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Lie algebras and 3-transpositions

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ABSTRACT

We describe a construction of an algebra over the field of order 2 starting from a conjugacy class of 3-transpositions in a group. In particular, we determine which simple Lie algebras arise by this construction. Among other things, this construction yields a natural embedding of the sporadic simple group Fi_{22} in the group ${}^2E_6(2)$.

1. Introduction

Let G be a group and D a conjugacy class of involutions generating G such that for all $d, e \in D$ the order of de is equal to 1, 2 or 3. Then D is called a class of 3-*transpositions* in G. Groups generated by 3-transpositions have been introduced by Fischer [7] and studied by various authors since; e.g., see [1,2,5,14,15].

Given a class D of 3-transpositions in a group G, we define the *Fischer space* $\Pi(D)$ to be the partial linear space with D as point set and as lines the triples of points of the form $\{d, e, e^d = d^e\}$, where $d, e \in D$ are non-commuting. Thus, three 3-transpositions on a line generate a subgroup isomorphic to Sym_3, and vice versa, every subgroup Sym_3 containing involutions from D produces a line.

The involutions from D on two intersecting lines in a Fischer space generate a subgroup isomorphic to Sym_4 or to a central quotient of the group $3^{1+2}:2$; e.g. see [5]. The subspace of the Fischer space generated by these two lines is then isomorphic to the dual of the affine plane of order 2 or to the affine plane of order 3, respectively. It was already noticed by Buekenhout, that Fischer spaces are characterized by the property that any two intersecting lines generate such subspaces; see for example [5].

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By $\mathbb{F}_2 D$ we denote the $\mathbb{F}_2$ vector space on the set of *finite* subsets of D, where addition of two sets is defined by the symmetric difference. We identify a point $d \in D$ with the vector $\{d\}$ of this vector space. Note that this makes D a basis of $\mathbb{F}_2 D$, so we can write vectors from $\mathbb{F}_2 D$ simply as linear combinations of the 3-transpositions from D. Let $\mathcal{A}(D)$ be the algebra on $\mathbb{F}_2 D$ whose multiplication $*$ is the linear expansion of the multiplication defined for $d, e \in D$ by

$$d * e := \begin{cases} d + e + f & \text{if } \{d, e, f\} \text{ is a line,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, if d and e commute (in particular, when $d = e$) then $d * e = 0$ and if d and e do not commute then $d * e$ is the line that passes through them. The group $G = \langle D \rangle$ acts on D by conjugation and it preserves lines. Hence it induces a group of automorphisms of both the Fischer space $\Pi(D)$ and the algebra $\mathcal{A}(D)$. Note that the action of G is faithful only if $Z(G) = 1$.

We refer to the algebras $\mathcal{A}(D)$ and their G-invariant quotients as 3-*transposition algebras*. Notice that Simon Norton [13] considered a similar class of algebras, but defined over the reals. Another related class of algebras (related to both Norton's algebras and our 3-transposition algebras) is the class of Majorana algebras introduced by Alexander Ivanov [11]. In the (2B, 3C)-case the Majorana algebras are related to a subclass of groups generated by 3-transpositions. They, furthermore, admit a natural basis with integral structure constants, and after reduction modulo two they produce our 3-transposition algebras.

The 3-transposition algebra $\mathcal{A}(D)$ is endowed with a natural bilinear form defined as follows: for $d, e \in D$ we set $\langle d|e \rangle = 0$, if d and e commute, and $\langle d|e \rangle = 1$, if d and e do not commute. Since D is a basis of $\mathcal{A}(D)$, this extends by linearity to the entire $\mathcal{A}(D)$. Note that $\langle d|d \rangle = 0$ for every $d \in D$, that is, this bilinear form is symplectic. Another important property of the form $\langle \cdot|\cdot \rangle$ is that it *associates* with the algebra product, that is,

$$\langle u | v * w \rangle = \langle u * v | w \rangle$$

for all $u, v, w \in \mathcal{A}(D)$. This will be verified in Proposition 2.1.

Clearly, the action of G leaves the form invariant. Hence the radical $\mathcal{V}(D)$ of $\langle \cdot|\cdot \rangle$ is G-invariant. More interestingly, because the form associates with the algebra product, $\mathcal{V}(D)$ is an ideal of $\mathcal{A}(D)$! We call $\mathcal{V}(D)$ the *vanishing ideal* of $\mathcal{A}(D)$ and we call the elements of $\mathcal{V}(D)$, viewed as subsets of D, the *vanishing sets*.

Note that with the exception of the trivial case, where G is a group of order 2, the form $\langle \cdot|\cdot \rangle$ is nonzero, which means that $\mathcal{V}(D)$ is a proper ideal of $\mathcal{A}(D)$. As we are mainly interested in simple algebras, we will study the algebra $\overline{\mathcal{A}} := \mathcal{A}(D)/\mathcal{V}(D)$, in which vanishing sets are reduced to zero. Note that in general $\overline{\mathcal{A}}$ does not need to be simple; however, it is very nearly so. We will show (see Proposition 2.4) that every G-invariant proper ideal of $\mathcal{A}(D)$ is contained in $\mathcal{V}(D)$. In particular, $\overline{\mathcal{A}}$ is always semisimple.

The following relation τ plays a key role in the theory of Fischer spaces. For an element $d \in D$, we denote by A_d the set of all points collinear to but distinct from d, i.e., $A_d := \{e \in D \mid o(de) = 3\}$. If, for $d, e \in D$, we have $A_d = A_e$, then we write $d\tau e$. The relation τ is an equivalence relation and it is related to the existence of a certain normal 2-subgroup of G; see [5]. Indeed, the subgroup $\tau(G) = \langle de \mid d, e \in D, d\tau e \rangle$ is normal in G. The image $\overline{D}$ of D in $\overline{G} = G/\tau(G)$ is a class of 3-transpositions of $\overline{G}$. Note that for any two elements $d, e \in D$ with $d\tau e$, we have $d + e \in \mathcal{V}(D)$ (that is, $\{d, e\}$ is a vanishing set in D). It follows that the algebras $\mathcal{A}(D)/\mathcal{V}(D)$ and $\mathcal{A}(\overline{D})/\mathcal{V}(\overline{D})$ are isomorphic. Thus, we can restrict our attention to the case where the relation τ is trivial. This is a strong condition that significantly simplifies the possible structure of G.

We will mainly focus on the case where the algebra $\overline{\mathcal{A}}$ is a Lie algebra. It turns out that $\overline{\mathcal{A}}$ is a Lie algebra if and only if the affine planes of the Fischer space are vanishing sets; see Proposition 2.9. This observation makes it possible to determine which simple Lie algebras are (quotients of) 3-transposition algebras.

The main result of this paper is as follows. We use the Atlas notation for groups; in particular, p^n stands for the elementary abelian group of order p^n. The colon : indicates a split extension (semidirect product), where the left side is normal. Furthermore, in all such extensions, the natural action of the complement on the normal subgroup is assumed. Finally, for a Dynkin diagram X_n, $W(X_n)$ stands for the corresponding Weyl group.

1.1. Theorem. *Let G be a nonabelian group generated by a class D of 3-transpositions. Suppose the relation τ on D is trivial and suppose further that $\overline{\mathcal{A}}$ is a simple Lie algebra.*

Then, up to the center of G, we have one of the following:

(a) *D is the unique class of 3-transpositions in $G = 3^n : W(A_n)$; the algebra $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2A_n(2)$.*
(b) *D is the unique class of 3-transpositions in $G = 3^n : W(D_n)$; the algebra $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2D_n(2)$ for odd n and of type $D_n(2)$ for even n.*
(c) *D is the unique class of 3-transpositions in the group $G = 3^n : W(E_n)$ with $n \in \{6, 7, 8\}$; the algebra $\overline{\mathcal{A}}$ is isomorphic to the Lie algebra of type ${}^2E_6(2)$ $(n = 6)$, $E_7(2)$ $(n = 7)$ or $E_8(2)$ $(n = 8)$.*
(d) *D is the class of transvections in $G = \mathrm{SU}_{n+1}(2)$; the algebra $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2A_n(2)$.*
(e) *D is one of the two classes of 126 reflections of $\mathrm{O}_6^-(3)$ and $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2A_5(2)$.*
(f) *D is one of the two classes of 117 reflections of $\mathrm{O}_6^+(3)$ and $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type $D_4(2)$.*
(g) *D is the unique class of 360 3-transpositions in $G = \mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$; the algebra $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type $D_4(2)$.*
(h) *D is the class of 351 reflections of $+$-type in $G = {}^+\Omega_7^+(3)$ and $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2E_6(2)$.*
(i) *D is the unique class of 3510 3-transpositions in Fi_{22} and $\overline{\mathcal{A}}$ is isomorphic to the simple Lie algebra of type ${}^2E_6(2)$.*

For the notation used in part (h) we refer the reader to Section 4.

The above result and its proof provide a geometric argument for the embedding of a central extension of $\mathrm{P}\Omega_6^-(3)$ into $\mathrm{SU}_6(2)$ and of Fi_{22} into ${}^2E_6(2)$. Indeed, we obtain the following result.

1.2. Corollary.

(a) *The group $\mathrm{PSU}_6(2)$ contains a subgroup isomorphic to $\mathrm{P}\Omega_6^-(3)$ generated by root elements (i.e., elations).*
(b) *The group ${}^2E_6(2)$ contains subgroups isomorphic to ${}^+\Omega^+{}_7(3)$ and Fi_{22} generated by root elements.*

The embedding of Fi_{22} into ${}^2E_6(2)$ was first established by Bernd Fischer. It led him to the discovery of the Baby Monster sporadic simple group and hence also to the discovery of the Monster, see [11].

We also notice that with a bit of extra effort we could have included the case where the group G is infinite. Indeed, in [5] all Fischer spaces, finite and infinite, have been classified. The infinite ones turn out to be limits of finite ones. So we only find the infinite dimensional, finitary versions of the unitary and orthogonal Lie algebras as in (a), (b), and (d) of the conclusion of Theorem 1.1, if we allow infinite groups G.

