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Reconstruction of Profinite Groups from the Closed Normal Hulls of Its Sylow Subgroups and Natural Actions

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INTRODUCTION

The concept of active sums of groups was initially proposed by Tomás in [6]. This construction was done for families of groups with normal actions. Ribenboim in [2] and [3] considered a construction for active families of groups and for active quivers of groups, respectively, and proposed the name of active sums. In [1] Díaz-Barriga and López, using the conditions of [2], considered the active pro-$\mathfrak{C}$-sums for active families of pro-$\mathfrak{C}$-groups with continuous actions. Ribenboim in [4] considered active quivers of pro-$\mathfrak{C}$-groups.

Tomás in [7] proved that if $p_1,..., p_r$ are all the primes that divide the order of a finite group G, and if for each i, N_i is the normal subgroup generated by the union of the p_i-subgroups of Sylow and H_i are disjoint groups isomorphic to N_i and the actions in the H_i are the ones induced by the conjugation of the N_i as normal subgroups of G, then G is isomorphic to the sum of the H_i with this action.

Using this result and the definition of active pro-$\mathfrak{C}$-sum given in [1], a proof of an analogous result for profinite groups is given in this paper.

1

Let G be a profinite group and, for each open normal subgroup U of G, and for each prime p, let W_{pU} be the normal hull of a Sylow p-subgroup of G/U.

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The following facts are obvious:

(i) W_{pU} does not depend on the choice of the Sylow p-subgroup of G/U

(ii) If $p \nmid \circ(G/U)$ then $W_{pU} = \{e\}$

(iii) W_{pU} is the subgroup of G/U generated by the p-elements.

(iv) If $U_1 \subset U_2$ then the epimorphism of G/U_1 onto G/U_2 maps W_{pU_1} onto W_{pU_2}.

Let $W_p = \varprojlim W_{pU}$ where U runs over open normal subgroup of G then by [5, pp. 46–52] we have:

(i) W_p is the closed normal hull of any Sylow p-subgroup of G

(ii) W_p is the closed subgroup generated by the p-elements of G.

Let $\mathfrak{S} = \{W_p\}_{p \in \mathfrak{P}}$ where $\mathfrak{P}$ is the set of primes with the discrete order. If $g \in W_p$ for $p \in \mathfrak{P}$ we define for every $q \in \mathfrak{P}$,

$$\pi_{g,p}(W_q) = W_q$$

and

$$\tau_{g,p}(h) = g^{-1}hg = h^g \qquad \text{for} \quad h \in W_q.$$

It is clear that $\pi_{g,p}$ and $\tau_{g,p}$ satisfy the conditions required in [1] and in consequence $\mathfrak{S}$ is an active family of profinite groups. Let $\boxed{\pi}\, W_p$ be its active pro-sum. As we have that the inclutions $\mathfrak{i}_p$: $W_p \to G$ satisfy the conditions (i) and (ii) of Proposition 1.2 in [1] then there exists a unique continuous homomorphism

$$\gamma: \quad \boxed{\pi}\, W_p \to G,$$

such that if

$$\gamma_p: \quad W_p \to \boxed{\pi}\, W_p$$

is the continuous homomorphism of each W_p in the active pro-sum then

$$\gamma \circ \gamma_p = \mathfrak{i}_p \qquad \text{for every } p \in \mathfrak{P}.$$

As the subgroup generated by $\bigcup_{p \in \mathfrak{P}} \mathfrak{i}_p(W_p)$ is dense in G then γ is surjective. In fact, γ is a continuous isomorphism, as will be proved in the following:

THEOREM. *Let G be a profinite group and $\mathfrak{S} = \{W_p\}_{p \in \mathfrak{P}}$ the active family of the closed normal hulls of the Sylow subgroups of G that were* *previously described. Then there exists a continuous isomorphism of G onto the active pro-sum of the family.*

Proof. We will prove that G has the universal property of the active pro-sum. Then, by proposition 1.2 in [1], G is isomorphic to the active pro-sum and γ is the isomorphism required.

