector spaces

$$0 \to (I_{Z'}/I_Z)_t \to (R/I_Z)_t \to (R/I_{Z'})_t \to 0$$

where $(M)_t$ denotes the vector space of degree t elements in M. Hence,

$$\dim_k (I_{Z'}/I_Z)_t = \dim_k (R/I_Z)_t - \dim_k (R/I_{Z'})_t \quad \text{for all } t \geqslant 0.$$

By Lemma 2.1, $\dim_k (R/I_Z)_t = \deg Z$, and $\dim_k (R/I_{Z'})_t = \deg Z'$ for all $t \gg 0$. Hence $\dim_k (I_{Z'}/I_Z)_t = \deg Z - \deg Z'$ for all $t \gg 0$. Fix a t such that $\dim_k (I_{Z'}/I_Z)_t = \deg Z - \deg Z'$ and set $t_i = t - \deg F_i$ for each $i = 1, \ldots, s$. If necessary, we can also take t large enough so that $t_i > 0$ for all i. Since $L = x_0$ is a nonzero divisor on R/I_Z, each $\overline{x_0^{t_i} F_i} \neq \bar{0}$ in R/I_Z. Also note that for each $i = 1, \ldots, s$, we have $\overline{x_0^{t_i} F_i} \in (I_{Z'}/I_Z)_t$.

We claim that $\{\overline{x_0^{t_1} F_1}, \ldots, \overline{x_0^{t_s} F_s}\}$ is a linearly independent set of forms in $(I_{Z'}/I_Z)_t$, whence $s \leqslant \deg Z - \deg Z'$. If necessary, relabel the F_i's so that $\deg F_1 \leqslant \deg F_2 \leqslant \cdots \leqslant \deg F_s$. Suppose that there exists $c_1, \ldots, c_s$ in k, not all zero, such that

$$c_1 \overline{x_0^{t_1} F_1} + \cdots + c_s \overline{x_0^{t_s} F_s} = \overline{c_1 x_0^{t_1} F_1 + \cdots + c_s x_0^{t_s} F_s} = \bar{0}.$$

Let r be the largest integer in $\{1, \ldots, s\}$ such that $c_r \neq 0$. Hence

$$\begin{aligned} \overline{c_1 x_0^{t_1} F_1 + \cdots + c_s x_0^{t_s} F_s} &= \overline{c_1 x_0^{t_1} F_1 + \cdots + c_r x_0^{t_r} F_r} \\ &= \overline{x_0^{t_r}}\, \overline{\left(c_1 x_0^{t_1 - t_r} F_1 + \cdots + c_r F_r\right)} = \bar{0}. \end{aligned}$$

Note that by our relabeling, $t_i - t_r \geqslant 0$ for $i = 1, \ldots, r$. Because x_0 is a nonzero divisor on R/I_Z, we must have $c_1 x_0^{t_1 - t_r} F_1 + \cdots + c_r F_r = H \in I_Z$. But this implies that

$$F_r = (c_r)^{-1} c_r F_r = (c_r)^{-1} \left(-c_1 x_0^{t_1 - t_r} F_1 - \cdots - c_{r-1} x_0^{t_{r-1} - t_r} F_{r-1} + H\right).$$

But then $\overline{F_r} \in (\overline{F}_1, \ldots, \overline{F}_{r-1}, \overline{F}_{r+1}, \ldots, \overline{F}_s)$, whence

$$(\overline{F}_1, \ldots, \overline{F}_{r-1}, \overline{F}_{r+1}, \ldots, \overline{F}_s) = (\overline{F}_1, \ldots, \overline{F}_r, \ldots, \overline{F}_s),$$

thus contradicting the minimality of s.

We now show that if $s < \deg Z - \deg Z'$, we can derive a contradiction, and hence $s = \deg Z - \deg Z'$. As above, fix t to be any integer such that $\dim_k (I_{Z'}/I_Z)_t = \deg Z - \deg Z'$. If $s < \deg Z - \deg Z'$, then there exists some $\overline{H} \in (I_{Z'}/I_Z)_t$ that is not in the span of $\{\overline{x_0^{t_1} F_1}, \ldots, \overline{x_0^{t_s} F_s}\}$. On the other hand, because $\overline{H} \in (I_{Z'}/I_Z)_t$, there exist homogeneous forms $G_1, \ldots, G_s$ in R such that

$$\overline{H} = \overline{G_1 F_1 + \cdots + G_s F_s} \quad \text{with } \deg G_i = t - \deg F_i.$$

Each G_i can be rewritten as

$$G_i = c_i x_0^{t-\deg F_i} + G_i'(x_0,\ldots,x_n)$$

where $G_i' = G_i'(x_0,\ldots,x_n) \in (x_1,\ldots,x_n) = I_P$. We then have

$$\overline{G_i F_i} = \overline{c_i x_0^{t-\deg F_i} F_i} + \overline{G_i' F_i} = \overline{c_i x_0^{t-\deg F_i} F_i}$$

since $G_i' F_i \in I_Z$ for all i. To see this, note that for any $P_j \in \operatorname{Supp}(Z)\setminus\{P\}$, we already have $F_i \in I_{P_j}^{m_j}$, and thus $G_i' F_i \in I_{P_j}^{m_j}$. On the other hand, since $G_i' \in I_P$ and $F_i \in I_P^{m-1}$, we get $G_i' F_i \in I_P^m$. As a consequence

$$\overline{H} = \overline{G_1 F_1 + \cdots + G_s F_s} = \overline{c_1 x_0^{t-\deg F_1} F_1 + \cdots + c_s x_0^{t-\deg F_s} F_s}.$$

But this implies that $\overline{H}$ is in the span of $\{\overline{x_0^{t_1} F_1},\ldots,\overline{x_0^{t_s} F_s}\}$, contradicting our choice of $\overline{H}$. Hence $s = \deg Z - \deg Z'$, as desired. $\square$

Remark 3.4. The number $\nu = \deg Z - \deg Z'$ can be computed directly from the degree formula; precisely

$$\begin{aligned}\deg Z - \deg Z' &= \deg m_i P_i - \deg(m_i - 1)P_i \\ &= \binom{m_i + n - 1}{n} - \binom{m_i - 1 + n - 1}{n} = \binom{m_i + n - 2}{n-1}.\end{aligned}$$

In light of the above theorem, we can introduce a minimal set of separators:

Definition 3.5. Let Z and Z' be as in Convention 3.1. If $\{F_1,\ldots,F_\nu\}$ is a set of polynomials that satisfies conditions (a) and (b) of Theorem 3.3, then we call $\{F_1,\ldots,F_\nu\}$ a *minimal set of separators of P_i of multiplicity m_i*.

Our next step is to use this minimal set of separators to develop a fat point analog for the degree of a point. We begin with a lemma.

Lemma 3.6. *Let Z and Z' be as in Convention* 3.1 *with associated ideals I_Z and I_Z', respectively. Suppose that $\{F_1,\ldots,F_\nu\}$ is a minimal set of separators of P_i of multiplicity m_i. Then, for all $t \geqslant 0$,*

$$\dim_k (I_{Z'}/I_Z)_t = \bigl|\{F_i \mid \deg F_i \leqslant t\}\bigr|.$$

Proof. Assume that $P := P_i = [1:0:\cdots:0]$ and that the hyperplane defined by $L = x_0$ is a nonzero divisor on R/I_Z. We can now argue as in the proof of Theorem 3.3 to get the conclusion. Indeed, fix any integer t, and let $F_1,\ldots,F_r$ be all the forms in the set $\{F_1,\ldots,F_\nu\}$ with $\deg F_i \leqslant t$. Then $\{\overline{x_0^{t-\deg F_1} F_1},\ldots,\overline{x_0^{t-\deg F_r} F_r}\}$ is a linearly independent set of elements in $(I_{Z'}/I_Z)_t$.