For Fischer spaces of symplectic type (see Section 4) a somewhat similar algebra product was considered by Irving Kaplansky [12] and the first author [4]. This leads to four series of simple Lie algebras in characteristic 2, none of which appears in the conclusion of our Theorem 1.1. We thank Bill Kantor for this observation.

The organization of the paper is as follows. In Section 2 we develop some general theory for 3-transposition algebras. In particular, we prove the algebra $\mathcal{A}$ to be a Lie algebra if and only if every affine plane of the Fischer space is vanishing. In Section 3 we prove various forms of simple Lie algebras of classical type A, D or E over the field $\mathbb{F}_2$ to be 3-transposition algebras. In Section 4 we start with the proof of our main result, Theorem 1.1. We determine those Fischer spaces that give rise to Lie algebras. In Section 5 we complete the proof of Theorem 1.1. In Section 6 we present some additional computational results on the dimensions of (arbitrary) 3-transposition algebras for relatively small groups.

2. 3-transposition algebras

Let G be a group and D a conjugacy class of 3-transpositions generating G. By $\mathcal{A}$ we denote the corresponding 3-transposition algebra $\mathcal{A}(D)$. We will study $\mathcal{A}$ and the action of G on $\mathcal{A}$. Since the center $Z(G)$ of G acts trivially on D and since $de \notin Z(G)$ for all $d, e \in D$, $d \neq e$, we can assume $Z(G) = 1$ whenever convenient.

Recall that the product in $\mathcal{A}$ is defined by

$$d * e := \begin{cases} d+e+f & \text{if } \{d, e, f\} \text{ is a line,} \\ 0 & \text{otherwise,} \end{cases}$$

and the symplectic form on $\mathcal{A}$ is defined by

$$\langle d|e\rangle := \begin{cases} 1 & \text{if } d \text{ and } e \text{ are collinear and distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

We first check that $\langle\cdot|\cdot\rangle$ associates with the product.

2.1. Proposition. *For $v, u, w \in \mathcal{A}$ we have*

$$\langle u|v * w\rangle = \langle u * v|w\rangle.$$

Proof. It suffices to prove that $\langle u|v * w\rangle = \langle u * v|w\rangle$ for $u, v, w \in D$.

Suppose $u, v, w \in D$. If u commutes with v, then $u * v = 0$ hence $\langle u * v|w\rangle = 0$. Thus we have to show that $\langle u|v * w\rangle = 0$. This is clear if w commutes with v, as then $w * v = v * w = 0$. Otherwise, let $t = w^v = v^w$ be the third point on the line through v and w. Note that u commutes either with both w and t, or with neither. In either case, we obtain $\langle u|v * w\rangle = 0$. (Recall that $\mathcal{A}$ is defined over $\mathbb{F}_2$ and hence the values of $\langle\cdot|\cdot\rangle$ are in $\mathbb{F}_2$, too.)

By symmetry, we can now restrict our attention to the case where both u and w do not commute with v. If all three points are together on a line, then clearly $\langle u|v * w\rangle = 0 = \langle u * v|w\rangle$. So we can assume that the three points span a plane. If this plane is an affine plane of order 3, then w is collinear to all three points on the line $u * v$ and u is collinear to all three points on $v * w$. Hence $\langle u|v * w\rangle = 1 = \langle u * v|w\rangle$. If the plane is the dual affine plane of order 2, then w is collinear to two points on the line $u * v$ and u to two points on $v * w$. Therefore $\langle u|v * w\rangle = 0 = \langle u * v|w\rangle$. □

Let $\mathcal{V} = \mathcal{V}(D)$ be the radical of $\langle\cdot|\cdot\rangle$.

2.2. Corollary. *The radical $\mathcal{V}$ is an ideal of $\mathcal{A}$.*

Proof. Since $\mathcal{V}$ is a linear subspace, it suffices to show that $\mathcal{V}$ is closed with respect to multiplication with elements of $\mathcal{A}$. Let $u \in \mathcal{V}$ and $v \in \mathcal{A}$. It follows from Proposition 2.1 that for every $w \in \mathcal{A}$ we have that $\langle u * v|w\rangle = \langle u|v * w\rangle = 0$, since u is in the radical $\mathcal{V}$. Thus, $\langle u * v|w\rangle = 0$ for all $w \in \mathcal{A}$, that is, $u * v$ is in $\mathcal{V}$. □

Recall that we call $\mathcal{V}$ the vanishing ideal and the elements of $\mathcal{V}$, viewed as subsets of D, vanishing sets. Clearly, a finite subset X of D is vanishing if and only if X is perpendicular to every $d \in D$, that is, if for every $d \in D$, the number of elements of X not commuting with d is even. Every finite Fischer space does contain nonempty vanishing subsets. Indeed, if D is finite, then D itself is a vanishing set, as every point d does not commute with an even number of points, two on every line through d. For each of the various types of Fischer spaces we can find more types of vanishing sets.

We now concentrate on G-invariant ideals and quotients of $\mathcal{A}$. It is clear that the symplectic form $\langle\cdot|\cdot\rangle$ is preserved by the action of G obtained by linearly extending the conjugation action of G on D. Thus $\mathcal{V}$ is G-invariant. We claim that $\mathcal{V}$ is the unique maximal among all G-invariant proper ideals of $\mathcal{A}$.

The following observation will be useful.

2.3. Lemma. *Let X be a finite subset of D and $d \in D$. Then*

$$d * X = \langle d|X\rangle d + X + X^d.$$

Proof. The expression in the right side is linear in X, so one only needs to verify it for X of size 1, in which case the claim follows directly from the definition of the product on $\mathcal{A}$. □

We now turn to the main claim.

2.4. Proposition. *Every G-invariant proper ideal of $\mathcal{A}$ is contained in $\mathcal{V}$.*

Proof. Suppose $\mathcal{I}$ is a G-invariant ideal containing an element X not in $\mathcal{V}$. Then we can find an element $d \in D$ with $\langle d|X\rangle \neq 0$. Since $\mathcal{I}$ is an ideal, $d * X \in \mathcal{I}$. On the other hand, by the above lemma, $d * X = d + X + X^d$. Note that $X \in \mathcal{I}$ and also $X^d \in \mathcal{I}$, since $\mathcal{I}$ is G-invariant. It follows that $d \in \mathcal{I}$. However, now by G-invariance, $d^G = D$ is contained in $\mathcal{I}$ and so $\mathcal{I} = \mathcal{A}$. □

The above result has the following important consequence. Recall that $\overline{\mathcal{A}} = \mathcal{A}(D)/\mathcal{V}(D)$.

2.5. Proposition. *If D is finite then $\overline{\mathcal{A}}$ is the direct product of isomorphic simple algebras. In particular, $\overline{\mathcal{A}}$ is semisimple.*

Proof. We first note that G acts on $\overline{\mathcal{A}}$ since $\mathcal{V}$ is G-invariant. Furthermore, in view of Proposition 2.4, the only G-invariant ideals of $\overline{\mathcal{A}}$ are the zero ideal and the entire $\overline{\mathcal{A}}$.

If D is finite then $\mathcal{A}$ (and hence also $\overline{\mathcal{A}}$) is finite dimensional and hence $\overline{\mathcal{A}}$ contains a minimal non-trivial ideal $\mathcal{I}$. For $g \in G$, if $\mathcal{I}^g \neq \mathcal{I}$ then $\mathcal{I} \cap \mathcal{I}^g = 0$ and hence $\mathcal{I}\mathcal{I}^g = 0$. It follows from here that the orbit of $\mathcal{I}$ under G generates an ideal $\mathcal{J}$ that is a direct sum of several conjugates of $\mathcal{I}$. Since $\mathcal{J}$ is manifestly G-invariant and nonzero, the remark at the beginning of the proof implies that $\mathcal{J} = \overline{\mathcal{A}}$. Finally, since $\mathcal{I}$ is a minimal ideal and since every factor of $\overline{\mathcal{A}}$, other than $\mathcal{I}$ itself, annihilates $\mathcal{I}$, it follows that $\mathcal{I}$ is a simple algebra. □

Note that we were careful in this proof not to state that $\overline{\mathcal{A}}$ is the direct product of all ideals $\mathcal{I}^g$. This is, however, almost always the case. Indeed, if some conjugate $\mathcal{I}^g$ does not appear in the direct product decomposition then every factor annihilates $\mathcal{I}^g$, which clearly means that $\mathcal{I}^g$ (and hence also $\mathcal{I}$ and the entire $\overline{\mathcal{A}}$) is a trivial algebra. It easily follows from the definition of the product of $\mathcal{A}$ that $\overline{\mathcal{A}}$ is trivial if and only if every line in the Fischer space $\Pi(D)$ is vanishing. An example of a Fischer space satisfying this property is the dual affine plane of order 2. For more examples see Lemma 5.1.

The above discussion yields the following.

2.6. Corollary. *If the finite Fischer space $\Pi(D)$ has at least one line that is not vanishing then $\overline{\mathcal{A}}$ is the direct product of all its minimal ideals. Furthermore, G transitively permutes the minimal ideals of $\overline{\mathcal{A}}$.*

We also note the following important case.

2.7. Proposition. *Suppose A is a simple quotient algebra of $\mathcal{A}$. If $A \cong \mathcal{A}/\mathcal{I}$ for some G-invariant ideal $\mathcal{I}$, then A is isomorphic to $\overline{\mathcal{A}}$.*

Proof. By Proposition 2.4, $\mathcal{I}$ is contained in $\mathcal{V}$. Simplicity of A implies now that $\mathcal{I} = \mathcal{V}$. □

The following lemma will be used in Section 5.

