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On the von Neumann and Wigner Potentials

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0. MAIN RESULTS

As a typical example of the existence and non-existence of positive eigenvalues embedded in the continuous spectrum, we consider von Neumann–Wigner potentials. This is a test of the effectivity of our result obtained in [2].

We consider an eigenvalue problem

$$H\psi := (-\Delta + q(x))\,\psi(x) = \lambda\psi(x) \qquad \text{in } \mathbf{R}^n, \tag{0.1}$$

where $q(x)$ satisfies

(Q.1) $q(x)$ is a real-valued continuous function on $\mathbf{R}^n$ and

(Q.2) $q(x) = -(k \sin 2r/r) + O(r^{-1-\varepsilon_0})$ as $r = |x| \to +\infty$ $(\varepsilon_0 > 0)$

with some real constant k. The potential $q(x)$ is not necessarily spherically symmetric.

Let H be the unique selfadjoint realization of the above operator in the Hilbert space $L^2(\mathbf{R}^n)$. It is well known that the essential spectrum of H coincides with the real half line $[0, +\infty)$. Our problem is as follows:

> What is the condition on $k \in \mathbf{R}$ and $\lambda > 0$, under which there exists $q(x)$ satisfying (Q.1) and (Q.2) such that λ is an eigenvalue of H?

In their famous paper [16], von Neumann and Wigner gave an example of $q(x)$ defined in $\mathbf{R}^3$ satisfying (Q.1) and (Q.2) with $k = 8$ such that $\lambda = 1$

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is an eigenvalue of H. Their example (corrected by B. Simon [14]) is

$$q(x) = \frac{-32 \sin r[g(r)^3 \cos r - 3g(r)^2 \sin^3 r + g(r) \cos r + \sin^3 r]}{[1+g(r)^2]^2},$$

$$g(r) = 2r - \sin 2r.$$

Then H has the eigenvalue $\lambda = 1$ with eigenfunction

$$\psi(x) = \frac{\sin r}{r[1+g(r)^2]} \in L^2(\mathbf{R}^3).$$

Moses and Tuan [13] gave another example of $q(x)$ defined in $\mathbf{R}^3$ satisfying (Q.1) and (Q.2) with $k=4$ such that $\lambda = 1$ is an eigenvalue of H. Their example is

$$q(x) = \frac{-32 \sin r[(r+1/2) \cos r - \sin r]}{[1+g(r)]^2}.$$

Then H has the eigenvalue $\lambda = 1$ with eigenfunction

$$\psi(x) = \frac{\sin r}{r[1+g(r)]} \in L^2(\mathbf{R}^3).$$

We can show

PROPOSITION 0.1. *For any $n \in \mathbf{N}$ and any $|k| > 2$, there exists a potential $q(x)$ satisfying* (Q.1) *and* (Q.2) *such that $\lambda = 1$ is an eigenvalue of H.*

PROPOSITION 0.2. *Let $|k| < 2$. Then for any $n \in \mathbf{N}$ and any $q(x)$ satisfying* (Q.1) *and* (Q.2), *$\lambda = 1$ is not an eigenvalue of H.*

Then what happens for $\lambda > 0$, $\neq 1$? In the half line case a complete answer (Proposition 0.3 below) is known [4]; see also [9, 8, pp. 93–95]. In case $n \geqslant 2$ we have Theorem 0.4 below, which is our main theorem.

PROPOSITION 0.3. *Let $\lambda > 0$ and k be real constant. Let $q(x)$ be a real-valued continuous function on $[0, \infty)$ satisfying*

(Q.2)′ $q(x) = -(k \sin 2x/x) + O(x^{-1-\varepsilon_0})$ *as* $x \to +\infty$ $(\varepsilon_0 > 0)$.

Then the equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right)\psi(x) = \lambda\psi(x) \qquad \text{in} \quad 0 \leqslant x < \infty \tag{0.2}$$

has a non-trivial solution $\psi \in L^2([0, +\infty))$ if and only if $|k| > 2$ and $\lambda = 1$.

Note that Proposition 0.1 concerns *some* q but Proposition 0.3 does with *any* q.

THEOREM 0.4. *Let $n \in \mathbf{N}$ be arbitrary. Let*

$$|k| < \begin{cases} \lambda + \sqrt{\lambda} & (\lambda \geqslant 1) \\ 1 + \sqrt{\lambda^2 - \lambda + 1} & (1 > \lambda > 0). \end{cases} \tag{0.3}$$

Then for any $q(x)$ satisfying (Q.1) *and* (Q.2), $\lambda > 0$ *is not an eigenvalue of H.*

Remark 0.5 A. Devinatz, R. Moeckel and P. Rejto [6] show (see also A. Devinatz and P. Rejto [7]) that any finite closed interval in $(0, 1) \cup (1, \infty)$ contains at most only a finite number of eigenvalues for any k. Also Ben Artzi and A. Devinatz [5] show that in case q is spherically symmetric there can exist at most one eigenvalue in $(0, \infty)$ at 1.

Proposition 0.2 is nothing but a corollary of Theorem 0.4.

Our main purpose of this paper is to show Theorem 0.4 applying our result obtained in [2]. This is an improvement of the result of the second author [15, Example 5.8], who showed that if

$$|k| < \sqrt{\lambda^2 + \lambda},$$

for any $q(x)$ satisfying (Q.1) and (Q.2), λ is not an eigenvalue of H.

Remark 0.6. Assume that in (0.1), $q(x)$ satisfies (Q.1) and

$$\text{(Q.2)}'' \quad q(x) = -(ka \sin 2ar/r) + O(r^{-1-\varepsilon_0}) \text{ as } r = |x| \to +\infty \ (\varepsilon_0 > 0)$$

instead of (Q.2) with some constant $a > 0$. Then putting $x' = ax$ and omitting the $'$ we have (0.1) with λ replaced by λ/a^2 and with $q(x)$ satisfying (Q.2). Thus we may replace (Q.2) and λ with (Q.2)$''$ and λ/a^2, respectively, in Propositions 0.1–0.3 and Theorem 0.4.

1. PROOF OF PROPOSITIONS 0.1 AND 0.3

Proposition 0.1 can be easily shown from Proposition 0.3 as follows.