Suppose that G' is a profinite group and that we have continuous homomorphisms

$$\chi_p: \quad W_p \to G'$$

such that

$$\chi_p(g^{-1})\,\chi_q(h)\,\chi_p(g) = \chi_q(g^{-1}hg)$$

for every $p, q \in \mathfrak{P}$, $h \in W_q$, $g \in W_p$

Let U' be an open normal subgroup of G'. For each $p \in \mathfrak{P}$ there exists an open normal subgroup U_p of G such that

$$U_p \cap W_p = \chi_p^{-1}(U')$$

and we have an injective homomorphism

$$\chi_{p,U'}: \frac{W_p}{W_p \cap U_p} \to G'/U'$$

as

$$W_{pU_p} = \frac{W_p U_p}{U_p} \simeq \frac{W_p}{W_p \cap U_p}$$

then if $W_{PU_p} \neq \{e\}$ we have that $p \mid \circ(G'/U')$. Thus if $p \nmid \circ(G'/U')$ then $U_p \supset W_p$.

Let $p_1, \ldots, p_r$ be the distinct primes that divide $\circ(G'/U')$ and let $U = \bigcap_{i=1}^{r} U_{p_i}$. By Tomás theorem, [7], we have an isomorphism

$$\psi_U: G/U \to \boxed{\pi}\ W_{pU}.$$

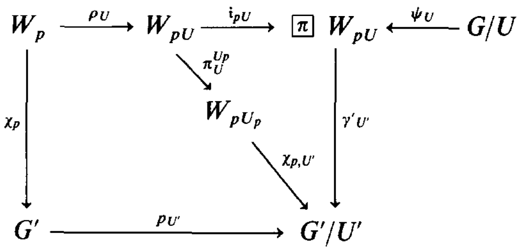
Besides, we have for each p an epimorphism

$$\pi_U^{Up}: W_{pU} \to W_{pU_p}$$

which composited with $\chi_{p,U'}$ satisfies conditions (i) and (ii) of Proposition 1.2 in [1] hence there exists a unique homomorphism

$$\gamma'_{U'}: \boxed{\pi}\ W_{pU} \to G'/U'$$

such that the following diagram is commutative:



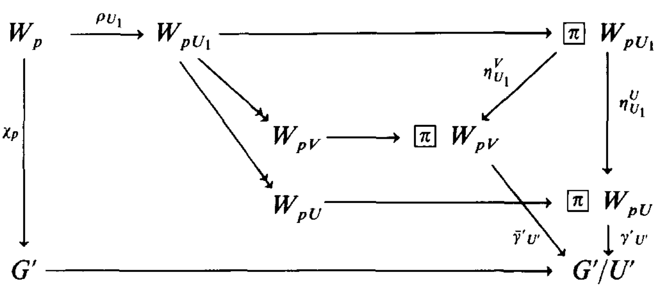
Suppose that V is an open normal subgroup of G and

$$\bar{\gamma}'_{U'}: G/V \to G'/U'$$

is such that

$$\bar{\gamma}'_{U'} \circ \psi_V^{-1} \circ \mathfrak{i}_{pV} \circ \rho_V = p_{U'} \circ \chi_p \qquad \text{for every } p \in \mathfrak{P}$$

Consider $U_1 = U \cap V$, then we have



where $\eta^V_{U_1} = \psi_V \circ \pi^V_{U_1} \circ \psi^{-1}_{V_1}$ and $\eta^U_{U_1} = \psi_U \circ \pi^U_{U_1} \circ \psi^{-1}_{U_1}$ therefore we have homomorphisms $\bar{\gamma}'_{U_1} \circ \eta^V_{U_1}$ and $\gamma'_{U'} \circ \eta^U_{U_1}$ such that

$$\bar{\gamma}'_{U'} \circ \eta^V_{U_1} \circ \mathfrak{i}_{PU_1} \circ \rho_{U_1} = \gamma'_{U'} \circ \eta^U_{U_1} \circ \mathfrak{i}_{pU_1} \circ \rho_{U_1}$$

thus

$$\bar{\gamma}'_{U'} \circ \eta^V_{U_1} \circ \mathfrak{i}_{pU_1} = \gamma'_{U'} \circ \eta^U_{U_1} \circ \mathfrak{i}_{pU_1}$$

and by the universal active pro-sum property we have that

$$\bar{\gamma}'_{U'} \circ \eta^V_{U_1} = \gamma'_{U'} \circ \eta^U_{U_1}. \tag{1}$$