2.8. Lemma. *Let E be a subspace of the Fischer space $\Pi(D)$ (i.e., E satisfies $d^e \in E$ for all $d, e \in E$). Then the algebra $\overline{\mathcal{A}}$ contains a subalgebra that has a quotient isomorphic to $\mathcal{A}(E)/\mathcal{V}(E)$.*

Proof. The elements of E generate a subalgebra $\mathcal{A}(E)$ of $\mathcal{A}(D)$. As every vanishing subset of D which is contained in E is also a vanishing subset of E, we find that $\mathcal{V}(D) \cap \mathcal{A}(E) \subseteq \mathcal{V}(E)$. Hence, $(\mathcal{A}(E) + \mathcal{V}(D))/\mathcal{V}(D)$ is the subalgebra we are looking for. □

Finally, we need a criterion which allows us to decide when $\overline{\mathcal{A}}$ is a Lie algebra.

2.9. Proposition. *Let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is a Lie algebra if and only if every plane of $\Pi(D)$ isomorphic to the affine plane of order 3 is in $\mathcal{I}$.*

Proof. Clearly the Jacobi identity holds in $\mathcal{A}$ if and only if it holds for any three elements of D. Let $d, e, f \in D$. We will check the Jacobi identity for these elements, that is, the equality

$$(d * e) * f + (e * f) * d + (f * d) * e = 0.$$

If two of the three elements are equal, say $d = e$, then

$$\begin{aligned}(d * e) * f + (e * f) * d + (f * d) * e &= (d * d) * f + (d * f) * d + (f * d) * d \\ &= 0 + (d * f) * d + (d * f) * f \\ &= 0.\end{aligned}$$

Hence, we can assume that the three elements are pairwise distinct.

If $\{d, e, f\}$ is a line, then $(d * e) * f = (d + e + f) * f = d + e + f + d + e + f = 0$. By symmetry, $(d * e) * f + (e * f) * d + (f * d) * e = 0 + 0 + 0 = 0$. Thus, assume that the elements d, e, f are not on a single line.

If there is a point (say d) among d, e, f which is neither collinear to e nor to f (i.e., d commutes with both e and f), then all three terms $(d * e) * f$, $(e * f) * d$ and $(f * d) * e$ are 0. Indeed, not only $d * e = f * d = 0$, but also $(e * f) * d = 0$, since either $e * f = 0$ or else $e * f = e + f + t$ for $t = e^f = f^e$ and so $(e * f) * d = e * d + f * d + t * d = 0 + 0 + 0$, as d is not collinear to t (indeed, if d commutes with e and f then it also commutes with $t = e^f$). So the Jacobi identity holds also in this case.

This leaves the situation where d, e, f are three points in a plane of the Fischer space $\Pi(D)$. Moreover, we can assume that e is collinear to both d and f. First suppose that d, e, f are in a dual affine plane. If $d * f = 0$, then $(e * f) * d$ equals the sum of the two lines on d. Also, $(f * d) * e = 0$ and $(d * e) * f$ equals the sum of the two lines on f. However, the sum of the two lines on d and the two lines on f equals zero, which establishes the Jacobi identity in this case.

If $d * f \neq 0$ then $(d * e) * f + (e * f) * d + (f * d) * e$ is the sum of the lines of the plane on d, e and f. Again this sum is equal to zero.

Now assume that d, e, f are three points inside an affine plane, say π. Then $(d * e) * f$ is equal to the sum of the three lines in the plane on f meeting the line through d and e. But then $(d * e) * f + (e * f) * d + (f * d) * e$, after canceling the lines through d and e, through d and f, and through e and f, each of which appears in the sum twice, equals the sum of the three lines passing through f and d^e, through d and e^f, and through e and f^d. This is the sum of three parallel lines in the plane π and hence it equals to π itself.

Thus, the Jacobi identity holds in $\mathcal{A}/\mathcal{I}$ if and only if every such π is in $\mathcal{I}$. □

Clearly, this gives us the following.

2.10. Corollary. *$\overline{\mathcal{A}}$ is a Lie algebra if and only if every affine plane of order* 3 *in $\Pi(D)$ is a vanishing subset of D.*

3. The normalizer of a Cartan subalgebra as a 3-transposition group

In this section we describe some examples of classical Lie algebras obtained as quotients of 3-transposition algebras.

Let $\mathbb{F}_4$ be the field with 4 elements. Let $\mathfrak{g}$ be a split classical Lie algebra over $\mathbb{F}_4$ of simply laced type, i.e., of type A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$ or E_n $(n = 6, 7, \text{ or } 8)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Φ the corresponding root system. We can then decompose $\mathfrak{g}$ as the sum of the Cartan subalgebra $\mathfrak{h}$ and the corresponding root spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space associated to the root $\alpha \in \Phi$.

For each $\alpha \in \Phi$, let $\mathfrak{x}_\alpha$ be a nonzero element in $\mathfrak{g}_\alpha$ such that the elements $\mathfrak{x}_\alpha$ together with some root elements $\mathfrak{h}_\alpha = [\mathfrak{x}_\alpha, \mathfrak{x}_{-\alpha}] \in \mathfrak{h}$ form a Chevalley basis for $\mathfrak{g}$. (The elements $\mathfrak{h}_\alpha$ are only included in this basis for *simple* roots α, although they make sense for arbitrary $\alpha \in \Phi$.) Taking into account that our field is of characteristic two and the diagram is simply laced, the elements $\mathfrak{x}_\alpha$ and $\mathfrak{h}_\alpha$ satisfy the following:

$$[\mathfrak{x}_\alpha, \mathfrak{x}_\beta] = \begin{cases} \mathfrak{h}_\alpha & \text{if } \beta = -\alpha, \\ \mathfrak{x}_{\alpha+\beta} & \text{if } \alpha + \beta \in \phi, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad [\mathfrak{h}_\alpha, \mathfrak{x}_\beta] = \begin{cases} \mathfrak{x}_\beta & \text{if } \alpha + \beta \in \phi \\ & \text{or } \alpha - \beta \in \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\alpha + \beta \in \phi$ if and only if the angle between α and β is $\frac{2\pi}{3}$ and, similarly, $\alpha - \beta$ is a root if and only if the angle is $\frac{\pi}{3}$. Note also that

$$\mathfrak{h}_\alpha = \mathfrak{h}_{-\alpha},$$

for all $\alpha \in \Phi$, and that

$$\mathfrak{h}_\alpha + \mathfrak{h}_\beta = \mathfrak{h}_{\alpha+\beta},$$

when $\alpha + \beta$ is a root. Using $\mathfrak{h}_\beta = \mathfrak{h}_{-\beta}$, we get that also $\mathfrak{h}_\alpha + \mathfrak{h}_\beta = \mathfrak{h}_{\alpha-\beta}$ when $\alpha - \beta$ is a root.

The Weyl group W of $\mathfrak{g}$ acts on $\mathfrak{g}$ by permuting the elements $\mathfrak{x}_\alpha$ and $\mathfrak{h}_\alpha$ according to how it permutes the roots α. Hence we can view W as a subgroup of the automorphism group of $\mathfrak{g}$ stabilizing $\mathfrak{h}$. Additionally, for $\alpha \in \Phi$ and $\omega \in \mathbb{F}_4^* = \mathbb{F}_4 \setminus \{0\}$, let $h(\alpha, \omega)$ be the automorphism of $\mathfrak{g}$ centralizing $\mathfrak{h}$ and such that

$$\mathfrak{x}_\beta^{h(\alpha,\omega)} = \begin{cases} \overline{\omega}\mathfrak{x}_\beta & \text{if } \beta = \alpha \text{ or } \alpha + \beta \text{ is a root}, \\ \omega\mathfrak{x}_\beta & \text{if } \beta = -\alpha \text{ or } \alpha - \beta \text{ is a root}, \\ \mathfrak{x}_\beta & \text{otherwise.} \end{cases}$$

Here the bar indicates the involutory automorphism of $\mathbb{F}_4$, that is, $\overline{\omega} = \omega^2$ for all $\omega \in \mathbb{F}_4$. Also, $\overline{\omega} = \omega^{-1}$, when $\omega \neq 0$.

It is straightforward to verify that $h(\alpha,\omega)$ is indeed an automorphism of $\mathfrak{g}$ for all $\alpha \in \Phi$ and all $\omega \in \mathbb{F}_4^*$. Clearly,

$$h(\alpha,\mu)h(\alpha,\nu) = h(\alpha,\mu\nu),$$

for all $\mu,\nu \in \mathbb{F}_4^*$. This means that, for a fixed root α, the three automorphisms $h(\alpha,\omega)$ form a cyclic group H_α of order 3. In particular, $h(\alpha,1)$ is the identity automorphism for all $\alpha \in \Phi$. Also, the relation

$$h(-\alpha,\omega) = h(\alpha,\overline{\omega})$$

shows that $H_\alpha = H_{-\alpha}$.

Manifestly, $h(\alpha,\mu)$ and $h(\beta,\nu)$ commute for all $\alpha,\beta \in \Phi$ and all $\mu,\nu \in \mathbb{F}_4^*$. Hence the subgroup H of $\mathrm{Aut}(\mathfrak{g})$ generated by all elements $h(\alpha,\omega)$ is elementary abelian. When $\alpha+\beta$ is a root, we compute that

$$h(\alpha,\omega)h(\beta,\omega) = h(\alpha+\beta,\omega),$$

which means that H is generated by its subgroups H_α for simple roots α, and this yields that H has order 3^n.

The Weyl group W acts on the elements $h(\alpha,\omega)$ via

$$h(\alpha,\omega)^w = h\big(\alpha^w,\omega\big),$$

for all $w \in W$. Hence W permutes the subgroups H_α and normalizes H. We set $G := WH$. Since $H \cong 3^n$ and $H \cap G = 1$, we have $G \cong 3^n : W$.

3.1. Lemma. *Let α be a root and let $w \in W$ be the corresponding reflection. Then $[H,w] = H_\alpha$.*

Proof. For this, it suffices to show that $[h(\beta,\omega),w] \in H_\alpha$ for all roots β and all $\omega \in \mathbb{F}_4^*$. Since $h(\beta,\omega) = h(-\beta,\overline{\omega})$, we can assume that the angle between α and β is at least $\frac{\pi}{2}$.