Proof of Proposition 0.1. Let $|k| > 2$ and let $\psi(x)$ be a spherically symmetric function and put

$$u(r) := r^{(n-1)/2} \psi(x).$$

Then Eq. (0.1) with $\lambda = 1$ and $q(x) = -(k \sin 2r/r)$ reduces to

$$-u''(r) + \left\{ -\frac{k \sin 2r}{r} + 4^{-1}(n-1)(n-3)\, r^{-2} \right\} u(r) = u(r),$$

which has a non-trivial solution $u \in L^2([1, +\infty))$ by Proposition 0.3 so that $\psi \in L^2(\{x \in \mathbf{R}^n \mid |x| > 1\})$. Let $R_1 > 1$ be a constant such that $\psi(x) \neq 0$ on $|x| = R_1$. Let $\tilde{\psi}(x)$ be a smooth function such that $\tilde{\psi}(x) \neq 0$ for $|x| \leqslant R_1$ and $\tilde{\psi}(x) = \psi(x)$ for $|x| > R_1$. Put

$$\tilde{q}(x) := \begin{cases} 1 + (\Delta\tilde{\psi})(x)/\tilde{\psi}(x) & \text{for} \quad |x| \leqslant R_1 \\ q(x) & \text{for} \quad |x| > R_1. \end{cases}$$

Then $\tilde{q}(x)$ and $\tilde{\psi}(x)$ satisfy all the requirements of Proposition 0.1.

Remark 1.1 We can prove Proposition 0.1 directly without the help of the *if* part of Proposition 0.3. Noting the argument given in the proof of Proposition 0.1, it suffices to construct a non-trivial function $u(r) \in L^2([1, \infty))$ and a real valued continuous function $q(r)$ in $[1, \infty)$ satisfying (Q.2)′ and

$$-u''(r) + \{q(r) + 4^{-1}(n-1)(n-3)\, r^{-2}\}\, u(r) = u(r), \qquad r \in [1, \infty). \tag{1.1}$$

This can be achieved as follows.

Let α be a positive constant. In case $k > 2$, we put

$$g(r) := 2r - \sin 2r = 4 \int_0^r \sin^2 t\, dt,$$

$$w(r) := (1 + g(r)^{\alpha})^{-k/(4\alpha)},$$

$$u(r) := w(r) \sin r.$$

Then by (1.1) we have

$$\begin{aligned} &q(r) + 4^{-1}(n-1)(n-3)\, r^{-2} \\ &\quad = 1 + \frac{u''}{u} = 2\,\frac{w'}{w}\frac{\cos r}{\sin r} + \frac{w''}{w} \\ &\quad = \frac{-4k \sin r}{(1 + g(r)^{\alpha})^2} \\ &\qquad \times [(g^{2\alpha-1} + g^{\alpha-1}) \cos r - (1 + k/4)\, g^{2\alpha-2} \sin^3 r + (\alpha - 1)\, g^{\alpha-2} \sin^3 r]. \end{aligned}$$

In case $k < -2$ we put

$$g(r) := 2r + \sin 2r = 4\int_0^r \cos^2 t\, dt,$$
$$w(r) := (1 + g(r)^\alpha)^{k/(4\alpha)},$$
$$u(r) := w(r) \cos r.$$

By a similar calculation, we have

$$q(x) + 4^{-1}(n-1)(n-3)\, r^{-2}$$
$$= \frac{-4k \cos r}{(1+g(r)^\alpha)^2}$$
$$\times [(g^{2\alpha-1} + g^{\alpha-1}) \sin r - (k/4 - 1)\, g^{2\alpha-2} \cos^3 r - (\alpha - 1)\, g^{\alpha-2} \cos^3 r].$$

In both cases, we have $q(r) = -kr^{-1} \sin 2r + O(r^{-1-\min\{\alpha,\, 1\}})$ as $r \to +\infty$ since $\alpha > 0$, and we have $u(r) \in L^2([1, \infty))$ since $\alpha > 0$ and $|k| > 2$.

Note that in case $n = 3$, the modification near the origin is not necessary. In this case, putting $k = 8$ and $\alpha = 2$, we have the von Neumann and Wigner example, and putting $k = 4$ and $\alpha = 1$, we have the Moses and Tuan example.

In order to prove Proposition 0.3, it is sufficient to show the following two propositions.

PROPOSITION 1.2. *Let $\lambda \neq 1$. Then Eq.* (0.2) *has no non-trivial solution in* $L^2([0, \infty))$.

PROPOSITION 1.3. *Let $\lambda = 1$. Then Eq.* (0.2) *has no non-trivial solution in* $L^2([0, \infty))$ *if and only if* $|k| \leqslant 2$.

The next lemma is a special case of [4, Theorem 2.1], from which Proposition 1.2 follows easily, since $q(x)$ satisfies the conditions (i)–(iii) of g if $\lambda \neq 1$. (If $\lambda = 1$, the function $g_2(x)$ does not exist.)

LEMMA 1.4. *Let $\lambda > 0$. Assume that*

(i) *$g(x)$ is a real valued continuous function in* $[1, \infty)$;

(ii) *the integrals*

$$\int_x^\infty g(t)\, dt, \quad g_1(x) =: \int_x^\infty g(t) \cos 2\sqrt{\lambda}\, t\, dt, \quad g_2(x) := \int_x^\infty g(t) \sin 2\sqrt{\lambda}\, t\, dt$$

exist;

(iii) $$\int_1^\infty |g(t)\, g_i(t)|\, dt < \infty \qquad (i = 1, 2).$$

Then for any non-trivial solution $\psi(x)$ *of the equation*

$$-\psi''(x) + g(x)\,\psi(x) = \lambda\psi(x) \qquad \text{in} \quad x \geqslant 1, \tag{1.2}$$

there exist constants A, B *with* $A > 0$ *such that*

$$\psi(x) = A\cos(\sqrt{\lambda}\,x + B) + o(1), \qquad \psi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x + B) + o(1)$$

as $x \to \infty$.

We owe the next lemma to Atkinson [4, Theorem 4.9, Remark in Sect. 1.3] to prove Proposition 1.3.