Furthermore, this set must span $(I_{Z'}/I_Z)_t$. Indeed, for any $\overline{H} \in (I_{Z'}/I_Z)_t$, there exists homogeneous forms $G_1,\ldots,G_r$ such that

$$\overline{H} = \overline{G_1 F_1 + \cdots + G_r F_r} \quad \text{with } \deg G_i = t - \deg F_i.$$

Note, by degree considerations, we do not need to concern ourselves with the forms $F_{r+1},\ldots,F_\nu$. Just as in proof of Theorem 3.3, we rewrite each G_i as $G_i = c_i x_0^{t-\deg F_i} + G_i'$ with $G_i' \in I_P$. This then implies that

$$\overline{H} = \overline{c_1 x_0^{t-\deg F_1} F_1 + \cdots + c_r x_0^{t-\deg F_r} F_r},$$

i.e., $\overline{H}$ is in the span of $\{\overline{x_0^{t-\deg F_1} F_1}, \ldots, \overline{x_0^{t-\deg F_r} F_r}\}$.

Because $\{\overline{x_0^{t-\deg F_1} F_1}, \ldots, \overline{x_0^{t-\deg F_r} F_r}\}$ is a basis for $(I_{Z'}/I_Z)_t$, the conclusion now follows. $\square$

Theorem 3.7. *Let Z and Z' be as in Convention* 3.1. *Suppose that $\{F_1, \ldots, F_\nu\}$ and $\{G_1, \ldots, G_\nu\}$ are two minimal sets of separators of P_i of multiplicity m_i. Relabel the F_i's so that* $\deg F_1 \leqslant \cdots \leqslant \deg F_\nu$*, and similarly for the G_i's. Then*

$$(\deg F_1, \ldots, \deg F_\nu) = (\deg G_1, \ldots, \deg G_\nu).$$

Proof. This follows immediately from Lemma 3.6 since we must have

$$\left|\{F_i \mid \deg F_i \leqslant t\}\right| = \left|\{G_i \mid \deg G_i \leqslant t\}\right|$$

for all integers $t \geqslant 0$. $\square$

Using Theorem 3.7, we can define a fat point analog for the degree of a point.

Definition 3.8. Let $\{F_1, \ldots, F_\nu\}$ be any minimal set of separators of P_i of multiplicity m_i, and relabel so that $\deg F_1 \leqslant \cdots \leqslant \deg F_\nu$. Then the *degree of the minimal separators of P_i of multiplicity m_i*, denoted $\deg_Z(P_i)$, is the tuple

$$\deg_Z(P_i) = (\deg F_1, \ldots, \deg F_\nu).$$

Remark 3.9. When all the multiplicities of Z are one, then $\nu = 1$, and $\deg_Z(P_i) = (\deg F_1)$ where F_1 is a minimal separator of P_i of multiplicity of $m_i = 1$. From the definition, we observe that F_1 passes through all the points of $Z = \operatorname{Supp}(Z)$ except the point P_i, i.e., F_1 is a minimal separator of P_i in the traditional sense.

We now illustrate some of the above ideas with the following two examples.

Example 3.10. Suppose that $Z = mP$ is a single fat point of multiplicity $m \geqslant 2$ in $\mathbb{P}^n$. We can therefore assume that $I_P = (x_1, \ldots, x_n)$, and $I_Z = I_P^m$. In this case, all the monomials of degree $m-1$ in the variables $\{x_1, \ldots, x_n\}$ form a minimal set of separators of P of multiplicity m since

$$I_{Z'}/I_Z = \left(\left\{\overline{M} \mid M = x_1^{a_1} \cdots x_n^{a_n} \text{ with } a_1 + \cdots + a_n = m-1\right\}\right).$$

Thus, $\deg_Z(P) = (\underbrace{m-1, \ldots, m-1}_{\binom{m+n-2}{n-1}})$.

Example 3.11. Let $F, G \in R = k[x, y, z]$ be two generic forms with $\deg F = 2$ and $\deg G = 3$. Then $I = (F, G)$ defines a complete intersection of six reduced points $\{P_1, \ldots, P_6\}$ in $\mathbb{P}^2$ of type $(2, 3)$. Because I is a complete intersection, the ideal $I^2 = (F, G)^2$ is the defining ideal of the set of double points:

$$Z = 2P_1 + \cdots + 2P_6 \subseteq \mathbb{P}^2.$$

Let $Z' = 1P_1 + 2P_2 + \cdots + 2P_6$, and let I_Z and $I_{Z'}$ be the associated ideals. The Hilbert functions of Z and Z' are, respectively,

t	0	1	2	3	4	5	6	7
$H_Z(t)$	1	3	6	10	14	17	18	$\to$
$H_{Z'}(t)$	1	3	6	10	14	16	16	$\to$

From the above Hilbert functions, we can determine $\dim_k(I_{Z'}/I_Z)_t = H_Z(t) - H_{Z'}(t)$ for all t. By appealing to Lemma 3.6, we then obtain $\deg_Z(P_1) = (5, 6)$. The connection between the Hilbert functions of H_Z and $H_{Z'}$ and the tuple $\deg_Z(P_1)$ will be highlighted in the next section.

As we shall see in the later sections, information about Z' can be obtained from Z and $\deg_Z(P_i)$. By reiterating this process, we can then start from any fat point scheme, and successively reduce the multiplicity of any fat point by one to obtain information about the subschemes of Z that are also fat point schemes. It therefore makes sense to develop some suitable notation and definitions to carry out this iteration. We end this section by working out these details.

We begin by introducing some more notation that describes the scheme after we have dropped the multiplicity of P_i by any integer $h \in \{0, \ldots, m_i\}$.

Definition 3.12. Let $Z = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$ whose support is $X = \{P_1, \ldots, P_s\}$. If we fix an $i \in \{1, \ldots, s\}$, then for every $h \in \{0, \ldots, m_i\}$ we define

$$Z_{m_i-h}(P_i) = m_1P_1 + \cdots + m_{i-1}P_{i-1} + (m_i - h)P_i + m_{i+1}P_{i+1} + \cdots + m_sP_s.$$

We shall write Z_{m_i-h} when P_i is understood.

Note that what we called Z and Z' in Convention 3.1 are denoted Z_{m_i} and Z_{m_i-1} with respect to the new notation. If $h = m_i$, then

$$Z_0 = Z_0(P_i) = m_1P_1 + \cdots + m_{i-1}P_{i-1} + m_{i+1}P_{i+1} + \cdots + m_sP_s$$

is a scheme of fat points whose support is $\mathrm{Supp}(Z) \setminus \{P_i\}$. We can now introduce the degree of the minimal separators at various levels, where the level keeps track of how much we have reduced the multiplicity.

Definition 3.13. Suppose that $Z = m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s$. For $h = 1, \ldots, m_i$, the *degree of the minimal separators of P_i of multiplicity m_i at level h*, is $\deg_{Z_{m_i-h+1}}(P_i)$.

When $h = 1$, $\deg_{Z_{m_i-h+1}}(P_i) = \deg_Z(P_i)$, so we can view $\deg_Z(P_i)$ as the degree of the minimal separators of P_i of multiplicity m_i at level 1. We can now combine all degrees at each level to define the minimal separating set of a fat point.

Definition 3.14. Let $Z = m_1P_1 + \cdots + m_iP_i + \cdots + m_sP_s$. The *minimal separating set of the fat point m_iP_i* is the set

$$\mathsf{DEG}_Z(m_iP_i) = \left\{\deg_{Z_1}(P_i), \ldots, \deg_{Z_{m_i}}(P_i)\right\}.$$

Remark 3.15. Note that $\deg_{Z_1}(P_i)$ has only one entry and it represents the minimal degree of a form that passes through all the points P_j of Z with multiplicity at least m_j with $j \neq i$, but not through P_i. When $m_i = 1$, the minimal separating set of the fat point $1P_i$, which in this case is a reduced point, is the set $\mathsf{DEG}_Z(1P_i) = \{\deg_{Z_1}(P_i)\}$, and this corresponds to the separator degree of a reduced point P_i as given in the introduction.