Let us consider the cases. If the angle is equal $\frac{\pi}{2}$ then $h(\beta,\omega)^w = h(\beta^w,\omega) = h(\beta,\omega)$. Hence in this case $[h(\beta,\omega),w] = 1$. If the angle between α and β is $\frac{2\pi}{3}$ then $h(\beta,\omega)^w = h(\beta^w,\omega) = h(\alpha+\beta,\omega) = h(\alpha,\omega)h(\beta,\omega)$. Hence $[h(\beta,\omega),w] = h(\alpha,\omega) \in H_\alpha$. Finally, if $\beta = -\alpha$ then $h(\beta,\omega)^w = h(\alpha,\omega)$. From this we deduce $[h(\beta,\omega),w] = h(-\alpha,\omega)^{-1}h(\alpha,\omega) = h(\alpha,\omega)^2 = h(\alpha,\overline{\omega}) \in H_\alpha$. Thus, $[H,w] = H_\alpha$, as claimed. □

Let D be the conjugacy class of G containing the class of reflections from W. It follows from our commutator computation that the intersection of D with the coset wH is wH_α. Hence D consists of all elements $d(\alpha,\omega) := wh(\alpha,\omega)$, where α is a root, w is the corresponding reflection, and $\omega \in \mathbb{F}_4^*$.

3.2. Proposition. *The class D is a class of 3-transpositions in G.*

Proof. Clearly, D generates G. Let $d = d(\alpha,\omega) = wh(\alpha,\omega)$ and $d' = d(\alpha',\omega') = w'h(\alpha',\omega')$ be two involutions from D. If $w = w'$ then $dd' \in H_\alpha = H_{\alpha'}$ and hence it has order dividing three. Suppose $w \neq w'$, that is, $\alpha \neq \pm\alpha'$. If w and w' commute then $(dd')^2$ is in H and this element is inverted by both d and d', which implies that it is inverted by both w and w'. Therefore, $(dd')^2 \in H_\alpha \cap H_{\alpha'} = 1$, proving that dd' has order 2.

Similarly, if w and w' do not commute then $(dd')^3$ is an element of H that is inverted by both w and w'. Again, this implies that $(dd')^3 \in H_\alpha \cap H_{\alpha'} = 1$; hence the order of dd' is three. □

Thus, G is a group of 3-transpositions with respect to D. Note that for different types of Lie algebra $\mathfrak{g}$, the group G does not always fall into the same case of the classification [5]. Indeed, following the notation in [5], if $\mathfrak{g}$ is of type A_n, then G has central type **PR 2**; if $\mathfrak{g}$ is of type D_n, then G has central type **PR 5**; and if $\mathfrak{g}$ is of type E_6, E_7, E_8, we get **PR 5**, **PR 9** and **PR 10**, respectively.

We now want to build in $\mathfrak{g}$ a quotient of the 3-transposition algebra $\mathcal{A} = \mathcal{A}(D)$. Since D is a basis of $\mathcal{A}$, to define an $\mathbb{F}_2$-linear mapping $\phi : \mathcal{A} \to \mathfrak{g}$, we can simply specify the image of each element $d(\alpha, \omega)$. We set

$$\phi\big(d(\alpha,\omega)\big) := x(\alpha,\omega) := \omega\mathfrak{x}_\alpha + \overline{\omega}\mathfrak{x}_{-\alpha} + \mathfrak{h}_\alpha,$$

where, as usual, $\alpha \in \Phi$, and ω is an element of $\mathbb{F}_4^*$. Note that $x(\alpha,\omega) = x(-\alpha,\overline{\omega})$, which means that ϕ is well defined. Let Δ be the set of all elements $x(\alpha,\omega)$.

3.3. Proposition. *The map ϕ is a homomorphism.*

Proof. Let $d = d(\alpha,\omega)$ and $d' = d(\alpha',\omega')$ be two elements of D. We need to check that $\phi(d * d') = [\phi(d), \phi(d')]$. If $d = d'$ then both sides of this equality are zero. Hence we can assume that $d \neq d'$. If $\alpha = \pm\alpha'$ then we can assume without loss of generality that $\alpha = \alpha'$ and then $\omega \neq \omega'$. Let $\omega'' \in \mathbb{F}_4^*$ so that $\mathbb{F}_4^* = \{\omega, \omega', \omega''\}$ holds, and set $d'' := d(\alpha, \omega'')$. Then $d * d' = d + d' + d''$ and hence $\phi(d * d') = x(\alpha,\omega) + x(\alpha,\omega') + x(\alpha,\omega'') = h_\alpha$, since $\omega + \omega' + \omega'' = 0$. On the other hand,

$$\begin{aligned}
\big[\phi(d), \phi\big(d'\big)\big] &= \big[x(\alpha,\omega),\, x\big(\alpha,\omega'\big)\big] \\
&= \big[\omega\mathfrak{x}_\alpha + \overline{\omega}\mathfrak{x}_{-\alpha} + \mathfrak{h}_\alpha,\, \omega'\mathfrak{x}_\alpha + \overline{\omega}'\mathfrak{x}_{-\alpha} + \mathfrak{h}_\alpha\big] \\
&= 0 + \omega\overline{\omega}'\mathfrak{h}_\alpha + 0 + \overline{\omega}\omega'\mathfrak{h}_\alpha + 0 + 0 + 0 \\
&= \big(\omega\overline{\omega}' + \overline{\omega}\omega'\big)\mathfrak{h}_\alpha = \mathfrak{h}_\alpha.
\end{aligned}$$

Here we used that $\omega\overline{\omega}' \neq 1$ if $\omega \neq \omega'$, and hence $\omega\overline{\omega}' + \overline{\omega}\omega' = 1$.

Thus, the claim holds when $\alpha = \pm\alpha'$. If α is perpendicular to α' then we proved in Proposition 3.2 that d and d' commute. Hence $d * d' = 0$. On the other hand, by expanding $[\phi(d), \phi(d')] = [x(\alpha,\omega), x(\alpha',\omega')]$, we see that each summand is zero and so $[\phi(d), \phi(d')]$ is also zero. So the claim holds again.

Finally, suppose that α and α' are not parallel and not perpendicular. Without loss of generality we can assume that the angle between α and α' is $\frac{2\pi}{3}$. In this case the order of dd' is three and so $d * d' = d + d' + d''$, where $d'' = d^{d'}$. Hence $d'' = d'dd' = w'h(\alpha',\omega')wh(\alpha,\omega)w'h(\alpha',\omega')$, where w and w' are the reflections with respect to α and α' respectively. Hence,

$$\begin{aligned}
d'' &= w'ww'h\big(\big(\alpha'\big)^{ww'}, \omega'\big)h\big(\alpha^{w'}, \omega\big)h\big(\alpha', \omega'\big) \\
&= w'ww'h\big(\alpha, \omega'\big)h\big(\alpha + \alpha', \omega\big)h\big(\alpha', \omega'\big) \\
&= w'ww'h\big(\alpha + \alpha', \omega\omega'\big) \\
&= d\big(\alpha + \alpha', \omega\omega'\big),
\end{aligned}$$

since $w'' = w'ww'$ is exactly the reflection with respect to $\alpha + \alpha'$. We can also write that $d(\alpha + \alpha', \omega\omega') = d(-\alpha - \alpha', \overline{\omega\omega'}) = d(-\alpha - \alpha', (\omega\omega')^{-1})$. Hence $d'' = d(\alpha'', \omega'')$, where α'' satisfies $\alpha + \alpha' + \alpha'' = 0$ and ω'' satisfies $\omega\omega'\omega'' = 1$.

We can now verify our equality. On the one hand, $\phi(d * d') = \phi(d + d' + d'') = x(\alpha,\omega) + x(\alpha',\omega') + x(\alpha'',\omega'')$. On the other hand,

$$\begin{aligned}\bigl[\phi(d),\phi\bigl(d'\bigr)\bigr] &= \bigl[x(\alpha,\omega),x\bigl(\alpha',\omega'\bigr)\bigr]\\ &= \bigl[\omega\mathfrak{x}_\alpha+\overline{\omega}\mathfrak{x}_{-\alpha}+\mathfrak{h}_\alpha,\omega'\mathfrak{x}_{\alpha'}+\overline{\omega}'\mathfrak{x}_{-\alpha'}+\mathfrak{h}_{\alpha'}\bigr]\\ &= \omega\omega'\mathfrak{x}_{\alpha+\alpha'}+0+\omega\mathfrak{x}_\alpha+0+\overline{\omega}\overline{\omega}'\mathfrak{x}_{-\alpha-\alpha'}+\overline{\omega}\mathfrak{x}_{-\alpha}+\omega'\mathfrak{x}_{\alpha'}+\overline{\omega}'\mathfrak{x}_{-\alpha'}+0\\ &= (\omega\mathfrak{x}_\alpha+\overline{\omega}\mathfrak{x}_{-\alpha})+\bigl(\omega'\mathfrak{x}_{\alpha'}+\overline{\omega}'\mathfrak{x}_{-\alpha'}\bigr)+\bigl(\omega''\mathfrak{x}_{\alpha''}+\overline{\omega}''\mathfrak{x}_{-\alpha''}\bigr).\end{aligned}$$

(Here we used that $\alpha''=-\alpha-\alpha'$ and $\omega''=(\omega\omega')^{-1}=\overline{\omega}\overline{\omega}'$.) The difference between this expression and $x(\alpha,\omega)+x(\alpha',\omega')+x(\alpha'',\omega'')$ is $\mathfrak{h}_\alpha+\mathfrak{h}_{\alpha'}+\mathfrak{h}_{\alpha''}=\mathfrak{h}_{-\alpha''}+\mathfrak{h}_{\alpha''}=0$, hence we again have equality, as claimed. □

As a by-product, our computation yields a description of the Fischer space $\Pi(D)$. The lines of $\Pi(D)$ are of two kinds: (1) For a fixed $\alpha\in\Phi$, the set $\{d(\alpha,\omega)\mid\omega\in\mathbb{F}_4^*\}$ is a line. Clearly, the lines corresponding to α and to $-\alpha$ are the same. (2) For a given triple of roots α, α', α'' satisfying $\alpha+\alpha'+\alpha''=0$, and a given triple of field elements ω, ω', ω'' satisfying $\omega\omega'\omega''=1$, the set $\{d(\alpha,\omega),d(\alpha',\omega'),d(\alpha'',\omega'')\}$ is a line. Again note that if we negate all α's and apply bar to the ω's, we will get the same line. Finally notice that the triples ω, ω', ω'' as above can be of two kinds: either $\omega=\omega'=\omega''$, or $\{\omega,\omega',\omega''\}=\mathbb{F}_4^*$.