Lemma 1.5. *Assume that*

(i) $h_1(x) > 0$ *is a real-valued* C^1 *function in* $[1, \infty)$;
(ii) $\int_1^\infty h_1(t)\,dt = \infty$, $\int_1^\infty h_1(t)^2\,dt < \infty$ *and* $\int_1^\infty |h_1'(t)|\,dt < \infty$;
(iii) $h_2(x)$ *is a real-valued continuous function of class* $L^1([1, \infty))$.

Then the equation

$$-\psi''(x) - \{h_1(x)\cos 2x + h_2(x)\}\,\psi(x) = \psi(x) \qquad \text{in} \quad x \geqslant 1 \tag{1.3}$$

has two solutions ψ_1, ψ_2 *such that*

$$\psi_1(x) = \exp\left[\frac{1}{4}\int_1^x h_1(t)\,dt\right]\left\{\cos\left(x + \frac{\pi}{4}\right) + o(1)\right\}$$

$$\psi_2(x) = \exp\left[-\frac{1}{4}\int_1^x h_1(t)\,dt\right]\left\{\cos\left(x - \frac{\pi}{4}\right) + o(1)\right\}$$

as $x \to \infty$.

Let $h_1(x) = |k|/(x + (\pi/4)\operatorname{sgn} k)$ in (1.3), where $\operatorname{sgn} k$ means the sign of k. Then the above lemma shows that Eq. (1.3) has no non-trivial solution in $L^2([1, \infty))$ if and only if $|k| \leqslant 2$. On the other hand changing the variable from $x + (\pi/4)\operatorname{sgn} k$ to new x in (1.2) with $\lambda = 1$, we have Eq. (1.3). Hence we have Proposition 1.3.

2. PROOF OF THEOREM 0.4

The following is a simplified version of [2, Theorems 1.1 and 1.3]; see also Remark 1.3(3) and Remark 1.4(2) there.

PROPOSITION 2.1. *Let $q_1(x) \in C^1(\mathbf{R}^n)$ and $q_2(x) \in C^0(\mathbf{R}^n)$ be real-valued bounded functions. We consider the equation*

$$(-\Delta + q_1(x) + q_2(x))\, \psi(x) = 0 \qquad \text{in } \mathbf{R}^n. \tag{2.1}$$

Assume that there exist some bounded smooth real-valued functions $\sigma(r)$ and $\eta(r)$ and some positive constants δ and τ satisfying

$$\sigma(r) \geqslant \delta; \tag{2.2}$$

$$\eta(r) \leqslant 2; \tag{2.3}$$

$$\limsup_{r \to \infty} \left[r\partial_r q_1(x) + \eta(r)\, q_1(x) + \sigma(r)^{-1}\, |rq_2(x) - Q'(r)|^2 \right] < 0, \tag{2.4}$$

where $\partial_r = \partial/(\partial r)$, $Q(r) := 4^{-1}(\eta(r) - \sigma(r))$;

$$rq_2(x) - Q'(r) \text{ is bounded}; \tag{2.5}$$

$$\lim_{r \to \infty} \exp\left(\int_1^r \frac{\tau - \eta(t)}{t}\, dt \right) = 0; \tag{2.6}$$

$$\exp\left(-\int_1^r \frac{\sigma(t) + \eta(t)}{2t}\, dt \right) \notin L^1(1, \infty). \tag{2.7}$$

Then Eq. (2.1) has no non-trivial $L^2(\mathbf{R}^n)$ solution.

Let σ_0, η_0, u, v be real constants satisfying

$$\begin{cases} \sigma_0 + \eta_0 \leqslant 2, & \eta_0 > 0, \quad \sigma_0 > 0, \\ 2\,|u| < \sigma_0, & \eta_0 + 2\,|v| \leqslant 2, \end{cases} \tag{2.8}$$

and let τ be a constant satisfying $0 < \tau < \eta_0$. We put

$$\sigma(r) := \sigma_0 + 2u \cos 2r, \qquad \eta(r) := \eta_0 + 2v \cos 2r.$$

Then

$$Q'(r) = (u - v) \sin 2r, \qquad \sigma(r) + \eta(r) = \sigma_0 + \eta_0 + 2(u + v) \cos 2r$$

and it is easy to see that the assumptions in Proposition 2.1 except (2.4) and (2.5) are satisfied since

$$\int_1^r t^{-1} \cos 2t\, dt = \left[\frac{1}{2t} \sin 2t \right]_1^r + \int_1^r \frac{1}{2t^2} \sin 2t\, dt$$

is a bounded function of $r \in [1, +\infty)$.

According to the idea of [15, Example 5.8], we put

$$\begin{cases} q_1 := -(k+s)\dfrac{\sin 2r}{r} - \lambda, \\ q_2 := s\dfrac{\sin 2r}{r} + O(r^{-1-\varepsilon_0}) \qquad \text{as} \quad r \to \infty \end{cases}$$

with some real constant s. Then Eq. (0.1) reduces to Eq. (2.1).

We can easily see that (2.5) holds.

Let us check the assumption (2.4). We may assume $0 < \varepsilon_0 \leqslant 1$ in (Q.2) without loss of generality.

$$\begin{aligned} & r\partial_r q_1 + \eta(r)\, q_1 + \sigma(r)^{-1} \, |rq_2 - Q'|^2 \\ &\quad = r\left[-2(k+s)\frac{\cos 2r}{r} + (k+s)\frac{\sin 2r}{r^2}\right] \\ &\qquad - (\eta_0 + 2v\cos 2r)\left[(k+s)\frac{\sin 2r}{r} + \lambda \right] \\ &\qquad + \sigma(r)^{-1} \, |s \sin 2r + O(r^{-\varepsilon_0}) - (u-v)\sin 2r|^2 \\ &\quad = -2(k+s)\cos 2r - \lambda(\eta_0 + 2v\cos 2r) \\ &\qquad + \sigma(r)^{-1} \, (s-u+v)^2 \sin^2 2r + O(r^{-\varepsilon_0}) \\ &\quad = -\sigma(r)^{-1} \, [(2u\cos 2r + \sigma_0)\{2(k+s+v\lambda)\cos 2r + \eta_0\lambda\} \\ &\qquad - (s-u+v)^2 \, (1-\cos^2 2r)] + O(r^{-\varepsilon_0}) \\ &\quad = -\sigma(r)^{-1} \, [\{(s-u+v)^2 + 4u(k+s+v\lambda)\} \cos^2 2r \\ &\qquad + 2\{(k+s+v\lambda)\,\sigma_0 + u\eta_0\lambda\}\cos 2r + \sigma_0\eta_0\lambda - (s-u+v)^2] + O(r^{-\varepsilon_0}). \end{aligned}$$