4. Hilbert functions and separators

In this short section we explain how to use $\deg_Z(P_i)$ to compare the Hilbert functions of Z and Z'. We continue to use Convention 3.1. Our main result specializes to a result of Geramita et al. [10] when all the multiplicities are one.

At the core of the following theorem is Lemma 3.6 which computes the dimension of $(I_{Z'}/I_Z)_t$ for all t. Recall that ΔH_Z denotes the first difference function. In what follows, we write $a \in (a_1, \dots, a_n)$ to mean that a appears in the tuple $(a_1, \dots, a_n)$.

Theorem 4.1. *Let Z and Z' be as in Convention* 3.1. *Suppose that $\deg_Z(P_i) = (d_1, \dots, d_\nu)$ where $\nu = \deg Z - \deg Z'$. Then for all $t \in \mathbb{N}$,*

$$\Delta H_{Z'}(t) = \Delta H_Z(t) - \bigl|\bigl\{d_j \in (d_1, \dots, d_\nu) \bigm| d_j = t\bigr\}\bigr|.$$

Proof. For each $t \in \mathbb{N}$, the Hilbert functions of Z and Z' in degree t are related via the following short exact sequence of vector spaces:

$$0 \to (I_{Z'}/I_Z)_t \to (R/I_Z)_t \to (R/I_{Z'})_t \to 0.$$

Thus,

$$\begin{aligned}\Delta H_{Z'}(t) &= H_{Z'}(t) - H_{Z'}(t-1)\\ &= \bigl(H_Z(t) - \dim_k(I_{Z'}/I_Z)_t\bigr) - \bigl(H_Z(t-1) - \dim_k(I_{Z'}/I_Z)_{t-1}\bigr)\\ &= \Delta H_Z(t) - \bigl(\dim_k(I_{Z'}/I_Z)_t - \dim_k(I_{Z'}/I_Z)_{t-1}\bigr).\end{aligned}$$

The conclusion now follows from Lemma 3.6 since

$$\begin{aligned}\dim_k(I_{Z'}/I_Z)_t - \dim_k(I_{Z'}/I_Z)_{t-1} &= \bigl|\bigl\{d_j \in \deg_Z(P_i) \bigm| d_j \leqslant t\bigr\}\bigr| - \bigl|\bigl\{d_j \in \deg_Z(P_i) \bigm| d_j \leqslant t-1\bigr\}\bigr|\\ &= \bigl|\bigl\{d_j \in (d_1, \dots, d_\nu) \bigm| d_j = t\bigr\}\bigr|,\end{aligned}$$

thus completing the proof. $\square$

Remark 4.2. Suppose that one knows two of the following three pieces of information: (1) H_Z, (2) $H_{Z'}$, and (3) $\deg_Z(P_i)$. It follows from Theorem 4.1 that one can also determine the third piece of information.

Example 4.3. In Example 3.10 we calculated $\deg_Z(P)$ when $Z = mP \subseteq \mathbb{P}^n$. We use this information to find the Hilbert function of $Z = 3P$ in $\mathbb{P}^2$. By Theorem 4.1

$$\Delta H_{2P}(t) = \begin{cases} \Delta H_{3P}(t) & \text{if } t \neq 2,\\ \Delta H_{3P}(t) - 3 & \text{if } t = 2\end{cases}$$

because $\deg_{3P}(P) = (2, 2, 2)$. We now need to find the Hilbert function of ΔH_{2P}. Again, appealing to Theorem 4.1, we get

$$\Delta H_P(t) = \begin{cases} \Delta H_{2P}(t) & \text{if } t \neq 1,\\ \Delta H_{2P}(t) - 2 & \text{if } t = 1\end{cases}$$

because $\deg_{2P}(P) = (1, 1)$. Since $H_P(t) = 1$ for all $t \in \mathbb{N}$, we use the above expressions to find

$$\Delta H_{3P} : 1\ 2\ 3\ 0\ \to .$$

It follows that this recursive procedure can be used to find the Hilbert function of any single fat point in any projective space. Indeed, when $Z = mP \subseteq \mathbb{P}^2$, this procedure recovers the well-known result that

$$\Delta H_{mP} : 1\ 2\ 3 \cdots m-1\ m\ 0 \to .$$

When we specialize to the case of reduced points we recover a result of Geramita, Maroscia, and Roberts.

Corollary 4.4. *(See [10, Lemma 2.3].) Let $X \subseteq \mathbb{P}^n$ be a reduced set of points, and suppose that $P \in X$. If $X' = X \setminus \{P\}$, then*

$$\Delta H_{X'}(t) = \begin{cases} \Delta H_X(t), & t \neq \deg_X(P), \\ \Delta H_X(t) - 1, & t = \deg_X(P). \end{cases}$$

In the same paper, Geramita et al. defined a *permissible value* (see [10, Definition 4.1]) and showed that the degree of every point P is X is a permissible value. We round out this section by generalizing the notion of a permissible value and show that the degree of a minimal set of separators of P of multiplicity m is also an example of this generalized permissible value.

Definition 4.5. Let $H = \{b_t\}$, $t \geqslant 0$ be a zero-dimensional differentiable O-sequence. That is, H is the Hilbert function of a zero-dimensional scheme, and its first difference is also an O-sequence (see [10], for example, for the definition of an O-sequence). Equivalently, if $b_1 = n+1$, then H is a zero-dimensional differentiable O-sequence if its first difference function ΔH is the Hilbert function of an artinian quotient of $k[x_1, \dots, x_n]$. Let $\underline{d} = (d_1, \dots, d_\tau)$ be any τ-tuple of positive integers with $\tau \geqslant 1$ and $d_1 \leqslant \cdots \leqslant d_\tau$. We say that $\underline{d}$ is a *permissible vector* of length τ for H if

$$H_{\underline{d}} = \left\{ b_t - \left| \left\{ d_j \in (d_1, \dots, d_\tau) \;\middle|\; d_j \leqslant t \right\} \right| \right\}$$

is again a zero-dimensional differentiable O-sequence. The set of all permissible vectors of length τ with respect to H shall be denoted by $S_{H,\tau}$.

Theorem 4.1 implies that $\deg_Z(P_i)$ is a permissible vector of H_Z.

Corollary 4.6. *Let Z and Z' be as in Convention* 3.1. *Suppose that* $\deg_Z(P_i) = (d_1, \dots, d_\nu)$ *where* $\nu = \deg Z - \deg Z'$. *Then*

$$\deg_Z(P_i) \in S_{H_Z,\nu}.$$

Proof. We use the formula for $\Delta H_{Z'}$ in Theorem 4.1 to calculate $H_{Z'}$:

$$H_{Z'}(t) = H_Z(t) - \left| \left\{ d_j \in \deg_Z(P_i) \;\middle|\; d_j \leqslant t \right\} \right|.$$

Since H_Z and $H_{Z'}$ are zero-dimensional differentiable O-sequences, it follows that $\deg_Z(P_i)$ is a permissible vector of length ν of H_Z. $\square$

5. The degree of a separator and the minimal resolution

As evident in the previous section, if one knows some information about Z and the tuple $\deg_Z(P_i)$, one can also obtain information about Z'. It is therefore useful to know how to find $\deg_Z(P_i)$. Abrescia, Bazzotti, and the second author [2] showed that in the case of reduced points in $\mathbb{P}^2$ (and extended to $\mathbb{P}^n$ in [19,3]), the degree of a point in X is related to a shift in the last syzygy module in the resolution of I_X. In this section we will prove a similar result about $\deg_Z(P_i)$: the entries in this tuple are related to the shifts in the last syzygy module of the resolution of I_Z.