Our next goal is to show that the homomorphism ϕ that we introduced behaves naturally with respect to the action of G. This will allow us later to invoke Proposition 2.7.

3.4. Proposition. *The homomorphism ϕ commutes with the action of G.*

Proof. Suppose $d=d(\alpha,\omega)=wh(\alpha,\omega)$. We need to show that $\phi(d^g)=\phi(d)^g$ for each $g\in G$. On the left, g acts by conjugation, while on the right it acts in the natural way, as an automorphism of $\mathfrak{g}$. Clearly, we can restrict g to a generating set of G, say, D. Let $g=d'=d(\alpha',\omega')=w'h(\alpha',\omega')$.

We can use the known structure of the Fischer space $\Pi(D)$ to compute $d^{d'}$, so it only remains to compute the right side of the claimed equality, $\phi(d)^{d'}=x(\alpha,\omega)^{d'}$. There are several cases. If $\alpha=\pm\alpha'$ then we can assume without loss of generality that $\alpha=\alpha'$. Then $d^{d'}=d'':=d(\alpha,\omega'')$, where $\omega''=\omega$ if $\omega'=\omega$ and, otherwise, $\{\omega,\omega',\omega''\}=\mathbb{F}_4^*$. (Note that in both cases ω'' satisfies $\omega\omega'\omega''=1$.) Hence the left hand side is $\phi(d'')=x(\alpha,\omega'')$. The right hand side is

$$\begin{aligned}x(\alpha,\omega)^{d'} &= (\omega\mathfrak{x}_\alpha+\overline{\omega}\mathfrak{x}_{-\alpha}+\mathfrak{h}_\alpha)^{wh(\alpha,\omega')}\\ &= (\omega\mathfrak{x}_{-\alpha}+\overline{\omega}\mathfrak{x}_\alpha+\mathfrak{h}_\alpha)^{h(\alpha,\omega')}\\ &= \omega\omega'\mathfrak{x}_{-\alpha}+\overline{\omega}\overline{\omega}'\mathfrak{x}_\alpha+\mathfrak{h}_\alpha\\ &= \overline{\omega}''\mathfrak{x}_{-\alpha}+\omega''\mathfrak{x}_\alpha+\mathfrak{h}_\alpha\\ &= x\bigl(\alpha,\omega''\bigr).\end{aligned}$$

Hence the claim holds in this case.

If the angle between α and α' is $\frac{\pi}{2}$ then $d^{d'}=d$. Correspondingly, in the right hand side, both w' and $d(\alpha',\omega')$ fix $x(\alpha,\omega)$ and so $x(\alpha,\omega)^{d'}=x(\alpha,\omega)$, yielding the equality.

Finally, if α and α' are neither parallel, nor perpendicular then we can assume without loss that the angle between α and α' is $\frac{2\pi}{3}$. In this case $d^{d'}=d'':=d(\alpha'',\omega'')$, where α'' satisfies $\alpha+\alpha'+\alpha''=0$ and ω'' satisfies $\omega\omega'\omega''=1$. Thus, the left hand side is $\phi(d'')=x(\alpha'',\omega'')$. Let us compute the right hand side:

$$\begin{aligned}x(\alpha,\omega)^{d'} &= (\omega\mathfrak{x}_\alpha+\overline{\omega}\mathfrak{x}_{-\alpha}+\mathfrak{h}_\alpha)^{w'h(\alpha',\omega')}\\ &= (\omega\mathfrak{x}_{-\alpha''}+\overline{\omega}\mathfrak{x}_{\alpha''}+\mathfrak{h}_{\alpha''})^{h(\alpha',\omega')}\end{aligned}$$

Table 1
Possible types and dimensions of the fixed subalgebra $\mathfrak{g}^\sigma$.

Type $\mathfrak{g}$	Type $\mathfrak{g}^\sigma$	Dimension $\mathfrak{g}^\sigma/Z(\mathfrak{g}^\sigma)$
A_n (n odd)	${}^2A_n$	$(n+1)^2-2$
A_n (n even)	${}^2A_n$	$(n+1)^2-1$
D_n (n odd)	${}^2D_n$	$2n^2-n-1$
D_n (n even)	D_n	$2n^2-n-2$
E_6	${}^2E_6$	78
E_7	E_7	132
E_8	E_8	248

$$\begin{aligned} &= \omega\omega' \mathfrak{x}_{-\alpha''} + \overline{\omega}\overline{\omega}' \mathfrak{x}_{\alpha''} + \mathfrak{h}_{\alpha''} \\ &= \overline{\omega}'' \mathfrak{x}_{-\alpha''} + \omega'' \mathfrak{x}_{\alpha''} + \mathfrak{h}_{\alpha''} \\ &= x\bigl(\alpha'', \omega''\bigr), \end{aligned}$$

and so the desired equality holds in all cases. □

As a consequence we have the following.

3.5. Corollary. *The ideal $\mathcal{I} := \ker\phi$ is G-invariant.*

It remains to discuss the image of ϕ. On the one hand, $\operatorname{im}\phi$ can be described as the $\mathbb{F}_2$-span of Δ. On the other hand, let σ be the composition of the Chevalley involution of $\mathfrak{g}$ (fixing $\mathfrak{h}_\alpha$ and sending $\mathfrak{x}_\alpha$ to $\mathfrak{x}_{-\alpha}$ for all $\alpha \in \Phi$) with the field bar automorphism applied to the coordinates with respect to our Chevalley basis of $\mathfrak{g}$. Then σ is a semilinear automorphism of $\mathfrak{g}$ and so the fixed subalgebra $\mathfrak{g}^\sigma := \{x \in \mathfrak{g} \mid x^\sigma = x\}$ of σ is an $\mathbb{F}_2$-form of $\mathfrak{g}$. It is well known that $\mathfrak{g}^\sigma$ is, up to its center, a simple Lie algebra (just like $\mathfrak{g}$ itself is simple up to the center); its type is given in Table 1.

It is immediate that $\Delta \subset \mathfrak{g}^\sigma$, since $x(\alpha,\omega)^\sigma = (\omega\mathfrak{x}_\alpha + \overline{\omega}\mathfrak{x}_{-\alpha} + \mathfrak{h}_\alpha)^\sigma = \overline{\omega}\mathfrak{x}_{-\alpha} + \omega\mathfrak{x}_\alpha + \mathfrak{h}_\alpha = x(\alpha,\omega)$. Therefore, $\operatorname{im}\phi \subseteq \mathfrak{g}^\sigma$.

3.6. Proposition. *We have* $\operatorname{im}\phi = \mathfrak{g}^\sigma$.

Proof. We already know that $\operatorname{im}\phi \subseteq \mathfrak{g}^\sigma$. Since $\mathfrak{g}^\sigma$ is an $\mathbb{F}_2$-form of $\mathfrak{g}$, the two algebras have the same dimension. Hence it suffices to show that Δ has the full rank, that is, Δ spans $\mathfrak{g}$ over $\mathbb{F}_4$.

Working in the A_1 subalgebra $\langle \mathfrak{x}_{-\alpha}, \mathfrak{h}_\alpha, \mathfrak{x}_\alpha\rangle$ for a single root $\alpha \in \Phi$, we find three vectors of Δ in this subalgebra, $x(\alpha,\omega)$, $x(\alpha,\omega')$, and $x(\alpha,\omega'')$, where as usual $\{\omega,\omega',\omega''\} = \mathbb{F}_4^*$. These three vectors are linearly independent, since the corresponding determinant,

$$\begin{vmatrix} \overline{\omega} & \overline{\omega}' & \overline{\omega}'' \\ 1 & 1 & 1 \\ \omega & \omega' & \omega'' \end{vmatrix},$$

is nonzero. (It is in essence a Vandermonde determinant.) Hence $\mathfrak{h}_\alpha$ and $\mathfrak{x}_\alpha$ are contained in the $\mathbb{F}_4$-span of Δ for each α. □

This means that the algebra $\mathfrak{g}^\sigma$ is a quotient of the 3-transposition algebra $\mathcal{A}(D)$. Since the ideal $\mathcal{I} = \ker\phi$ is G-invariant by Corollary 3.5 and since $\mathfrak{g}^\sigma$ modulo its center is simple, an application of Proposition 2.7 yields the following.

3.7. Proposition. *Suppose G and D are as defined in this section. Then $\mathcal{A}(D)/\mathcal{V}(D)$ is isomorphic to the Lie algebra $\mathfrak{g}^\sigma$ modulo its center.*

4. Fischer spaces in which affine planes vanish

The 3-transposition algebras giving rise to Lie algebras come from Fischer spaces in which affine planes are vanishing; see Proposition 2.9. Thus, in order to find all Lie algebras among the 3-transposition algebras, we can restrict our attention to Fischer spaces in which affine planes are vanishing. It is the purpose of this section to classify all such spaces. For this we make use of (parts of) the classification of Fischer spaces as presented in [5].

We introduce some notation. Denote by D_d the set of all points in D not collinear with d, i.e., $D_d := \{e \in D \mid o(de) = 2\}$ (observe that $D = A_d \cup D_d \cup \{d\}$). As we already saw in the introduction, the relation τ on D defined by $d\tau e$ for $d, e \in D$ if and only if $A_d = A_e$ is related to the existence of a normal 2-subgroup in G. The relation θ on D, defined by $d\theta e$ if and only if $D_d = D_e$ is related to normal 3-subgroups. Indeed, the subgroup $\theta(G) = \langle de \mid d, e \in D, d\theta e\rangle$ is normal in G and it is a 3-group.