We put

$$\begin{aligned} f(X) := {} & \{(s-u+v)^2 + 4u(k+s+v\lambda)\} \, X^2 \\ & + 2\{(k+s+v\lambda)\,\sigma_0 + u\eta_0\lambda\} \, X + \sigma_0\eta_0\lambda - (s-u+v)^2. \end{aligned}$$

Then (2.4) is satisfied if and only if

$$f(X) > 0 \qquad \text{for any} \quad X \in [-1, 1]. \tag{2.9}$$

We claim

PROPOSITION 2.2. *Let real numbers* k *and* $\lambda > 0$ *be given. The following three assertions are equivalent.*

(1) *There exist real constants* s, u, v, σ_0 *and* η_0 *such that*

$$q_1 = -(k+s)\frac{\sin 2r}{r} - \lambda,$$

$$q_2 = s\frac{\sin 2r}{r} + O(r^{-1-\varepsilon_0}) \qquad as \quad r \to \infty,$$

$$\sigma(r) = \sigma_0 + 2u\cos 2r \qquad and \qquad \eta(r) = \eta_0 + 2v\cos 2r$$

satisfy the assumptions of Proposition 2.1.

(2) *There exist real constants* s, u, v, σ_0, *and* η_0 *saysfying* (2.8) *and* (2.9).

(3) λ *and* k *satisfy* (0.3).

The preceding argument guarantees the equivalence of the assertions (1) and (2). We will show the equivalence of the assertions (2) and (3) in the followings. Then as a corollary, we have Theorem 0.4.

We put $f(X)$ in the form

$$f(X) = AX^2 + 2BX + C,$$

where

$$A := (s-u+v)^2 + 4u(k+s+v\lambda),$$

$$B := (k+s+v\lambda)\,\sigma_0 + u\eta_0\lambda,$$

$$C := \sigma_0\eta_0\lambda - (s-u+v)^2.$$

LEMMA 2.3. *The inequality* (2.9) *holds if and only if one of the following holds*:

(a) $f(1) > 0$, $f(-1) > 0$ *and* $A \leqslant C$

(b) $C > 0$ *and* $B^2 < AC$.

Note that in case (b), the condition $f(\pm 1) > 0$ holds automatically, and that (a) and (b) are not mutually exclusive.

Proof. We have the following:

(i) Let $A \leqslant 0$. Then $f(X)$ is concave and we can see that (2.9) holds if and only if $f(1) > 0$ and $f(-1) > 0$.

(ii) Let $0 < A \leqslant B$ $(0 < A \leqslant -B)$. Then the axis of the parabola $y = f(X)$ lies in the left(right) side of the line $X = -1(X = 1)$. (2.9) holds if and only if $f(-1) > 0$ $(f(1) > 0)$, and then $f(1) > 0$ $(f(-1) > 0)$ also holds (respectively).

(iii) Let $|B| < A$. Then the axis of the parabola lies in $-1 < X < -1$, and (2.9) holds if and only if $B^2 < AC$, and then $C = f(0) > 0$.

Assume that (2.9) holds. If $A \leqslant |B|$, then by (i) and (ii), $f(\pm 1) = A + C \pm 2B > 0$, which with $A \leqslant |B|$ implies $A \leqslant C$. Thus (a) holds. If $A > |B|$, then by (iii), (b) holds.

Conversely, assume (a). If $A \leqslant |B|$, then by (i) and (ii), (2.9) holds. If $A > |B|$, then by $A \leqslant C$ we have $C > 0$ and $B^2 < AC$, so by (iii), (2.9) holds. It is obvious that (b) implies (2.9).

We will show the following two lemmas, which yield the equivalence of the assertions (2) and (3) in Proposition 2.2.

LEMMA 2.4. *There exist real constants s, u, v, σ_0, and η_0 satisfying* (2.8) *and* (a) *in Lemma* 2.3 *if and only if λ and k satisfy* (0.3).

LEMMA 2.5. *There exist real constants s, u, v, σ_0, and η_0 satisfying* (2.8) *and* (b) *in Lemma* 2.3 *if and only if λ and k satisfy* (0.3).

In the proof of the above two lemmas, we will use implicitly

LEMMA 2.6. *Let $f_1(x)$ and $f_2(x)$ be real valued continuous functions defined in an open interval I and satisfy*

$$f_1(x) < f_2(x) \qquad \text{for any} \quad x \in I. \tag{2.10}$$

Let μ be a real constant. Then there exists $x \in I$ such that $f_1(x) < \mu < f_2(x)$ if and only if

$$\inf\{f_1(x) \mid x \in I\} < \mu < \sup\{f_2(x) \mid x \in I\}. \tag{2.11}$$

Proof. The *only if* part is obvious. In order to prove the *if* part, assume (2.11). Then the sets $G_1 = \{x \in I \mid f_1(x) < \mu\}$ and $G_2 = \{x \in I \mid \mu < f_2(x)\}$ are non-empty and open by the continuity of $f_1(x)$ and $f_2(x)$. Since $G_1 \cup G_2 = I$ by (2.10), we have $G_1 \cap G_2 \neq \emptyset$ by the connectivity of I.