Before arriving at our main result, we will require a technical lemma that will be used in the induction step of our next theorem.

Lemma 5.1. *Let Z and Z' be as in Convention* 3.1. *Let $\{F_1, \ldots, F_\nu\}$ be a minimal set of separators of P_i of multiplicity m_i, and furthermore, suppose that the separators have been relabeled so that* $\deg F_1 \leqslant \cdots \leqslant \deg F_\nu$. *Then*

(a) *For $j = 1, \ldots, \nu$, $(I_Z, F_1, \ldots, F_{j-1}) : (F_j) = I_{P_i}$.*
(b) *For $j = 1, \ldots, \nu$, $(I_Z, F_1, \ldots, F_j)$ is a saturated ideal.*

Proof. We set $d_j := \deg F_j$ for $j = 1, \ldots, \nu$.

(a) To prove the inclusion $I_{P_i} \subseteq (I_Z, F_1, \ldots, F_{j-1}) : (F_j)$, note that $F_j \in I_{P_l}^{m_l}$ for all $l \neq i$, and for $l = i$, $F_j I_{P_i} \subseteq I_{P_i}^{m_i}$ since $F_j \in I_{P_i}^{m_i - 1}$. Hence $F_j I_{P_i} \subseteq I_Z \subseteq (I_Z, F_1, \ldots, F_{j-1})$.

To prove the other inclusion, we do a change of coordinates so that $P_i = [1 : 0 : \cdots : 0]$, and so that x_0 is a nonzero divisor on R/I_Z. Note that $I_{P_i} = (x_1, \ldots, x_n)$. Suppose that $G \in (I_Z, F_1, \ldots, F_{j-1}) : (F_j)$. So, $GF_j \in (I_Z, F_1, \ldots, F_{j-1})$. Then there are forms $A_1, \ldots, A_{j-1} \in R$ and $A \in I_Z$ such that

$$GF_j = A + A_1F_1 + \cdots + A_{j-1}F_{j-1} \quad \Leftrightarrow \quad GF_j - (A_1F_1 + \cdots + A_{j-1}F_{j-1}) = A \in I_Z. \tag{5.1}$$

We can take $G, A_1, \ldots, A_{j-1}$ to be homogeneous. Furthermore, if $\deg A = d$, then $\deg G = d - d_j$ and $\deg A_l = d - d_l$ for $l = 1, \ldots, j-1$. Furthermore, $d - d_l \geqslant 0$ for $l = 1, \ldots, j-1$ by our ordering of the minimal separators. We can also write

$$G = cx_0^{d-d_j} + G' \quad \text{and} \quad A_l = a_l x_0^{d-d_l} + A_l'$$

where $G', A_1', \ldots, A_{j-1}' \in I_{P_i} = (x_1, \ldots, x_n)$. Our goal is to show that $c = 0$, whence $G \in I_{P_i}$.

It follows that $G'F_{j-1} \in I_{P_i}^{m_i}$, and similarly $A_l'F_l \in I_{P_i}^{m_i}$ for $l = 1, \ldots, j-1$. Because $F_1, \ldots, F_j \in I_{P_l}^{m_l}$ for $l \neq i$, we get

$$G'F_j - \left(A_1'F_1 + \cdots + A_{j-1}'F_{j-1}\right) \in I_Z.$$

If we subtract this expression from (5.1), we get

$$cx_0^{d-d_j}F_j - \left(a_1x_0^{d-d_1}F_1 + \cdots + a_{j-1}x_0^{d-d_{j-1}}F_{j-1}\right) \in I_Z.$$

But then in $(I_{Z'}/I_Z)_d$ we have

$$\overline{cx_0^{d-d_j}F_j - \left(a_1x_0^{d-d_1}F_1 + \cdots + a_{j-1}x_0^{d-d_{j-1}}F_{j-1}\right)} = \bar{0}. \tag{5.2}$$

But by adapting the proof given in Theorem 3.3 (this is where you require that x_0 to be a nonzero divisor) the elements $\{\overline{x_0^{d-d_1}F_1}, \ldots, \overline{x_0^{d-d_j}F_j}\}$ are linearly independent in $(I_{Z'}/I_Z)_d$. Thus Eq. (5.2) holds only if $c = 0$. But this means that $G = G' \in I_{P_i}$, as desired.

To prove (b), we do a proof by contradiction. So, suppose that there exists a j such that $(I_Z, F_1, \ldots, F_j)$ is not saturated. As above, we take $P_i = [1:0:\cdots:0]$ and x_0 to be a nonzero divisor. The saturation of $(I_Z, F_1, \ldots, F_j)$, denoted $(I_Z, F_1, \ldots, F_j)^{\mathrm{sat}}$, is given by

$$(I_Z, F_1, \ldots, F_j)^{\mathrm{sat}} = (I_Z, F_1, \ldots, F_j) : (x_0, \ldots, x_n)^\infty.$$

Now suppose that there exists a $G \in (I_Z, F_1, \ldots, F_j)^{\mathrm{sat}} \setminus (I_Z, F_1, \ldots, F_j)$. It then follows that $Gx_0^t \in (I_Z, F_1, \ldots, F_j)$ for $t \gg 0$. For any $P_l \in \mathrm{Supp}(Z) \setminus \{P_i\}$, we have $Gx_0^t \in I_{P_l}^{m_l}$ since $(I_Z, F_1, \ldots, F_j) \subseteq I_{P_l}^{m_l}$. Because x_0 is a nonzero divisor on R/I_Z, $x_0 \notin I_{P_l}$. Thus, no power of x_0 belongs to any $I_{P_l}^{m_l}$. This means no power of x_0^t belongs to $I_{P_l}^{m_l}$, and thus, by Lemma 2.2, $G \in I_{P_l}^{m_l}$ since $I_{P_l}^{m_l}$ is a primary ideal.

On the other hand, since $(I_Z, F_1, \ldots, F_j) \subseteq I_{P_i}^{m_i-1}$, we have $Gx_0^t \in I_{P_i}^{m_i-1}$, and arguing as above, we must have $G \in I_{P_i}^{m_i-1}$. Thus, $G \in I_{Z'}$, or in other words, $\overline{G} \neq \bar{0}$ in $(I_{Z'}/I_Z)$. (If $\overline{G} = \bar{0}$, that would mean $G \in I_Z \subseteq (I_Z, F_1, \ldots, F_j)$, contradicting our choice of G.)

We then have

$$\overline{G} = \overline{c_1 x_0^{d-d_1} F_1 + \cdots + c_\nu x_0^{d-d_\nu} F_\nu}$$

for some constants $c_1, \ldots, c_\nu$, where the constant is zero if $d - d_\nu < 0$. There then must exist some $A \in I_Z$, such that

$$G - \left(c_1 x_0^{d-d_1} F_1 + \cdots + c_\nu x_0^{d-d_\nu} F_\nu\right) = A \in I_Z.$$

Rearranging gives us

$$G = A + \left(c_1 x_0^{d-d_1} F_1 + \cdots + c_\nu x_0^{d-d_\nu} F_\nu\right) \tag{5.3}$$

and thus,

$$Gx_0^t = Ax_0^t + \left(c_1 x_0^{d-d_1+t} F_1 + \cdots + c_\nu x_0^{d-d_\nu+t} F_\nu\right). \tag{5.4}$$

But $Gx_0^t \in (I_Z, F_1, \ldots, F_j)$, so we can also write it as

$$Gx_0^t = B + B_1 F_1 + \cdots + B_j F_j$$

with $B \in I_Z$ and $B_1, \ldots, B_j \in R$.