Besides, if U'_1 and U'_2 are open normal subgroups of G' such that $U'_1 \subset U'_2$, then

$$U_{p,1} \cap W_p = \chi_p^{-1}(U'_1) \subset \chi_p^{-1}(U'_2) = U_{p,2} \cap W_p \quad \text{for every } p \in \mathfrak{P},$$

and we have the following commutative diagram:

$$\begin{array}{ccc} \dfrac{W_p}{W_r \cap U_{p,1}} & \xrightarrow{\chi_{p,U'_1}} & G'/U'_1 \\ \downarrow & & \downarrow \\ \dfrac{W_p}{W_p \cap U_{p,2}} & \xrightarrow{\chi_{p,U'_2}} & G'/U'_2 \end{array}$$

Suppose $p_1, \ldots, p_r$ are all the distinct primes which divide $\circ(G'/U'_2)$ and $p_1, \ldots, p_r, q_1, \ldots, q_s$ are the primes which divide $\circ(G'/U'_1)$. If

$$U_1 = \left[\bigcap_{i=1}^{r} U_{p_i,1}\right] \cap \left[\bigcap_{j=1}^{s} U_{q_j,1}\right]$$

and

$$U_2 = \bigcap_{i=1}^{r} U_{p_i,2},$$

then making

$$V = U_1 \cap U_2$$

and as

$$\begin{aligned} &\pi'^{U'_2}_{U'_1} \circ \gamma'_{U'_1} \circ \psi_{U_1} \circ \psi^{-1}_{U_1} \circ \mathrm{i}_{pU_1} \circ \rho_{U_1} \\ &\quad = \pi'^{U'_2}_{U'_1} \circ \gamma'_{U'_1} \circ \mathrm{i}_{pU_1} \circ \rho_{U_1} = \pi'^{U'_2}_{U'_1} \circ \rho_{U'_2} \circ \chi_p = \rho_{U'_2} \circ \chi_p \end{aligned}$$

then by (1) we have

$$\pi'^{U'_2}_{U'_1} \circ \gamma'_{U'_1} \circ \eta^{U_1}_V = \gamma'_{U'_2} \circ \eta^{U_2}_V$$

therefore

$$\pi'^{U'_2}_{U'_1} \circ \gamma'_{U'_1} \circ \psi_{U_1} \circ \pi^{U_1}_V = \gamma'_{U'_2} \circ \psi_{U_2} \circ \pi^{U_2}_V,$$

i.e., the diagram

$$\begin{array}{cccc} & G/U_1 & \xrightarrow{\gamma'_{U_1} \circ \psi_{U_1}} & G'/U' \\ G/V \begin{array}{c} \nearrow^{\pi^{U_1}_V} \\ \searrow_{\pi^{U_2}_V} \end{array} & \downarrow & & \downarrow \pi'^{U_2}_{U'_1} \\ & G/U & \xrightarrow{\gamma'_{U_2} \circ \psi_{U_2}} & G'/U'_2 \end{array}$$

commutes. As a consequence, if we define

$$\gamma': G \to G'$$

such that if $a \in G$ then

$$\rho_{U'}(\gamma'(a)) = \gamma'_{U'}(\psi_U(\rho_U(a))),$$

where U is the previously described subgroup. γ' is a continuous homorphism and if $a \in W_p$ then

$$p_{U'}(\gamma'(\mathrm{i}_p(a))) = \gamma'_{U'}(\psi_U(\rho_U(\mathrm{i}_p(a))) = (\gamma'_{U'} \circ \mathrm{i}_{pU} \circ \rho_U)(a) = (p_{U'} \circ \chi_p)(a)$$

hence

$$\gamma'(\mathrm{i}_p(a)) = \chi_p(a).$$

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