Proof of Lemma 2.4. We put

$$S := s + v \qquad \text{and} \qquad K := k + v\lambda - v.$$

Then the arbitrariness of s reflects to the arbitrariness of S and we have

$$\begin{aligned} f(X) &= AX^2 + 2BX + C, \\ A &= (S-u)^2 + 4u(K+S), \\ B &= (K+S)\,\sigma_0 + u\eta_0\lambda, \\ C &= \sigma_0\eta_0\lambda - (S-u)^2. \end{aligned}$$

Since

$$f(\pm 1) = (\sigma_0 \pm 2u)\{\eta_0\lambda \pm 2(K+S)\}$$

and $\sigma_0 \pm 2u > 0$ by (2.8), the condition $f(\pm 1) > 0$ is equivalent to

$$-\eta_0\lambda/2 - K < S < \eta_0\lambda/2 - K. \tag{2.12}$$

The condition $C \geqslant A$ is equivalent to

$$S^2 \leqslant \sigma_0\eta_0\lambda/2 - 2uK - u^2. \tag{2.13}$$

The real number S satisfying (2.12) and (2.13) exists if and only if one of the followings holds:

$$K + \eta_0\lambda/2 \leqslant 0 \qquad \text{and} \qquad (K+\eta_0\lambda/2)^2 < \sigma_0\eta_0\lambda/2 - 2uK - u^2, \tag{2.14}$$

$$|K| < \eta_0\lambda/2 \qquad \text{and} \qquad \sigma_0\eta_0\lambda/2 - 2uK - u^2 \geqslant 0, \tag{2.15}$$

$$K - \eta_0\lambda/2 \geqslant 0 \qquad \text{and} \qquad (K-\eta_0\lambda/2)^2 < \sigma_0\eta_0\lambda/2 - 2uK - u^2. \tag{2.16}$$

In case (2.14), we have $K + \eta_0\lambda/2 \leqslant 0$ and

$$\psi_{1-}(u; \sigma_0, \eta_0) < K + \eta_0\lambda/2 < \psi_{1+}(u; \sigma_0, \eta_0),$$

where we put

$$\psi_{1\pm}(u; \sigma_0, \eta_0) := -u \pm \sqrt{\eta_0\lambda(\sigma_0/2 + u)}.$$

We can see

$$\begin{aligned} &\inf\{\psi_{1-}(u; \sigma_0, \eta_0) \mid 2\,|u| < \sigma_0\} = \psi_{1-}(\sigma_0/2; \sigma_0, \eta_0) = -\sigma_0/2 - \sqrt{\sigma_0\eta_0\lambda}, \\ &\sup\{\psi_{1+}(u; \sigma_0, \eta_0) \mid 2\,|u| < \sigma_0\} \geqslant \psi_{1+}(0; \sigma_0, \eta_0) > 0 \end{aligned}$$

so that there exists u satisfying $2\,|u| < \sigma_0$ and (2.14) if and only if

$$-(\sigma_0 + \eta_0\lambda)/2 - \sqrt{\sigma_0\eta_0\lambda} < K \leqslant -\eta_0\lambda/2.$$

The second condition of (2.15) holds with $u = 0$ so that there exists u satisfying $2\,|u| < \sigma_0$ and (2.15) if and only if $|K| < \eta_0\lambda/2$.

In case (2.16), we have $K - \eta_0\lambda/2 \geqslant 0$ and

$$\psi_{2-}(u;\sigma_0,\eta_0) < K - \eta_0\lambda/2 < \psi_{2+}(u;\sigma_0,\eta_0),$$

where we put

$$\psi_{2\pm}(u;\sigma_0,\eta_0) := -u \pm \sqrt{\eta_0\lambda(\sigma_0/2 - u)}.$$

We can see

$$\inf\{\psi_{2-}(u;\sigma_0,\eta_0) \mid 2\,|u| < \sigma_0\} \leqslant \psi_{2-}(0;\sigma_0,\eta_0) < 0,$$
$$\sup\{\psi_{2+}(u;\sigma_0,\eta_0) \mid 2\,|u| < \sigma_0\} = \psi_{2+}(-\sigma_0/2;\sigma_0,\eta_0) = \sigma_0/2 + \sqrt{\sigma_0\eta_0\lambda}$$

so that there exists u satisfying $2\,|u| < \sigma_0$ and (2.16) if and only if

$$\eta_0\lambda/2 \leqslant K < (\sigma_0 + \eta_0\lambda)/2 + \sqrt{\sigma_0\eta_0\lambda}.$$

Therefore the real number u satisfying $2\,|u| < \sigma_0$ and one of (2.14)–(2.16) exists if and only if

$$-\psi_3(\sigma_0;\eta_0) < K < \psi_3(\sigma_0;\eta_0), \tag{2.17}$$

where we put

$$\psi_3(\sigma_0;\eta_0) := (\sigma_0 + \eta_0\lambda)/2 + \sqrt{\sigma_0\eta_0\lambda}.$$

Since ψ_3 is a monotone increasing function of σ_0, there exists a real number σ_0 satisfying $0 < \sigma_0 \leqslant 2 - \eta_0$ and (2.17) if and only if (2.17) holds with $\sigma_0 = 2 - \eta_0$. Remembering the definition $K = k + v\lambda - v$, we have

$$-\psi_3(2-\eta_0;\eta_0) - (\lambda - 1)v < k < \psi_3(2-\eta_0;\eta_0) - (\lambda - 1)v.$$

There exists a real number v satisfying $|v| \leqslant 1 - \eta_0/2$ and the above inequality if and only if

$$\begin{aligned} |k| &< \psi_3(2-\eta_0;\eta_0) + |\lambda - 1|\,(1 - \eta_0/2) \\ &= \begin{cases} \lambda + \sqrt{(2-\eta_0)\,\eta_0\lambda} & \text{for } \lambda \geqslant 1, \\ 2 - \eta_0 - \lambda + \eta_0\lambda + \sqrt{(2-\eta_0)\,\eta_0\lambda} & \text{for } 1 > \lambda > 0. \end{cases} \end{aligned}$$

When η_0 runs over $0<\eta_0<2$, the last side of the above formula takes its maximum value $\lambda+\sqrt{\lambda}$ or $\sqrt{\lambda^2-\lambda+1}+1$ at $\eta_0=1$ or at $\eta_0=1-(1-\lambda)/\sqrt{\lambda^2-\lambda+1}$, if $\lambda\geqslant 1$ or $1>\lambda>0$, respectively. Thus, there exists a real number η_0 satisfying $0<\eta_0<2$ and the above inequality if and only if k satisfies (0.3), and we complete the proof of Lemma 2.4.