We can rewrite each B_l for $l = 1, \ldots, j$ as

$$B_l = b_l x_0^{d-d_l+t} + B_l' \quad \text{with } B_l' \in I_{P_i} = (x_1, \ldots, x_n).$$

Since $B_l' F_l \in I_{P_i}^{m_i}$ and $F_l \in I_{P_r}^{m_r}$ for all $P_r \in \mathrm{Supp}(Z) \setminus \{P_i\}$, we can write Gx_0^t has

$$Gx_0^t = B' + b_1 x_0^{d-d_i+t} F_1 + \cdots + b_j x_0^{d-d_j+t} F_j \quad \text{with } B' \in I_Z, \tag{5.5}$$

that is, the terms $B_l' F_l$ get absorbed into the B'. Setting the expressions (5.4) and (5.5) equal to each other and rearranging, we get

$$(c_1 - b_1) x_0^{d-d_1+t} F_1 + \cdots + (c_j - b_j) x_0^{d-d_j+t} F_j + c_{j+1} x_0^{d-d_{j+1}+t} F_{j+1} + \cdots + c_\nu x_0^{d-d_\nu+t} F_\nu \in I_Z.$$

But if we now consider the class of this element in $I_{Z'}/I_Z$, this element is $\bar{0}$. However the elements $\{\overline{x_0^{d-d_1+t}F_1}, \dots, \overline{x_0^{d-d_\nu+t}F_\nu}\}$ form a linear independent set (we omit any term with $d-d_i+t<0$). So $c_1-b_1=\cdots=c_j-b_j=c_{j+1}=\cdots=c_\nu=0$, or in other words, $c_l=b_l$ for $l=1,\dots,j$, and zero for the remaining c_l's. But by (5.3), this implies that $G\in(I_Z,F_1,\dots,F_j)$ contradicting our choice of G. $\square$

Remark 5.2. A different proof of Lemma 5.1(b), can be obtained by using Proposition 3.13 and Remark 3.14 in [12].

Remark 5.3. Although $(I_Z,F_1,\dots,F_j)$ is saturated for all j, it does not define a fat point scheme. It, however, defines a scheme of degree $\deg Z-j$. If we let W_j define the scheme defined by this ideal, then $W_0,\dots,W_{\binom{m+n-2}{n-1}}$ are all the "intermediate" schemes between Z' and Z, i.e.,

$$Z'=W_{\binom{m+n-2}{n-1}}\subset\cdots\subset W_1\subset W_0=Z.$$

We will now prove the main theorem of this section: given a minimal graded free resolution of I_Z, the entries of $\deg_Z(P_i)$ appear among the degrees of the last syzygies after shifting by n.

Theorem 5.4. *Let Z,Z' be the fat point schemes of $\mathbb{P}^n$ as in Convention 3.1, and suppose that $\deg_Z(P)=(d_1,\dots,d_\nu)$ where $\nu=\deg Z-\deg Z'$. If*

$$0\to\mathbb{F}_{n-1}\to\cdots\to\mathbb{F}_0\to I_Z\to 0 \tag{5.6}$$

is the minimal graded free resolution of I_Z, then the last syzygy module has the form

$$\mathbb{F}_{n-1}=\mathbb{F}'_{n-1}\oplus R(-d_1-n)\oplus R(-d_2-n)\oplus\cdots\oplus R(-d_\nu-n).$$

Proof. Let $F_1,\dots,F_\nu$ be a minimal set of separators of P_i of multiplicity m_i and let $d_r=\deg F_r$ for $r=1,\dots,\nu$. Let $\mathcal{H}_0$ denote the minimal graded free resolution of I_Z. We will proceed by induction on r.

When $r=1$, we have the short exact sequence

$$0\to R/\big((I_Z):(F_1)\big)(-d_1)\xrightarrow{\times F_1}R/I_Z\to R/(I_Z,F_1)\to 0. \tag{5.7}$$

By Lemma 5.1 we have $R/((I_Z):(F_1))=R/I_{P_i}$. Thus, by Lemma 2.2, the minimal graded free resolution of $R/((I_Z):(F_1))(-d_1)$ has the form

$$\mathcal{K}_1:0\to R(-d_1-n)\to R^{\binom{n}{n-1}}(-d_1-n+1)\to\cdots\to R(-d_1)\to R/I_{P_i}(-d_1)\to 0.$$

If we now apply the mapping cone construction to (5.7), using the resolutions $\mathcal{K}_1$ and $\mathcal{H}_0$, we construct a graded resolution of (I_Z,F_1):

$$\mathcal{H}:0\to R(-d_1-n)\to\mathbb{F}_{n-1}\oplus R^{\binom{n}{n-1}}(-d_1-n+1)\to\cdots\to R\to R/(I_Z,F_1)\to 0.$$

The mapping cone construction gives a resolution that, in general, is not minimal. Since the ideal (I_Z,F_1) is saturated by Lemma 5.1, its projective dimension is at most $n-1$, and thus $\mathcal{H}$ is a non-minimal resolution. So $\mathcal{H}=\mathcal{F}\oplus\mathcal{G}$ where $\mathcal{F}$ is the minimal resolution of $R/(I_Z,F_1)$ and $\mathcal{G}$ is isomorphic to the trivial complex[1] (see [7, Theorem 20.2]). In particular $R(-d_1-n)$ must be part of

[1] A *trivial complex* is the direct sum of complexes of the form $0\to R\xrightarrow{1}R\to 0\to 0\to\cdots$.

the trivial complex $\mathcal{G}$, and thus, to obtain a minimal resolution, the term $R(-d_1-n)$ must cancel with something in

$$\mathbb{F}_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1).$$

By degree considerations, we cannot cancel with any of the terms of $R^{\binom{n}{n-1}}(-d_1-n+1)$. Thus, $\mathbb{F}_{n-1} = \mathbb{F}'_{n-1} \oplus R(-d_1-n)$, i.e., the term $R(-d_1-n)$ must cancel with something in $\mathbb{F}_{n-1}$. Note that after we cancel $R(-d_1-n)$, we get a resolution of (I_Z, F_1), that may or may not be minimal. We let

$$\mathcal{H}_1 : 0 \to \mathbb{F}'_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1) \to \cdots \to R \to R/(I_Z, F_1) \to 0,$$

denote this resolution; we shall also require this resolution at the induction step.

Now suppose that $1 < r \leqslant \nu$, and assume by induction that we have shown that $\mathbb{F}_{n-1} = \mathbb{F}'_{n-1} \oplus R(-d_1-n) \oplus \cdots \oplus R(-d_{r-1}-n)$, and that a resolution of $(I_Z, F_1, \ldots, F_{r-1})$ is given by

$$\begin{aligned}\mathcal{H}_{r-1} : 0 \to \mathbb{F}'_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1) \oplus \cdots \oplus R^{\binom{n}{n-1}}(-d_{r-1}-n+1) \to \cdots\\ \to R \to R/(I_Z, F_1, \ldots, F_{r-1}) \to 0.\end{aligned}$$

We have a short exact sequence

$$\begin{aligned}0 \to R/\big((I_Z, F_1, \ldots, F_{r-1}) : (F_r)\big)(-d_r) \xrightarrow{\times F_r} R/(I_Z, F_1, \ldots, F_{r-1})\\ \to R/(I_Z, F_1, \ldots, F_r) \to 0.\end{aligned} \tag{5.8}$$

By Lemma 5.1, $R/((I_Z, F_1, \ldots, F_{r-1}) : (F_r))(-d_r) \cong R/I_{P_i}(-d_r)$, so its resolution is given by

$$\mathcal{K}_r : 0 \to R(-d_r-n) \to R^{\binom{n}{n-1}}(-d_r-n+1) \to \cdots \to R(-d_r) \to R/I_{P_i}(-d_r) \to 0.$$

Using the resolutions $\mathcal{K}_r$ and $\mathcal{H}_{r-1}$, the short exact sequence (5.8), and the mapping cone construction, we have a resolution of $R/(I_Z, F_1, \ldots, F_r)$ of the following form:

$$\begin{aligned}0 \to R(-d_r-n) \to \mathbb{F}'_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1) \oplus \cdots \oplus R^{\binom{n}{n-1}}(d_r-n+1) \to \cdots\\ \to R \to R/(I_Z, F_1, \ldots, F_r) \to 0.\end{aligned}$$