The space Π is said to be of *symplectic* type if it contains a dual affine plane, but no affine planes. It is called of *orthogonal* type if it does contain an affine plane, but every point not in the plane is collinear with 0, 6 or all points in the plane. This excludes the case where G contains a subgroup generated by elements from D isomorphic to a central quotient of $2^{1+6} : \mathrm{SU}_3(2)$. Such subgroups appear in the unitary groups over $\mathbb{F}_4$. Thus, continuing in this vein, we say that Π is of *unitary* type when it contains an affine plane and a point outside of the plane that is collinear to exactly 8 points of the plane, but does not contain a subspace isomorphic to the Fischer space of the unique class of 3-transpositions in $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$ (a subgroup of all five sporadic examples from [5]). Finally if Π does contains a subspace isomorphic to the Fischer space of $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$, then it is of *sporadic* type.

The Fischer space $\Pi(D)$ (as well as the group $G = \langle D\rangle$) will be called *irreducible* if and only if $\Pi(D)$ is connected, and both relations τ and θ are trivial. We recall the first main theorem from [5].

4.1. Theorem. *(See [5, Theorem 1.1].) Let G be a group generated by a conjugacy class D of* 3*-transpositions. If G is irreducible, then, up to a center, we may identify D with one of the following*:

(a) *the transposition class of a symmetric group*;
(b) *the transvection class of the isometry group of a nondegenerate orthogonal space over* $\mathbb{F}_2$;
(c) *the transvection class of the isometry group of a nondegenerate symplectic space over* $\mathbb{F}_2$;
(d) *a reflection class of the isometry group of a nondegenerate orthogonal space over* $\mathbb{F}_3$;
(e) *the transvection class of the isometry group of a nondegenerate unitary space over* $\mathbb{F}_4$;
(f) *a unique class of involutions in one of the five groups* $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$, $\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}_3$, Fi_{22}, Fi_{23}, *or* Fi_{24}.

The 3-transposition classes of cases (a)–(c) of Theorem 4.1 are of symplectic type, those described in case (d) of orthogonal type, the ones described in (e) of unitary type, and finally, those in (f) of sporadic type.

In all cases except for (d), the class of 3-transpositions is unique. An orthogonal group over $\mathbb{F}_3$, however, contains two classes of reflections. Let (V, Q) be a nondegenerate orthogonal space of dimension n over $\mathbb{F}_3$ and suppose f is the associated bilinear form. If n is finite, then, up to isometry, there are two choices for the form Q, distinguished by their discriminant $\Delta = \pm 1$. This discriminant is, in even dimension, determined by the Witt sign ϵ of Q, which is defined as $+1$ if the Witt index (that is, the dimension of maximal isotropic subspaces) equals $\frac{n}{2}$ and as -1 if the Witt index equals $\frac{n}{2} - 1$. Indeed, for even n we have $\epsilon\Delta = (-1)^{\frac{(n+1)n}{2}}$. We use this formula to define the Witt sign ϵ also in odd dimensions. We write $\epsilon = \pm$ for $\epsilon = \pm 1$. For all n, we denote $\mathrm{O}(V, Q)$ by $\mathrm{O}_n^\epsilon(3)$ and its derived subgroup by $\Omega_n^\epsilon(3)$. For odd n, the ϵ is often left out, since then $\mathrm{O}_n^+(3)$ and $\mathrm{O}_n^-(3)$ are isomorphic.

For each vector $x \in V$ with $Q(x) \neq 0$, the reflection $r_x : v \mapsto v + f(v, x)Q(x)x$ is an element of the orthogonal group $\mathrm{O}(V, Q)$. There are two classes of reflections in $\mathrm{O}(V, Q)$; those of $+$-type with $Q(x) = 1$ and those of minus type with $Q(x) = -1$.

By ${}^\gamma\Omega_n^\epsilon(3)$ we denote the subgroup of $\mathrm{O}(V, Q)$ generated by the reflections of type γ.

The aim of this section is to prove the following result.

4.2. Proposition. *Let Π be a finite connected Fischer space with trivial relation τ. Then the affine planes in Π are vanishing if and only if one of the following holds*:

(a) *Π is of symplectic type.*
(b) *Π is isomorphic to the Fischer space on the unique class of* 3*-transpositions in* $3^n : W(X_n)$*, where X_n is A_n ($n \geqslant 2$), D_n ($n \geqslant 4$) or E_n ($n = 6, 7, 8$).*
(c) *Π is isomorphic to the Fischer space of transvections in the group* $\mathrm{SU}_n(2)$ *with $n \geqslant 3$.*
(d) *Π is isomorphic to the Fischer space of the class of* 126 $+$*-reflections in* $^{+}\Omega_6^{-}(3)$.
(e) *Π is isomorphic to the Fischer space of the class of* 117 $+$*-reflections in* $^{+}\Omega_6^{+}(3)$.
(f) *Π is isomorphic to the Fischer space on the* 351 $+$*-reflections in* $^{+}\Omega^{+}{}_7(3)$.
(g) *Π is isomorphic to the Fischer space of the unique class of* 360 3*-transpositions of* $\mathrm{P}\Omega_8(2) : \mathrm{Sym}_3$.
(h) *Π is isomorphic to the Fischer space of the unique class of* 3*-transpositions of* Fi_{22}.

The proof of this proposition is given in the remainder of this section.

Assume that Π is a Fischer space of some class D of 3-transpositions in a group $G = \langle D\rangle$. Moreover, assume Π satisfies the hypothesis of Proposition 4.2.

If Π is of symplectic type, there is nothing to prove. So, we assume that Π does contain affine planes. The following observation is very useful.

4.3. Lemma. *A Fischer space in which affine planes are vanishing does not contain affine* 3*-spaces.*

Proof. Inside an affine 3-space, an affine plane is not vanishing. □

As already noticed in the introduction, we focus on the Fischer spaces for which the relation τ is trivial. However, let us first assume that not only τ, but also θ is trivial.

Suppose Π is of orthogonal type. In this case D can be identified with a class of reflections in an orthogonal group $\mathrm{O}(W, q)$ for some nondegenerate orthogonal space (W, q) over the field $\mathbb{F}_3$, as in case (d) of Theorem 4.1. However, since by Lemma 4.3 there are no affine 3-spaces in Π, the dimension of W is restricted by $5 \leqslant \dim W \leqslant 7$.

If $\dim(W) = 5$, then only one class of reflections provides a Fischer space with affine planes, namely those reflections, which centralize an orthogonal 4-space of maximal Witt index. This Fischer space is then isomorphic to the Fischer space of the unique class of 3-transpositions in $\mathrm{SU}_4(2)$. Hence we are in case (c) of the proposition.

If $\dim(W) = 6$, then we have case (d) or (e) of Proposition 4.2.

Finally, if $\dim(W) = 7$, then by the condition that a point always centralizes at least a line in a plane not containing the point, only the class of reflections centralizing an orthogonal 6-space with non-maximal Witt index satisfies our assumptions. This class is the class of $+$-reflections in the group $^{+}\Omega_7^{+}(3)$, as described under (f) of Proposition 4.2.

Next assume that Π is of unitary type. Then we are in case (e) of Theorem 4.1 and hence in case (c) of Proposition 4.2. The fact that the affine planes inside all these unitary spaces are vanishing can easily be checked within $\mathrm{SU}_4(2)$ and $\mathrm{SU}_5(2)$.

It remains to consider the case where the Fischer space is sporadic, as in case (g) of Theorem 4.1.

The commuting graph of the Fischer space of $\mathrm{P}\Omega_8(3) : \mathrm{Sym}_3$ consists of three parts, each forming a Fischer space for $\mathrm{P}\Omega_8(3)$. Points of one part are collinear to all points of the other two parts. As each part contains affine planes, we conclude that these planes are not vanishing. Since the Fischer spaces related to Fi_{23} and Fi_{24} contain subspaces isomorphic to the Fischer space of $\mathrm{P}\Omega_8(3) : \mathrm{Sym}_3$, they contain non-vanishing affine planes, too.

The Fischer spaces of $\mathrm{P}\Omega_8(2) : \mathrm{Sym}_3$ and Fi_{22} occur in (f) and (g) of Proposition 4.2, respectively. It remains to prove that affine planes in these Fischer spaces are vanishing.

4.4. Lemma. *Let Π be the Fischer space of* $\mathrm{P}\Omega_8(2) : \mathrm{Sym}_3$*. Then all affine planes in Π are vanishing.*

Proof. The non-collinearity graph of Π partitions into three parts, each part being a Fischer space for $\mathrm{P}\Omega_8(2)$. The latter space is of symplectic type and does not contain affine planes.

Therefore, every affine plane π of Π meets each part of the partition in a line. The three lines form a parallel class of lines in the plane. If d is a point of Π, then d centralizes one or three points of the line of π that lies in the same part as d, and no other point of π. Hence π is vanishing. □

Now assume Π to be the Fischer space of Fi_{22}. To show that the affine planes in Π are vanishing, we consider its subspace on the 693 points not collinear to a fixed point d. This latter space is the Fischer space related to $\mathrm{SU}_6(2)$; e.g., see [1,2,5]. This space has the following properties.

4.5. Lemma. *In the Fischer space of* $\mathrm{SU}_6(2)$ *we have the following*:

(a) *each point is on* 256 *lines*;
(b) *each line is in* 40 *affine planes and* 135 *dual affine planes*;
(c) *there are in total* 59 136 *lines and* 197 120 *affine planes*;
(d) *each point lies in* 2560 *affine planes.*

Proof. The collinearity graph of the Fischer space of $\mathrm{SU}_6(2)$ is strongly regular with parameters $(v, k, \lambda, \mu) = (693, 512, 376, 384)$ (see for example [1,2]). So, each point in this Fischer space is collinear to 512 points. As any pair of collinear points determines a unique line, there are $512/2 = 256$ lines per point. This proves (a).

Fix collinear points d and e. Then each dual affine plane on the line l through d and e contains a unique point collinear to e, but not to d. Moreover, each point with this property determines a unique dual affine plane containing l. Hence, there are $k - \lambda - 1 = 135$ dual affine planes on d and e. Inside these planes we find $1 + 135 = 136$ common neighbors of d and e. So, the remaining $376 - 136 = 240$ common neighbors of d and e are in $240/6 = 40$ affine planes, proving (b).