Proof of Lemma 2.5. We put $\tilde{S}:=S-u=s-u+v$. Then the arbitrariness of s reflects to the arbitrariness of $\tilde{S}$. The condition $C>0$ is written as

$$\tilde{S}^2<\sigma_0\eta_0\lambda. \tag{2.18}$$

The condition $B^2<AC$ is written as

$$\sigma_0^2(K+\tilde{S}+u)^2-2u(\sigma_0\eta_0\lambda-2\tilde{S}^2)(K+\tilde{S}+u)+\tilde{S}^4-\sigma_0\eta_0\lambda\tilde{S}^2+(u\eta_0\lambda)^2<0,$$

that is,

$$F_-(u;\tilde{S},\sigma_0,\eta_0)<K<F_+(u;\tilde{S},\sigma_0,\eta_0), \tag{2.19}$$

where we put

$$F_\pm(u;\tilde{S},\sigma_0,\eta_0):=\sigma_0^{-2}\{(\sigma_0(\eta_0\lambda-\sigma_0)-2\tilde{S}^2)u\pm\sqrt{D}\}-\tilde{S},$$

$$D(u;\tilde{S},\sigma_0,\eta_0):=\tilde{S}^2(\sigma_0^2-4u^2)(\sigma_0\eta_0\lambda-\tilde{S}^2).$$

In order for K satisfying (2.19) to exist, it must be $D>0$, which is equivalent to $\tilde{S}\neq 0$ by (2.8) and (2.18).

We put

$$\begin{aligned} u_\pm &:= \pm\frac{\sigma_0(\eta_0\lambda-\sigma_0)-2\tilde{S}^2}{2\sqrt{(\eta_0\lambda-\sigma_0)^2+4\tilde{S}^2}}\\ &=\pm\frac{\sigma_0}{2}\cdot\frac{\sigma_0(\eta_0\lambda-\sigma_0)-2\tilde{S}^2}{\sqrt{4\tilde{S}^2(\sigma_0\eta_0\lambda-\tilde{S}^2)+\{\sigma_0(\eta_0\lambda-\sigma_0)-2\tilde{S}^2\}^2}}.\end{aligned}$$

Then $|u_\pm|<\sigma_0/2$ by (2.18) and $\tilde{S}\neq 0$. Now vary u on $|u|<\sigma_0/2$ by (2.8), and $F_\pm(u;\tilde{S},\sigma_0,\eta_0)$ takes at $u=u_\pm$ its supremum/infimum value

$$G_\pm(\tilde{S};\sigma_0,\eta_0):=\pm\tfrac{1}{2}\sqrt{(\eta_0\lambda-\sigma_0)^2+4\tilde{S}^2}-\tilde{S},$$

respectively. So we have

$$G_-(\tilde{S};\sigma_0,\eta_0)<K<G_+(\tilde{S};\sigma_0,\eta_0).$$

Since $G_{\pm}(\tilde{S}; \sigma_0, \eta_0)$ is a non-increasing function of $\tilde{S}$, its supremum/infimum is given at $\tilde{S} = \mp\sqrt{\sigma_0 \eta_0 \lambda} \neq 0$ by (2.18), respectively, and

$$G_{\pm}(\mp\sqrt{\sigma_0 \eta_0 \lambda}; \sigma_0, \eta_0) = \pm\{(\sigma_0 + \eta_0 \lambda)/2 + \sqrt{\sigma_0 \eta_0 \lambda}\}$$

leads (2.17). The argument following (2.17) in the proof of Lemma 2.4 proves the present lemma, again.

3. RESULTS ANALOGOUS TO PREVIOUS CRITERIA

Let

$$q(x) = V_1(x) + V_2(x) + V_3(r), \tag{3.1}$$

where $V_1(x) \in C^1(\mathbf{R}^n)$, $V_2(x) \in C^0(\mathbf{R}^n)$, and $V_3(r) \in C^0(\mathbf{R})$ are real-valued functions. We put

$$Q(r) = \int_1^r t V_3(t)\, dt.$$

We assume

$$V_1(x) \text{ is a bounded function,} \tag{3.2}$$

$$\limsup_{|x| \to \infty} V_1(x) = 0, \tag{3.3}$$

$$L := \limsup_{r \to \infty} r \partial_r V_1(x) < \infty, \tag{3.4}$$

$$K := \limsup_{r \to \infty} |r V_2(x)| < \infty, \tag{3.5}$$

$$M := \limsup_{r \to \infty} Q(r) - \liminf_{r \to \infty} Q(r) < 1. \tag{3.6}$$

Note that we have $L \geqslant 0$ by (3.3) and (3.4).

Many authors gave a number $\Lambda \geqslant 0$ as a function of K, L, and M such that if $\lambda > \Lambda$, then λ is not an eigenvalue of the operator H defined by (0.1) and (3.1); see [3, Remark 1.2]. So we remark that the smaller Λ, the better results for non-existence of eigenvalue of H we have.

Kato [11] considered the case $V_1(x) \equiv V_3(r) \equiv 0$ and gave

$$\Lambda_K = K^2.$$

Agmon [1] considered the case $V_3(r) \equiv 0$ and $K = 0$. Applying his result we have

$$\Lambda_A = \frac{L}{2}.$$

Eastham and Kalf [8, p. 187] considered the case $V_3(r) \equiv 0$ and gave

$$\Lambda_{EK} = \frac{1}{2}\{K^2 + L + \sqrt{K^2(K^2+2L)}\} = \left[\frac{K + \sqrt{K^2+2L}}{2}\right]^2.$$

Khosrovshahi *et al.* [12] gave under the condition $M < 4^{-1}$

$$\Lambda_{KLP} = \max\left\{\left[\frac{K + \sqrt{K^2 + 2L(1-2M)}}{2(1-2M)}\right]^2, \frac{2K^2 + L(1-4M)}{2(1-4M)^2}\right\}.$$

Kalf and Kumar [10] gave under the condition $M < 2^{-1}$

$$\Lambda_{KK} = \left[\frac{K + \sqrt{K^2 + 2L(1-2M)}}{2(1-2M)}\right]^2.$$

The authors [3] have given

$$\Lambda_{AU} = \frac{1}{2} \cdot \frac{1}{1-M^2}\,[K^2 + L + \sqrt{K^2(K^2+2L) + L^2M^2}].$$

We can show that $\Lambda_{KLP} \geqslant \Lambda_{KK} \geqslant \Lambda_{AU}$.