By Lemma 5.1, the ideal $(I_Z, F_1, \ldots, F_{r-1})$ is saturated, so the ideal can have projective dimension at most $n-1$. In other words, the above resolution, which has length n, is too long. As argued above, $R(-d_r-n)$ must be part of the trivial complex and cancel with some term in

$$\mathbb{F}'_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1) \oplus \cdots \oplus R^{\binom{n}{n-1}}(-d_r-n+1).$$

Recall that the definition of $\deg_Z(P_i) = (d_1, \ldots, d_\nu)$ implies that $d_1 \leqslant \cdots \leqslant d_r \leqslant \cdots \leqslant d_\nu$. So $d_r + n > d_j + n - 1$ for all $j = 1, \ldots, r$. Thus, $R(-d_r-n)$ must cancel with some term in $\mathbb{F}'_{n-1}$, i.e., $\mathbb{F}'_{n-1} = \mathbb{F}''_{n-1} \oplus R(-d_r-n)$. Thus, $\mathbb{F}_n = \mathbb{F}''_{n-1} \oplus R(-d_1-n) \oplus \cdots \oplus R(-d_r-n)$, and

$$\begin{aligned}\mathcal{H}_r : 0 \to \mathbb{F}''_{n-1} \oplus R^{\binom{n}{n-1}}(-d_1-n+1) \oplus \cdots \oplus R^{\binom{n}{n-1}}(-d_r-n+1) \to \cdots\\ \to R \to R/(I_Z, F_1, \ldots, F_r) \to 0\end{aligned}$$

is a resolution of $R/(I_Z, F_1, \ldots, F_r)$. This now completes the induction step. $\square$

Remark 5.5. When all the m_i's are one, that is, Z is a set of reduced points, our result recovers the results of Abrescia, Bazzotti, and Marino [2,3,19].

Definition 5.6. Let Z be a scheme of fat points with minimal graded free resolution of type (5.6) with $\mathbb{F}_{n-1} = \bigoplus_{j \in B_{n-1}} R(-j)^{\beta_{(n-1),j}}$. If $B_{n-1} = \{j_1, \dots, j_t\}$, then associate to $\mathbb{F}_{n-1}$ the vector

$$\mathcal{B}_{n-1} = (\underbrace{j_1, \dots, j_1}_{\beta_{n-1,j_1}}, \dots, \underbrace{j_t, \dots, j_t}_{\beta_{n-1,j_t}}).$$

For each integer $\tau \geqslant 1$, let

$$S_{\mathcal{B}_Z,\tau} = \bigl\{(j_{i_1} - n, \dots, j_{i_\tau} - n) \bigm| j_{i_1} \leqslant \dots \leqslant j_{i_\tau} \text{ and } j_{i_1}, \dots, j_{i_\tau} \in \mathcal{B}_{n-1}\bigr\},$$

that is, the set of τ-tuples whose entries are non-decreasing and appear among the shifts of $\mathbb{F}_{n-1}$. We call $S_{\mathcal{B}_Z,\tau}$ the *socle-vectors* of length τ associated to the Betti numbers of Z.

An example of the set of socle-vectors can be found in the example following the next theorem. Our next theorem shows that the set of socle-vectors is a subset of the set of permissible vectors.

Theorem 5.7. *Let Z, Z' be as in Convention 3.1, let $\nu = \deg Z - \deg Z'$ and let H_Z be the Hilbert function of Z. Then*

$$\deg_Z(P_i) \in S_{\mathcal{B}_Z,\nu} \subseteq S_{H_Z,\nu}.$$

Proof. By Theorem 5.4, $\deg_Z(P_i) \in S_{\mathcal{B}_Z,\nu}$.

For each ν-tuple $\underline{d} \in S_{\mathcal{B}_Z,\nu}$, we want to show that $\underline{d} \in S_{H_Z,\nu}$. Note that to show that $\underline{d}$ is a permissible vector of length ν of H_Z, it suffices to show that the sequence

$$\bigl\{\Delta H_Z(t) - \bigl|\bigl\{d_i \in (d_1, \dots, d_\nu) \bigm| d_i = t\bigr\}\bigr|\bigr\}$$

is the Hilbert function of an artinian quotient of $k[x_1, \dots, x_n]$. Then, by "integrating" this sequence, we obtain the sequence

$$\bigl\{H_Z(t) - \bigl|\bigl\{d_i \in (d_1, \dots, d_\nu) \bigm| d_i \leqslant t\bigr\}\bigr|\bigr\},$$

which will be the Hilbert function of a zero-dimensional scheme.

Let $S = R/I_Z$ be the coordinate ring of Z. Since S is a Cohen–Macaulay ring of dimension one, we can pass to an artinian reduction S' of S. That is, there exists a nonzero divisor L of degree one such that $S' \cong R/(I_Z, L)$ is an artinian ring. Furthermore, since $R/(I_Z, L) \cong (R/(L))/((I_Z, L)/(L))$, we can assume that S' is an artinian quotient of $k[x_1, \dots, x_n]$. So $S' = k[x_1, \dots, x_n]/J$ for some ideal J.

Because S' is artinian, we can rewrite S' as

$$S' = k \oplus S'_1 \oplus \dots \oplus S'_t \quad \text{with } S'_t \neq 0$$

where S'_i is the set of homogeneous elements of S' of degree i. The maximal ideal of S' is then $m = \bigoplus_{i=1}^t S'_i$. The socle of S', denoted $\operatorname{soc}(S')$, is the annihilator of m. In particular, $\operatorname{soc}(S')$ is a homogeneous ideal which we can write as the direct sum of its graded pieces: $\operatorname{soc}(S') = T_1 \oplus \dots \oplus T_t$

where $T_t = S'_t$. The dimension of each T_i is then related to the graded Betti numbers of J. In particular,

$$\dim_k T_i = \beta^{R'}_{n-1,n+i}(J) \quad \text{where } R' = k[x_1, \dots, x_n].$$

For more information about the socle and for proofs of these facts, see [8].

But because we are passing to an artinian reduction, the graded Betti numbers of I_Z and J are the same, that is,

$$\beta^{R}_{n-1,n+i}(I_Z) = \beta^{R'}_{n-1,n+i}(J) \quad \text{for all } i \in \mathbb{N}.$$

Thus, if $\underline{d} = (d_1, \dots, d_\nu) \in S_{\mathcal{B}_Z,\nu}$, we can pick an element $G_i \in \operatorname{soc}(S')$ such that $\deg G_i = d_i$. Moreover, if $d_i = d_{i+1} = \cdots = d_{i+b}$, we can pick elements $G_i, \dots, G_{i+b}$ that are linearly independent socle elements since $b \leqslant \beta^{R}_{n-1,n+d_i}(I_Z) = \beta^{R'}_{n-1,n+d_i}(J) = \dim_k T_{d_i}$. That is, we take $G_i, \dots, G_{i+b}$ to be basis elements of T_{d_i}.

Thus, to $\underline{d} = (d_1, \dots, d_\nu)$ we can associate ν socle elements $\{G_1, \dots, G_\nu\}$ of S' such that $\deg G_i = d_i$, and if any subset of elements has the same degree, then these elements are linearly independent over k.

