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Note

On an Upper Bound of a Graph's Chromatic Number, Depending on the Graph's Degree and Density

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Grünbaum's conjecture on the existence of k-chromatic graphs of degree k and girth g for every $k \geqslant 3$, $g \geqslant 3$ is disproved. In particular, the bound obtained states that the chromatic number of a triangle-free graph does not exceed $[3(\sigma + 2)/4]$, where σ is the graph's degree.

All graph-theoretic concepts, which are not defined further, are taken from [6, 11]. By graphs we mean nondirected graphs without loops and multiple edges.

Let $G(\sigma, g, \chi)$ be a graph of degree $\sigma(G(\sigma, g, \chi)) = \sigma$, girth $g(G(\sigma, g, \chi)) = g$ and chromatic number $\chi(G(\sigma, g, \chi)) = \chi$. Zykov [12], Tutte [3], and Mycielsky [9] showed that an upper bound for $\chi(G)$ depending only on the graph's density (clique number) $\varphi(G)$ does not exist. Moreover, Erdös [4], and afterward Lovasz [7] constructively, have proved that for every $\chi \geqslant 2$, $g \geqslant 3$, and some σ there exists a $G(\sigma, g, \chi)$.

Chvatal [2] constructed an example of a $G(4, 4, 4)$. Grünbaum conjectured [5] that given any $\chi = \sigma \geqslant 3$, $g \geqslant 3$, there exists a $G(\sigma, g, \chi)$; i.e., the bound given by Brooks' theorem cannot be sharpened for graphs of arbitrarily large girth. In the same paper [5] Grünbaum has constructed an example of a $G(4, 5, 4)$.

But further, we prove a bound for $\chi(G)$ depending on $\sigma(G)$ and $\varphi(G)$, which leads to a contradiction with Grünbaum's conjecture when $\sigma \geqslant 7$ and $g \geqslant 4$.

In recent years a series of papers has appeared (see [1]) dealing with the generalization of the chromatic number notion as the point partition numbers $\alpha_k(G)$ of the graph G.

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DEFINITION 1. Graph G is called k degenerated if its Vizing–Wilf number $W(G) = \max_{G' \subset G} \min_{v \in V(G')} \sigma_{G'}(v) < k$; i.e., every (induced) subgraph G' of G contains a vertex of degree less than k.

DEFINITION 2. $\alpha_k(G)$ is the smallest number of k-degenerated subgraphs which cover the $V(G)$.

Obviously $\alpha_1(G) = \chi(G)$. The quantity $\alpha_2(G)$ is known as the point arboricity of the graph G.

Further, beside the trivial bound $\alpha_k(G) \leqslant \lfloor \sigma(G)/k \rfloor + 1$ we need the following.

LEMMA 1 [1]. *If $\varphi(G) \leqslant \sigma(G)$, $3 \leqslant \sigma(G) \geqslant 2k$, then $\alpha_k(G) \leqslant \lceil \sigma(G)/k \rceil$.*

LEMMA 2 [8]. *Let $\sigma(G) + 1 = \sum_{i=1}^{n} (\sigma_i + 1)$, σ_i being nonnegative integers, and $n \geqslant 1$. Then there exists a covering of $V(G)$ by n subgraphs G_i of the graph G, such that $\sigma(G_i) \leqslant \sigma_i$, $1 \leqslant i \leqslant n$.*

Remark 1. $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, lower and upper integers of x.

Remark 2. At first we did not know about Lovasz's result, and in proving our theorem used a similar but more general result, obtained independently.

LEMMA 2′. *Let $\sum_{i \in I} f_i(v) > \sigma(v)$ for every $v \in V(G)$; then there exists a coloring c: $V(G) \mapsto I$ such that every vertex v is adjacent, with fewer than $f_{c(v)}(v)$ vertices colored by $c(v)$.*

THEOREM. *Let $\sigma(G) \geqslant 3$, $k \geqslant 1$. Then*

$$\alpha_k(G) \leqslant \lfloor \{\sigma(G) - \lfloor [\sigma(G) + 1]/(t + 1) \rfloor / k\} \rfloor + 1,$$

where

$$t = \max\{3, 2k, \lfloor \varphi(G)/k \rfloor \cdot k\}.$$

Proof. Let $s = \lfloor [\sigma(G) + 1]/(t + 1) \rfloor$, $r = \sigma(G) + 1 - s(t + 1)$. Then $\sigma(G) + 1 = s(t + 1) + r$ and, by Lemma 2, the vertices of G can be covered by $s + 1$ subgraphs $G_1, G_2, \ldots, G_{s+1}$ which have

$$\begin{aligned} \sigma(G_i) &\leqslant t, && \text{if } 1 \leqslant i \leqslant s; \\ &\leqslant r - 1, && \text{if } i = s + 1. \end{aligned}$$

By Lemma 1, using $\varphi(G) \leqslant t$, we have

$$\begin{aligned} \alpha_k(G_i) &\leqslant \lceil t/k \rceil, && \text{if } 1 \leqslant i \leqslant s; \\ &\leqslant 1 + \lfloor (r - 1)/k \rfloor, && \text{if } i = s + 1. \end{aligned}$$

Consequently,

$$\alpha_k(G) \leqslant \sum_{i=1}^{s+1} \alpha_k(G_i)$$

$$\leqslant \left\lceil \frac{t}{k} \right\rceil \cdot \left\lfloor \frac{\sigma(G)+1}{t+1} \right\rfloor + \left\lfloor \frac{\sigma(G) - (t+1) \cdot \lfloor [\sigma(G)+1]/(t+1) \rfloor}{k} \right\rfloor + 1$$

$$= \left\lfloor \frac{\sigma(G)}{k} - \left(\frac{t+1}{k} - \left\lceil \frac{t}{k} \right\rceil\right) \left\lfloor \frac{\sigma(G)+1}{t+1} \right\rfloor \right\rfloor + 1.$$

It is easily seen that t/k is an integer, therefore $[(t+1)/k] - \lceil t/k \rceil = 1/k$, which completes the proof.

COROLLARY 1. *If G is a connected graph and is not an odd cycle* ($\sigma(G) \geqslant 2$) *then*

$$\chi(G) \leqslant \sigma(G) - \left\lfloor \frac{\sigma(G) - \max\{3, \varphi(G)\}}{1 + \max\{3, \varphi(G)\}} \right\rfloor.$$

This disproves the conjecture made in [5]:

COROLLARY 2. *If $\sigma(G) \geqslant 7$ and $\varphi(G) \leqslant \lfloor [\sigma(G) - 1]/2 \rfloor$, then $\chi(G) < \sigma(G)$.*

COROLLARY 3. *If $\varphi(G) \leqslant 3$, then $\chi(G) \leqslant \lceil 3[\sigma(G) + 1]/4 \rceil$.*

Remark 3. After this article was submitted for publication we learned that the results of Corollaries 1-3 had been obtained independently by P. A. Catlin.

The authors would like to draw attention to the following general problem, already mentioned by Vizing [10]: Find the exact upper bound for $\chi(G)$ depending on $\sigma(G)$ and $\varphi(G)$ or $g(G)$ and describe all the extremal graphs. It seems to us that the following is true.

Conjecture. If $\sigma(G) \geqslant 9$, and $\varphi(G) < \sigma(G)$, then $\chi(G) < \sigma(G)$.

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