Let $q(x)$ be the one satisfying (Q.1) and (Q.2). We put

$$V_1(x) = -(k+s+t)\,\frac{\sin 2r}{r},$$

$$V_2(x) = \frac{s \sin 2r}{r} + O(r^{-1-\varepsilon_0}),$$

$$V_3(r) = \frac{t \sin 2r}{r}$$

with some real constants s and t satisfying $|t| < 1$. Then (3.1)–(3.6) are satisfied by $L = 2\,|k+s+t|$, $K = |s|$ and $M = |t|$.

Remembering that we aim at getting a small Λ, we denote by $\Lambda_* = \Lambda_*(s, t; k)$ one of the above Λ's coresponding to the above decomposition of $q(x)$ and by $\Lambda_*^0(k)$ the $\inf_{s,t} \Lambda_*(s, t; k)$, where s and t run over a set specified below and * stands for one of K, A, EK, KLP, KK and AU. Even in the case that s and/or t are fixed (e.g., Λ_K), we use the notations $\Lambda_*(s, t; k)$ in order to unify the notations and in order to recognize the points (s, t) where the infimums are attained. We remark again that *if* $\lambda > \Lambda_*^0(k)$, *then* λ *is not an eigenvalue of* H *with any* $q(x)$ *satisfying* (Q.1) *and* (Q.2).

We will show

$$\Lambda_K^0(k) = \Lambda_K(-k, 0; k) = |k|^2, \tag{3.7}$$

$$\Lambda_A^0(k) = \Lambda_A(0, 0; k) = |k|, \tag{3.8}$$

$$\Lambda_{EK}^0(k) = \begin{cases} \Lambda_{EK}(-k, 0; k) = k^2 & \text{for} \quad 1 \geqslant |k|, \\ \Lambda_{EK}(0, 0; k) = |k| & \text{for} \quad |k| \geqslant 1, \end{cases} \tag{3.9}$$

$$\Lambda_{KLP}^0(k) = \begin{cases} \Lambda_{KLP}(0, -k; k) = 0 & \text{for} \quad |k| < \frac{1}{4}, \\ \Lambda_{KLP}(-k, 0; k) = k^2 & \text{for} \quad \frac{1}{4} \leqslant |k| \leqslant 1, \\ \Lambda_{KLP}(0, 0; k) = |k| & \text{for} \quad |k| \geqslant 1, \end{cases} \tag{3.10}$$

$$\Lambda_{KK}^0(k) = \begin{cases} \Lambda_{KK}(0, -k; k) = 0 & \text{for} \quad |k| < \frac{1}{2}, \\ \Lambda_{KK}(-k, 0; k) = k^2 & \text{for} \quad \frac{1}{2} \leqslant |k| \leqslant 1, \\ \Lambda_{KK}(0, 0; k) = |k| & \text{for} \quad |k| \geqslant 1, \end{cases} \tag{3.11}$$

$$\Lambda_{AU}^0(k) = \begin{cases} \Lambda_{AU}(0, -k; k) = 0 & \text{for} \quad |k| < 1, \\ \lim_{t \to -1/k \pm 0} \Lambda_{AU}(-k - t, t; k) = k^2 - 1 = 0 & \\ & \text{for} \quad k = \pm 1, \\ \Lambda_{AU}(-k + 1/k, -1/k; k) = k^2 - 1 & \\ & \text{for} \quad 1 < |k| \leqslant \dfrac{1 + \sqrt{5}}{2}, \\ \Lambda_{AU}(0, 0; k) = |k| & \text{for} \quad |k| \geqslant \dfrac{1 + \sqrt{5}}{2}. \end{cases} \tag{3.12}$$

Remark 3.1. Noting that the minimum of the right hand side of (0.3) is $1 + \sqrt{3}/2$, we can compare the first lines of (3.10)–(3.12) as follows: any $\lambda > 0$ is not an eigenvalue of H with any $q(x)$ satisfying (Q.1) and (Q.2) if $|k| < 1/4$ according to KLP, if $|k| < 1/2$ according to KK, if $|k| \leqslant 1$ according to AU, and if $|k| < 1 + \sqrt{3}/2$ according to our Theorem 0.4.

The results are illustrated in the following Fig. 1.

Let us show (3.7)–(3.12).

In Kato's case $s = -k$, $t = 0$ and $K = |k|$ so that we have (3.7).

In Agmon's case $s = t = 0$ and $L = 2\,|k|$ so that we have (3.8).

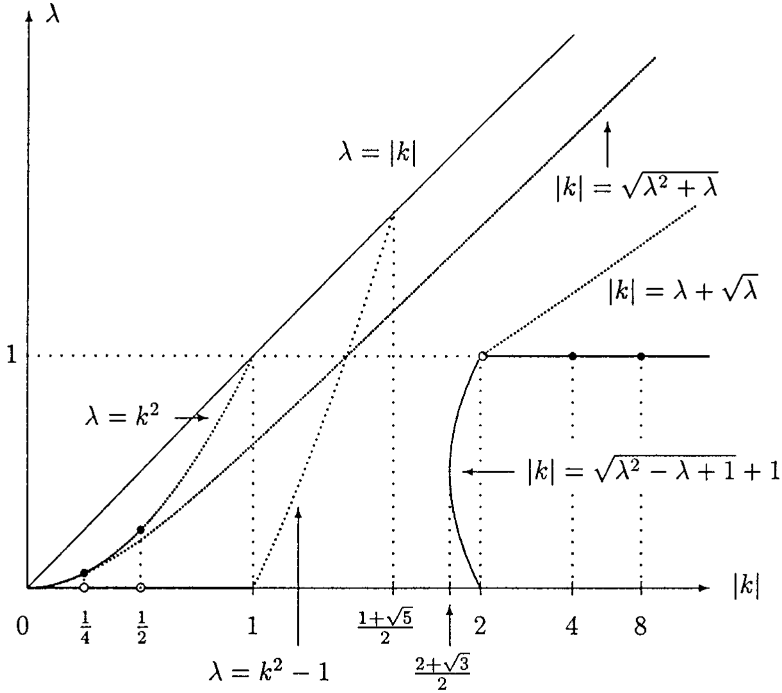

FIGURE 1

In the sequel, we may assume $k \geqslant 0$ without loss of generality since $\Lambda_*(-s, -t; -k) = \Lambda_*(s, t; k)$ for $* = EK$, KLP, KK, and AU.