We now want to compute the Hilbert function of $S'/(G_1, \dots, G_\nu)$. We claim that for all $t \in \mathbb{N}$,

$$\dim_k (G_1, \dots, G_\nu)_t = \left|\left\{G_i \in \{G_1, \dots, G_\nu\} \;\middle|\; \deg G_i = t\right\}\right|.$$

We partition the elements of $\{G_1, \dots, G_\nu\}$ into three sets, some of which may be empty:

$$\begin{aligned}
\mathcal{G}_< &= \{G_1, \dots, G_a\} = \left\{G_i \in \{G_1, \dots, G_\nu\} \;\middle|\; \deg G_i < t\right\}, \\
\mathcal{G}_t &= \{G_{a+1}, \dots, G_b\} = \left\{G_i \in \{G_1, \dots, G_\nu\} \;\middle|\; \deg G_i = t\right\}, \\
\mathcal{G}_> &= \{G_{b+1}, \dots, G_\nu\} = \left\{G_i \in \{G_1, \dots, G_\nu\} \;\middle|\; \deg G_i > t\right\}.
\end{aligned}$$

By our choice of the G_i's, the elements of $\mathcal{G}_t$ are linearly independent, so $\dim_k (G_1, \dots, G_\nu)_t \geqslant |\mathcal{G}_t|$. Now let F be any element of $(G_1, \dots, G_\nu)_t$. By degree considerations, the elements of $\mathcal{G}_>$ do not contribute to $(G_1, \dots, G_\nu)_t$. So,

$$F = G_1 A_1 + \cdots + G_a A_a + c_{a+1} G_{a+1} + \cdots + c_b G_b$$

where $G_1, \dots, G_a \in \mathcal{G}_<$, $G_{a+1}, \dots, G_b \in \mathcal{G}_t$, $A_1, \dots, A_a \in S'$, and $c_{a+1}, \dots, c_b \in k$. But since $\deg F = t$, $\deg A_i = t - \deg G_i > 0$ for $i = 1, \dots, a$. This means that each A_i is in m, which implies that $G_i A_i = 0$ since each G_i is a socle element. Hence

$$F = c_{a+1} G_{a+1} + \cdots + c_b G_b.$$

So, F is in the vector space spanned by $\{G_{a+1}, \dots, G_b\}$, whence $\dim_k (G_1, \dots, G_\nu)_t \leqslant |\mathcal{G}_t|$. We have thus shown that for all $t \in \mathbb{N}$

$$\begin{aligned}
H_{S'/(G_1,\dots,G_\nu)}(t) &= H_{S'}(t) - \left|\left\{G_i \in \{G_1, \dots, G_\nu\} \;\middle|\; \deg G_i = t\right\}\right| \\
&= H_{S'}(t) - \left|\left\{d_i \in (d_1, \dots, d_\nu) \;\middle|\; d_i = t\right\}\right|.
\end{aligned}$$

Because $H_{S'}(t) = \Delta H_Z(t)$ for all t, this now completes the proof since $H_{S'/(G_1,\dots,G_\nu)}$ is the Hilbert function of an artinian quotient of $k[x_1, \dots, x_n]$. $\square$

Example 5.8. In $\mathbb{P}^2$ let us consider two totally reducible forms F and G such that $\deg F = 3$ and $\deg G = 7$, i.e., $F = L_1L_2L_3$ and $G = L'_1 \cdots L'_7$ where the L_is and L'_is are linear forms. Let $X = CI(3,7)$ be the complete intersection of type $(3,7)$ defined by $I_X = (F, G)$. The 21 points of X are the 21 points of intersection of the L_is and L'_is, i.e., $P_{ij} = L_i \cap L'_j$ for $i = 1,2,3$ and $j = 1,\dots,7$. Set $Y = CI(3,7) \setminus \{P_{37}\}$ and let Z be the scheme of double points whose support is the 20 points of Y, i.e.,

$$Z = 2P_{11} + \cdots + 2P_{36}.$$

We will now find the minimal separating set $\mathsf{DEG}_Z(2P_{36})$. We let $Z_2 = Z$, and

$$Z_1 = 2P_1 + \cdots + 2P_{35} + P_{36} \quad \text{and} \quad Z_0 = 2P_1 + \cdots + 2P_{35}.$$

By results found in [13,14], the minimal graded free resolution of I_{Z_2} is

$$0 \to R^2(-12) \oplus R(-15) \oplus R(-16) \to R(-6) \oplus R(-10) \oplus R(-11) \oplus R^2(-14) \to I_{Z_2} \to 0.$$

Furthermore, the Hilbert function of R/I_{Z_2} is

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$H_{Z_2}(t)$	:	1	3	6	10	15	21	27	33	39	45	50	53	56	59	60	$\to$	$\to$
$\Delta H_{Z_2}(t)$	:	1	2	3	4	5	6	6	6	6	6	5	3	3	3	1	0	$\to$

By Theorem 5.4, the degree of the minimal separators of P_{36} of multiplicity 2 must be one of the elements of $S_{B_Z,2}$. From the resolution of I_Z, we compute the vector

$$\mathcal{B}_{2-1} = (12, 12, 15, 16).$$

Then the set of socle vectors of length 2 is

$$S_{B_Z,2} = \big\{(10,10), (10,13), (10,14), (13,14)\big\},$$

i.e., $\deg_Z(P_{36})$ is one of these four tuples. We use CoCoA [6] to compute the minimal graded free resolution of I_{Z_1}:

$$0 \to R(-11) \oplus R(-12) \oplus R(-14) \oplus R(-16) \to R(-6) \oplus R^2(-10) \oplus R(-13) \oplus R(-14) \to I_{Z_1} \to 0$$

and its first difference Hilbert function is

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta H_{Z_1}(t)$	:	1	2	3	4	5	6	6	6	6	6	4	3	3	2	1	0	$\to$

By comparing ΔH_{Z_1} and ΔH_{Z_2}, Theorem 4.1 reveals that $\deg_{Z_2}(P_{36}) = (10, 13)$. Furthermore, by Theorem 5.4, $\deg_{Z_1}(P_{36})$ must be one of $\{(11-2), (12-2), (14-2), (16-2)\} = \{(9), (10), (12), (14)\}$.

If we compute the Hilbert function of R/I_{Z_0} we get

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta H_{Z_0}(t)$	:	1	2	3	4	5	6	6	6	6	6	4	3	2	2	1	0	$\to$

which reveals that $\deg_{Z_1}(P_{36}) = (12)$.

Thus, the minimal separating set of the fat point $2P_{36}$ is the set $\mathsf{DEG}_Z(2P_{36}) = \{(12), (10, 13)\}$.

As an interesting corollary to Theorem 5.4, we get a bound on the rank of the last syzygy module in terms of the m_is and n.

Corollary 5.9. *Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$ be a set of fat points, and let $m = \max\{m_1, \ldots, m_s\}$. If*

$$0 \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_0 \to I_Z \to 0$$

is a minimal graded free resolution of I_Z, then

$$\operatorname{rk} \mathbb{F}_{n-1} \geqslant \binom{m+n-2}{n-1}.$$

Proof. Suppose P_i has multiplicity m. Then by Theorem 5.4, the syzygy module $\mathbb{F}_{n-1}$ must have at least $\nu = \deg Z - \deg Z' = \binom{m+n-2}{n-1}$ shifts. The conclusion now follows. □

6. Application: a Cayley–Bacharach type of result

We use Theorem 5.4 to produce a Cayley–Bacharach (to be defined below) type of result for homogeneous sets of fat points in $\mathbb{P}^n$ whose support is a complete intersection (see [13,14] and references there within, for more on these special configurations). In particular, we show that if Z is a homogeneous fat point scheme whose support is a complete intersection, then $\deg_Z(P)$ is the same for every point $P \in \operatorname{Supp}(Z)$. We prove this result by showing that the last syzygy module of I_Z only permits one possible choice for $\deg_Z(P)$. We also show how to calculate $\deg_Z(P)$ in this situation.

Let $X \subseteq \mathbb{P}^n$ be a complete intersection of points of type $(\delta_1, \ldots, \delta_n)$. This means that $I_X = (F_1, \ldots, F_n)$ where $F_1, \ldots, F_n$ define a complete intersection with $\deg F_i = \delta_i$ for all $i = 1, \ldots, n$. Without loss of generality, we can assume that $\delta_1 \leqslant \cdots \leqslant \delta_n$. We now recall a result which is a special case of a classical result of Zariski and Samuel [23, Lemma 5, Appendix 6].