Double counting now implies (c) and (d). □

The collinearity graph of the Fischer space of Fi_{22} is strongly regular with parameters $(v, k, \lambda, \mu) = (3510, 2816, 2248, 2304)$; see for example [1,2] or [5]. By similar arguments as used in the proof of the previous lemma we obtain:

4.6. Lemma. *In the Fischer space of* Fi_{22} *we have the following*:

(a) *each point is on* 1408 *lines*;
(b) *each line is in* 280 *affine planes and* 567 *dual affine planes*;
(c) *there are in total* 1 647 360 *lines and* 38 438 400 *affine planes*;
(d) *each point lies in* 98 560 *affine planes.*

We are now in a position to prove the following:

4.7. Lemma. *The affine planes in the Fischer space of* Fi_{22} *are vanishing.*

Proof. Fix a point d. As we have seen above, there are 98 560 affine planes containing d, and by Lemma 4.5, there are 197 120 affine planes containing no point collinear to d.

Now fix a point e not collinear to d and a line l through e consisting of points not collinear to d. By Lemma 4.5 there are 693 such points e and 256 such lines on each e. Of the 280 affine planes containing l, there are 40 planes having no point collinear to d. The other 240 planes contain 6 points collinear to d. So, e is in $256 \cdot 240 = 61\,440$ affine planes containing just a line of points not collinear to d. As e lies in 2560 affine planes that contain no point collinear with d (see Lemma 4.5), there are $98\,560 - 61\,440 - 2560 = 34\,560$ planes through e containing 8 points collinear with d. So, we find $\frac{693 \cdot 256 \cdot 240}{3} = 14\,192\,640$ affine planes containing just a line of points not collinear to d, and $693 \cdot 34\,560 = 23\,950\,080$ planes containing a unique point not collinear to d.

However, by now we have accounted for $98\,560 + 197\,120 + 14\,192\,640 + 23\,950\,080 = 38\,438\,400$ affine planes. As this is in fact the total number of affine planes in Π, we conclude that the point d is collinear to 8, 6, or 0 points of any affine plane. This finishes the proof. □

We have covered the case where the relation θ on the set D is trivial. Now assume θ to be non-trivial. In this case the quotient $G/\theta(G)$ is generated by a class $\overline{D}$ of 3-transpositions whose Fischer space does not contain affine planes. For otherwise, we would find affine 3-spaces in the space on D, contradicting Lemma 4.3. Moreover, each θ-equivalence class consists of exactly 3 elements, or else the Fischer space on $\overline{D}$ contains no lines. In the latter case, Π is an affine plane, the Fischer space of $\mathrm{SU}_3(2)'$, as in case (c) of Proposition 4.2.

So we can assume that the Fischer space $\Pi(\overline{D})$ has lines but no affine planes, and furthermore, that every θ-equivalence class has size 3. This implies that, in the notation of [5], the Fischer space $\Pi(D)$ is of orthogonal type with the property that in each affine plane π there is a line l such that each point d collinear with a point of π is also collinear to a point on l. Such spaces are classified in [5, Theorem 6.13]. By this theorem and the condition that each θ-equivalence class has size 3 we find the examples described in case (b) of Proposition 4.2.

Indeed, if, in the notation of [5, Theorem 6.13], G is of type **PR1**, then the absence of an affine 3-space leads to the case where Π is the Fischer space related to $3^n : W(A_n)$ with $n = 1, 2$. If G is of type **PR5**, **PR9**, or **PR10**, then we find the cases where Π is the Fischer space of $3^n : W(E_n)$ with $n = 6$, 7, or 8, respectively. Hence, there only remains the case where G is isomorphic to a group $W(K, \Omega)$, a subgroup of the wreath product of a strong $\{2, 3\}$-group K and Sym_Ω, with Ω a set of size at least 4, as described in **PR2** of [5]. If K is a 2-group, then the corresponding Fischer space does not contain affine planes. If K contains subgroup of order 3^n with $n \geqslant 2$, then the Fischer space will contain affine 3-spaces, which is against our assumptions. Thus, K is either cyclic of order 3 or isomorphic to Sym_3. The first case leads to the Fischer spaces related to $3^n : W(A_n)$, $n \geqslant 4$, the second case to the Fischer spaces related to $3^n : W(D_n)$, $n \geqslant 4$, as described in case (b) of Proposition 4.2.

5. Proof of Theorem 1.1

In this section we prove the main result of this paper, Theorem 1.1. We keep the notation as in the previous sections. In particular, $\overline{\mathcal{A}} = \mathcal{A}(D)/\mathcal{V}(D)$.

5.1. Lemma. *If the Fischer space $\Pi(D)$ is symplectic, then $\overline{\mathcal{A}}$ is an abelian Lie algebra.*

Proof. In this case the lines of the Fischer space are vanishing sets. Indeed, any point $d \in D$ is collinear to 0 or 2 points (different from d) on any line. So in $\overline{\mathcal{A}}$ the product of any two elements is 0. □

The next lemma covers part (d) of Theorem 1.1.

5.2. Lemma. *Suppose D is the class of transvections in the unitary group $\mathrm{SU}_{n+1}(2)$, $n \geqslant 2$. Then $\overline{\mathcal{A}}$ is isomorphic to the Lie algebra of type ${}^2A_n(2)$ modulo its center.*

Proof. The long root elements in the Lie algebra of type ${}^2A_n(2)$ modulo its center satisfy the same relations as the elements of D. Indeed, we can identify these long root elements with the rank 1 matrices in the unitary Lie algebra $\mathfrak{su}_{n+1}(2)$, which are in a one-to-one correspondence with the transvections in $\mathrm{SU}_{n+1}(2)$. In view of simplicity of the Lie algebra of type ${}^2A_n(2)$ modulo its center, the result follows from Proposition 2.7. □

We now turn our attention to the exceptional cases (e)–(i) of Theorem 1.1.

5.3. Lemma. *Suppose D is one of the two classes of reflections in $\mathrm{O}_6^-(3)$. Then the algebra $\overline{\mathcal{A}}$ is isomorphic to the Lie algebra of type ${}^2A_5(2)$ modulo its center.*

Proof. Let (M, Q) be a 6-dimensional orthogonal space over $\mathbb{F}_3$ of Witt index -1. Up to isomorphism, we can identify the elements of D with the non-isotropic 1-spaces $\langle m\rangle$ in M, with $Q(m) = 1$. The 3-transpositions in a parabolic subgroup of $G = \langle D\rangle$ stabilizing an isotropic point of M generate a subgroup of G which, up to its center, is isomorphic to $3^5 : \mathrm{Sym}_6 \cong 3^5 : W(A_5)$. The 3-transposition subalgebra of $\mathbb{F}_2 D$ generated by the 3-transpositions in this subgroup is, modulo its vanishing ideal, isomorphic to the Lie algebra of type ${}^2A_5(2)$ modulo its center; see Proposition 3.7. Hence $\overline{\mathcal{A}}$ contains a subalgebra which has a quotient isomorphic to ${}^2A_5(2)$ modulo its center. Note that this latter algebra has dimension 34.

On the other hand, a computer with the computer algebra system GAP [8] reveals that the dimension of $\overline{\mathcal{A}}$ is 34; see Table 3. Combined, this implies that $\overline{\mathcal{A}}$ is isomorphic to the Lie algebra of type ${}^2A_5(2)$ modulo its center. □

Note that a computer free verification of this can be found in [6].

5.4. Lemma. *Suppose D is one of the two classes of reflections in $\mathrm{O}_6^+(3)$. Then the related algebra $\overline{\mathcal{A}}$ is isomorphic to a Lie algebra of type $D_4(2)$ modulo its center.*

Proof. The existence of a 3-transposition subgroup of $G = \langle D\rangle$ isomorphic to the group $3^4 : W(D_4)$ (as found inside a parabolic subgroup of G) implies that the dimension of $\overline{\mathcal{A}}$ is at least 26. Indeed, Lemma 2.8 and Proposition 3.7 imply that $\overline{\mathcal{A}}$ contains a subalgebra having a quotient isomorphic to the 26-dimensional simple Lie algebra of type $D_4(2)$.

On the other hand, G is a subgroup of $F_4(2)$ generated by long root elements; see [3]. These root elements are the 3-transpositions in G. The Lie algebra of type $F_4(2)$ is not simple. It has a G-invariant ideal of dimension 26 generated by the short roots. The quotient algebra is a simple Lie algebra of type $D_4(2)$. Hence, the algebra $\overline{\mathcal{A}}$ is isomorphic to this latter algebra. □

5.5. Lemma. *Suppose D is the class of $+$-reflections in ${}^+\Omega_7^+(3)$ or the unique class of 3-transpositions in Fi_{22}. Then $\overline{\mathcal{A}}$ is the Lie algebra of type ${}^2E_6(2)$.*

Proof. We note that ${}^+\Omega_7^+(3)$ is a 3-transposition subgroup of Fi_{22}; see [1,3]. Inside the group ${}^+\Omega_7^+(3)$, we find that the reflections in D that stabilize a fixed isotropic vector in the natural orthogonal module for ${}^+\Omega_7^+(3)$ form a Fischer space isomorphic to that of $3^6 : W(E_6)$. This shows that (in both cases) the algebra $\overline{\mathcal{A}}$ contains a subalgebra which has a quotient isomorphic to the Lie algebra of type ${}^2E_6(2)$ of dimension 78.

With help from the computer algebra system GAP [8], we verified that in both cases, the dimension of $\overline{\mathcal{A}}$ equals 78; see Table 4. Thus, $\overline{\mathcal{A}}$ itself must be isomorphic to the simple Lie algebra of type ${}^2E_6(2)$. □

A computer free version of the last part of this proof can be found in [6].