Lemma 6.1. *Suppose that $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is a complete intersection of reduced points. For any integer $m > 1$, the defining ideal of the homogeneous fat point scheme $Z = mP_1 + \cdots + mP_s$ is given by $I_Z = I_X^m$.*

The ideal of Z is then a power of a complete intersection. In [13], the first and third authors described the graded Betti numbers in the graded minimal free resolution of the power of any complete intersection in terms of the type. As a special case, we can describe all the shifts at the end of the resolution of I_Z.

Theorem 6.2. *Suppose that $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is a complete intersection of reduced points of type $(\delta_1, \ldots, \delta_n)$. For any integer $m > 1$, the minimal graded free resolution of the ideal I_Z defining the homogeneous fat point scheme $Z = mP_1 + \cdots + mP_s$ has the form*

$$0 \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_0 \to I_Z = I_X^m \to 0$$

where

$$\mathbb{F}_{n-1} = \bigoplus_{(a_1,\ldots,a_n)\in\mathcal{M}_{n,m+n-1}} R(-a_1\delta_1 - \cdots - a_n\delta_n).$$

Here, the set

$$\mathcal{M}_{n,m+n-1} := \left\{ (a_1, \ldots, a_n) \in \mathbb{N}^n \;\middle|\; \begin{array}{l} a_1 + \cdots + a_n = m+n-1 \text{ and} \\ a_i \geqslant 1 \text{ for all } i \end{array} \right\}.$$

Proof. See Theorem 2.1 in [13]. □

Corollary 6.3. *With the hypotheses as in Theorem* 6.2,

$$\operatorname{rk} \mathbb{F}_{n-1} = \binom{m+n-2}{n-1}.$$

Proof. By Theorem 6.2, the set of integer solutions to $a_1 + \cdots + a_n = m + n - 1$ with all $a_i \geqslant 1$ is in bijection with the generators of the free module $\mathbb{F}_{n-1}$. The number of integer solutions to this equation is $\binom{m+n-2}{n-1}$. □

Every fat point in a homogeneous fat point scheme whose support is a complete intersection must now have the same degree:

Theorem 6.4. *Let $Z = mP_1 + \cdots + mP_s \subseteq \mathbb{P}^n$ be a homogeneous fat point scheme such that* $\operatorname{Supp}(Z)$ *is a complete intersection. Then, for every $P_i \in \operatorname{Supp}(Z)$, the tuple $\deg_Z(P_i)$ is the same. In particular, for every $P_i \in Z$, the schemes $Z' = mP_1 + \cdots + (m-1)P_i + \cdots + mP_s$ all have the same Hilbert function.*

Proof. By Theorem 5.4, each of the $\nu = \deg Z - \deg Z'$ entries of $\deg_Z(P_i) = (d_1, \ldots, d_\nu)$ appear as shifts of the form $-d_i - n$ among the shifts of the $(n-1)$-th syzygy module of I_Z. But by Corollary 6.3, there are exactly ν such shifts in $\mathbb{F}_{n-1}$ when Z is a homogeneous fat point scheme whose support is a complete intersection. Thus, for each $P_i \in \operatorname{Supp}(Z)$, there is only choice for $\deg_Z(P_i)$. □

The above result can be interpreted as saying that homogeneous fat point schemes whose support is a complete intersection have a property similar to the Cayley–Bacharach property for reduced points. We recall this definition:

Definition 6.5. A set of reduced points $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is said to have the *Cayley–Bacharach property* (CBP) if for every $P \in X$, the Hilbert function $H_{X \setminus \{P\}}$ is the same.

Using Corollary 4.4, one can prove:

Theorem 6.6. *Let $X = \{P_1, \ldots, P_s\}$ be a set of reduced points in $\mathbb{P}^n$. Then X has the CBP if and only if* $\deg_X(P_1) = \cdots = \deg_X(P_s)$.

In Theorem 6.4, we showed that $\deg_Z(P_1) = \cdots = \deg_Z(P_s)$ when $\operatorname{Supp}(Z) = \{P_1, \ldots, P_s\}$ is a complete intersection. By analogy with Theorem 6.6, this suggests that homogeneous fat point schemes whose support is a complete intersection have a property similar to the reduced sets of points with the CBP. There are many examples of reduced sets points with the CBP: level sets of points, Gorenstein sets of points, and complete intersections (the last two are examples of the first). It would be interesting to find other classes of fat point schemes Z which have the property that $\deg_Z(P_1) = \cdots = \deg_Z(P_s)$ when $\operatorname{Supp}(Z) = \{P_1, \ldots, P_s\}$.

Remark 6.7. In her PhD thesis [11], the first author introduced the definition of a Cayley–Bacharach property for homogeneous schemes of fat points in $\mathbb{P}^2$ whose support is a complete intersection of type (a, b).

Definition 6.8. In $\mathbb{P}^2$, a homogeneous scheme of fat points Z whose support is a complete intersection of type (a, b) has the *Cayley–Bacharach property* if for all $i = 1, \ldots, ab$, the subschemes of Z of type

$$Y_i = mP_1 + \cdots + \widehat{mP_i} + \cdots + mP_{ab} \quad \text{with } \deg(Y) = \deg(X) - \binom{m+1}{2},$$

have the same Hilbert function.

Theorem 3.5.4 and Corollary 3.5.5 in [11] showed that when $m = 2$, all the homogeneous schemes of double points with support a complete intersection have the Cayley–Bacharach property. Note that the point-of-view taken in this definition is different from the one we have used in this paper. The schemes being studied in [11] have "removed" the entire fat point, while in this paper we have focused on what happens when we "reduce" the multiplicity of a point.

Using Theorems 6.2 and 6.4 we can actually calculate $\deg_Z(P) = (d_1, \ldots, d_\nu)$ when Z is a homogeneous fat point scheme supported on a complete intersection directly from the type of the complete intersection. We illustrate this behavior via an example.

Example 6.9. Consider a complete intersection of points X in $\mathbb{P}^3$ of type $(2, 3, 4)$, and consider the homogeneous scheme of fat points Z of multiplicity $m = 3$ supported on X. Then

$$\mathcal{M}_{3,3+3-1} := \left\{ (a_1, a_2, a_3) \in \mathbb{N}^3 \;\middle|\; \begin{array}{l} a_1 + a_2 + a_3 = 3 + 3 - 1 = 5 \text{ and} \\ a_i \geqslant 1 \text{ for all } i \end{array} \right\}.$$

This set only contains six elements:

$$\mathcal{M}_{3,5} = \big\{(1,1,3), (1,3,1), (1,1,3), (1,2,2), (2,1,2), (2,2,1)\big\}.$$

Thus, by Theorem 6.2, the last syzygy module $\mathbb{F}_2$ in the resolution of $I_Z = I_X^3$ has the form:

$$\begin{aligned} &R(-1\cdot 2 - 1\cdot 3 - 3\cdot 4) \oplus R(-1\cdot 2 - 3\cdot 3 - 1\cdot 4) \oplus R(-3\cdot 2 - 1\cdot 3 - 1\cdot 4) \\ &\quad \oplus R(-1\cdot 2 - 2\cdot 3 - 2\cdot 4) \oplus R(-2\cdot 2 - 1\cdot 3 - 2\cdot 4) \oplus R(-2\cdot 2 - 2\cdot 3 - 1\cdot 4) \\ &= R(-13) \oplus R(-14) \oplus R^2(-15) \oplus R(-16) \oplus R(-17). \end{aligned}$$

Thus, for any $P \in \operatorname{Supp}(Z)$, Theorems 5.4 and 6.4 give

$$\deg_Z(P) = (13-3, 14-3, 15-3, 15-3, 16-3, 17-3) = (10, 11, 12, 12, 13, 14).$$

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