We remark that, in order to obtain the upper bounds for the dimension of $\overline{\mathcal{A}}$ in the above results, we could have used the fact that $\mathrm{P}\Omega_6^-(3)$ embeds into $\mathrm{PSU}_6(2)$ and that both ${}^+\Omega_7^+(3)$ and Fi_{22} embed in ${}^2E_6(2)$. However, with our present approach, these embeddings become consequences of the above results.

Indeed, we can use them to prove Corollary 1.2:

5.6. Corollary.

(a) *The group $\mathrm{PSU}_6(2)$ contains a subgroup isomorphic to ${}^+\Omega_6^-(3)$ generated by root elements.*
(b) *The group ${}^2E_6(2)$ contains subgroups generated by root elements, which are isomorphic to Fi_{22} respectively ${}^+\Omega_7^+(3)$.*

Proof. By the above results, we find that the group $H = \mathrm{PO}_6^-(3)$ embeds into the automorphism group of the unitary Lie algebra $\mathfrak{psu}_6(2)$. Moreover, by construction, the 3-transpositions of G correspond to root elements in $\mathfrak{psu}_6(2)$. However, this implies that H embeds into $G = \mathrm{SU}_6(2)$. Under this embedding, the 3-transpositions of H are root elements in G. Indeed, as we have seen in Section 3, the

3-transpositions in the subgroup $3^5 : \mathrm{Sym}_6$ of H correspond to root elements in $\mathfrak{psu}_6(2)$; and this clearly implies (a).

Similarly, we find Fi_{22} and ${}^+\Omega_7^+(3)$ to be subgroups of the automorphism group of the ${}^2E_6(2)$ Lie algebra defined by their Fischer spaces. Again, the 3-transpositions correspond to root elements in the Lie algebra. This proves (b). □

5.7. Lemma. *Let D be the unique class of 3-transpositions in $\Omega_8^+(2) : \mathrm{Sym}_3$. Then $\overline{\mathcal{A}}$ is isomorphic to the 26-dimensional Lie algebra of type $D_4(2)$.*

Proof. The Fischer space on D can be partitioned into three parts, D_1, D_2 and D_3, each forming a Fischer space of type $\Omega_8^+(2)$, such that any point in one part is collinear with all points in the other parts. This implies that any vanishing set of D_i of even size is also a vanishing set of D.

As the lines of the Fischer subspace D_i are vanishing in D_i, we find that they generate a subspace of $\mathbb{F}_2 D_i$ of codimension 8. Indeed, modulo this subspace we obtain the natural embedding of the Fischer space in the orthogonal space $\mathrm{O}_8^+(2)$; see [9]. However, this implies that the vanishing sets of even size in D_i generate a subspace of codimension 9. As every affine plane is a vanishing set of D of odd size, we find that $\mathcal{V}(D)$ has codimension at most $9+9+9-1=26$ in $\mathcal{A}(D)$.

Now, $G = \mathrm{P}\Omega_8^+(2) : \mathrm{Sym}_3$ embeds in $F_4(2)$ in such a way that the 3-transpositions are long root elements. The Lie algebra of type $F_4(2)$ is not simple. It has a G-invariant ideal of dimension 26 generated by the short root elements. The quotient algebra is a simple Lie algebra of type $D_4(2)$. See also 5.4. This proves that the latter algebra is isomorphic to $\overline{\mathcal{A}}$. □

We completed our proof of Theorem 1.1.

6. Some computational results

In this final section we present some computational results on algebras defined by 3-transpositions. A few of these computations repeat the results obtained above.

We retain the notation of the previous sections. Thus, assume that G is a group generated by its class of 3-transpositions D, and denote by $\mathcal{A}$ the 3-transposition algebra of D. The vanishing ideal is denoted by $\mathcal{V}$. In addition, let $\mathcal{I}_{\mathrm{Aff}}$ denote the ideal of $\mathcal{A}$ generated by the set Aff of affine planes in the Fischer space $\Pi(D)$.

To determine the dimension of $\mathcal{V}$ and hence of $\overline{\mathcal{A}}$ we have used the following proposition.

6.1. Proposition. *Suppose D is finite. Then the dimension of $\overline{\mathcal{A}}$ equals the $\mathbb{F}_2$-rank of the adjacency matrix of the collinearity graph of the Fischer space on D.*

Proof. Note that this matrix, viewed over $\mathbb{F}_2$, coincides with the Gram matrix of the form $\langle\cdot|\cdot\rangle$. Hence the rank of the matrix is equal to the codimension of the radical. □

With this, and given a pair (G, D) as above, implementing a computer program that determines all relevant dimensions is a relatively routine application of linear algebra. Indeed, we implemented this with the help of GAP [8], and computed the dimensions of two quotients of $\mathcal{A}$, one being $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{V}$ and the other $L := A/\mathcal{I}_{\mathrm{Aff}}$. Note that L is the largest quotient of $\mathcal{A}$ that is a Lie algebra.

The source code for our implementation of this can be found on the second author's homepage at http://www.icm.tu-bs.de/~mhorn/. We summarize some results obtained this way in Tables 2–5.

6.2. Some 3-transposition groups with normal 3-group

We consider the examples **PR9-16** from [5]. To perform our calculations, we made use of the presentations of these groups given in [10].

Table 2
Dimensions of algebras induced by unitary groups.

Group G	$\|D\|$	$\dim L$	$\dim \overline{\mathcal{A}}$
$SU_2(2)$	3	3	2
$SU_3(2)$	9	8	8
$SU_4(2)$	45	30	14
$SU_5(2)$	165	45	24
$SU_6(2)$	693	78	34
$SU_7(2)$	2709	119	48
$SU_8(2)$	10 965	176	62
$SU_9(2)$	43 605	249	80
$SU_{10}(2)$	174 933	340	98

Table 3
Dimensions of algebras induced by orthogonal groups over $\mathbb{F}_3$.

Group G	$\|D\|$	$\dim L$	$\dim \overline{\mathcal{A}}$
${}^+\Omega_3^+(3)$	3	3	0
${}^+\Omega_3^-(3)$	6	3	2
${}^+\Omega_4^+(3)$	12	12	2
${}^+\Omega_4^-(3)$	15	15	4
${}^+\Omega_5^+(3)$	36	36	6
${}^+\Omega_5^-(3)$	45	30	14
${}^+\Omega_6^+(3)$	117	52	26
${}^+\Omega_6^-(3)$	126	56	34
${}^+\Omega_7^+(3)$	351	78	78
${}^+\Omega_7^-(3)$	378	0	104
${}^+\Omega_8^+(3)$	1080	0	260
${}^+\Omega_8^-(3)$	1107	0	286
${}^+\Omega_9^+(3)$	3240	0	780
${}^+\Omega_9^-(3)$	3321	0	860
${}^+\Omega_{10}^+(3)$	9801	0	2420
${}^+\Omega_{10}^-(3)$	9882	0	2500

Table 4
Dimensions of algebras induced by sporadic 3-transposition groups.

Group G	$\|D\|$	$\dim L$	$\dim \overline{\mathcal{A}}$
$P\Omega_8^+(2):Sym_3$	360	52	26
$P\Omega_8^+(3):Sym_3$	3240	0	782
Fi_{22}	3510	78	78
Fi_{23}	31 671	0	782
Fi_{24}	306 936	0	3774

We include $\dim \mathcal{V}$ in Table 5, to highlight that it is the same for certain related examples. E.g., the two cases of **PR11** both have $\dim \mathcal{V} = 141$. Both are extensions of $SU_5(2)$, and there we also have $\dim \mathcal{V} = 141$. Also, in the case **PR12**, for $|I| = 1$ or 2 we find that $\dim \mathcal{V} = 28$. Finally, **PR13** is an extension of ${}^+\Omega_5^-(3)$, and for both we have $\dim \mathcal{V} = 31$.

Another interesting tidbit is that $\dim \overline{\mathcal{A}}$ is the same for ${}^+\Omega_9^+(3)$ and **PR15**, and for ${}^+\Omega_{10}^-(3)$ and **PR16**. It would be interesting to know whether the respective semisimple algebras are isomorphic.

The data in the above tables leads to some further interesting questions. For example, the 3-transposition algebra $\overline{\mathcal{A}}$ of $P\Omega_8^+(3):Sym_3$ has the same dimension as the one associated to Fi_{23}. This implies that they are the same. Can this observation lead to a new construction of the group Fi_{23} as a (sub)group of the automorphism group of the 3-transposition algebra associated to $P\Omega_8^+(3):Sym_3$? Is Fi_{23} the full group of automorphisms of this algebra?

Table 5
Dimensions of algebras induced by examples from [5].

Ref. [5]	Group G	$\lvert D\rvert$	$\dim L$	$\dim \overline{\mathcal{A}}$	$\dim \mathcal{V}$
PR9, $\lvert I\rvert = 1$	$3^7 : W(E_7)$	189	133	132	57
PR10, $\lvert I\rvert = 1$	$3^8 : W(E_8)$	360	248	248	112
PR11, $\lvert I\rvert = 1$	$3^{10} : (2 \times \mathrm{SU}_5(2))$	1485	0	1344	141
PR11, $\lvert I\rvert = 2$	$3^{10+10} : (2 \times \mathrm{SU}_5(2))$	13 365	0	13 224	141
PR12, $\lvert I\rvert = 1$	$3^8 : (2^{1+6} : \mathrm{SU}_3(2)')$	324	0	296	28
PR12, $\lvert I\rvert = 2$	$3^{8+8} : (2^{1+6} : \mathrm{SU}_3(2)')$	2916	0	2888	28
PR13, $\lvert I\rvert = 1$	$(3^5 \cdot 3^5) : {}^+\Omega_5^-(3)$	405	0	374	31
PR14, $\lvert I\rvert = 1$	$(3^6 \cdot 3^6) : (3 \cdot {}^+\Omega_6^-(3))$	1134	0	1042	92
PR15, $\lvert I\rvert = 1$	$3^7 \cdot {}^+\Omega_7^-(3)$	1134	0	860	274
PR16, $\lvert I\rvert = 1$	$3^8 \cdot {}^+\Omega_8^-(3)$	3321	0	2500	821

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