In the calculation of Λ_{EK}, Λ_{KLP}, and Λ_{KK}, we will use the following lemma, whose proof can be seen by means of elementary consideration of the calculus.

LEMMA 3.1. (1) *Let*

$$g_1(x, y) = \tfrac{1}{2}\{|x| + \sqrt{x^2 + 4\,|x+y|}\}, \qquad y \geqslant 0.$$

Then we have

$$\inf_{x \in \mathbf{R}} g_1(x, y) = \begin{cases} g_1(-y, y) = y & \text{for} \quad 1 \geqslant y \geqslant 0, \\ g_1(0, y) = \sqrt{y} & \text{for} \quad y \geqslant 1. \end{cases} \tag{3.13}$$

(2) *Let*

$$g_2(x, y) = x^2 + |x + y|, \qquad y \geqslant 0.$$

Then we have

$$\inf_{x\in\mathbf{R}} g_2(x, y) = \begin{cases} g_2(-y, y) = y^2 & \textit{for} \quad 1/2 \geqslant y \geqslant 0, \\ g_2(-1/2, y) = y - 1/4 & \textit{for} \quad y \geqslant 1/2. \end{cases} \tag{3.14}$$

Let us treat Eastham and Kalf's Λ, where $-\infty < s < \infty$ and $t = 0$. Since

$$\Lambda_{EK} = \Lambda_{EK}(s, 0; k) = g_1(s, k)^2,$$

we have (3.9) by virtue of (3.13).

Let us consider the case of Khosrovshahi, Levine, and Payne, where $-\infty < s < \infty$ and $|t| < 1/4$. We put

$$a(t) = (1 - 2\,|t|)^{-1}, \qquad b(t) = (1 - 4\,|t|)^{-1}.$$

Then $a(t) > 0$, $b(t) > 0$ and

$$\Lambda_{KLP}(s, t; k) = \max\{ g_1(a(t)s, a(t)(k+t))^2, g_2(b(t)s, b(t)(k+t))\}.$$

Since $\Lambda_{KLP}(s, t; k) \geqslant 0$ and

$$\Lambda_{KLP}(0, -k; k) = 0 \qquad \text{for} \quad |k| < \tfrac{1}{4},$$

$$\Lambda_{KLP}(-k, 0; k) = k^2 \qquad \text{for any } k,$$

$$\Lambda_{KLP}(0, 0; k) = |k| \qquad \text{for any } k,$$

we have

$$\Lambda^0_{KLP} = 0 \qquad \text{for} \quad |k| < \tfrac{1}{4}$$

and

$$\Lambda^0_{KLP} \leqslant \begin{cases} k^2 & \text{for} \quad \tfrac{1}{4} \leqslant |k| \leqslant 1, \\ |k| & \text{for} \quad |k| \geqslant 1. \end{cases} \tag{3.15}$$

We will show that the reverse inequality holds in (3.15). Then we have (3.10). First, we assume $1/4 \leqslant k \leqslant 1/2$. In this case we use

$$\Lambda_{KLP}(s, t; k) \geqslant g_2(b(t)s, b(t)(k+t)).$$

Under our assumptions $|t| < 1/4$ and $k \geqslant 1/4$, we have $b(t)(k+t) > 0$ and by (3.14) we have

$$\inf_{s\in\mathbf{R}} \Lambda_{KLP}(s, t; k) \geqslant \begin{cases} (b(t)(k+t))^2 & \text{if} \quad b(t)(k+t) \leqslant 1/2, \\ (b(t)(k+t)) - 1/4 \geqslant 1/4 & \text{if} \quad b(t)(k+t) \geqslant 1/2, \end{cases}$$

so that we have

$$\Lambda^0_{KLP} \geqslant \min\left\{\inf\left\{\left(\frac{k+t}{1-4\,|t|}\right)^2 \,\middle|\, \frac{2k-1}{2} \leqslant t \leqslant \frac{1-2k}{6}\right\}, \frac{1}{4}\right\}$$
$$= \min\left\{\left(\frac{k+t}{1-4\,|t|}\right)^2 \Bigg|_{t=0}, \frac{1}{4}\right\}$$
$$= \min\left\{k^2, \frac{1}{4}\right\} = k^2$$

since $k \leqslant 1/2$.

Next, let $k \geqslant 1/2$. In this case we use

$$\Lambda_{KLP}(s, t; k) \geqslant g_1(a(t)s, a(t)(k+t))^2.$$

By (3.13), we have

$$\inf_{s \in \mathbf{R}} g_1(a(t)s, a(t)(k+t))^2 = \begin{cases} a(t)^2\,(k+t)^2 & \text{if} \quad 1 \geqslant a(t)(k+t), \\ a(t)(k+t) & \text{if} \quad a(t)(k+t) \geqslant 1, \end{cases}$$

that is,

$$\inf_{s \in \mathbf{R}} \Lambda_{KLP}(s, t; k) \geqslant \begin{cases} \left(\dfrac{k+t}{1-2\,|t|}\right)^2 & \text{if} \quad k+t \leqslant 1-2\,|t|, \\ \dfrac{k+t}{1-2\,|t|} & \text{if} \quad k+t \geqslant 1-2\,|t|. \end{cases}$$

If $1/2 \leqslant k \leqslant 1$, we have

$$\Lambda^0_{KLP} \geqslant \min\left\{\inf\left\{\left(\frac{k+t}{1-2\,|t|}\right)^2 \,\middle|\, k+t \leqslant 1-2\,|t|,\ |t| < \frac{1}{4}\right\},\right.$$
$$\left.\inf\left\{\frac{k+t}{1-2\,|t|} \,\middle|\, k+t \geqslant 1-2\,|t|,\ |t| < \frac{1}{4}\right\}\right\}$$
$$\geqslant \min\left\{\inf\left\{\left(\frac{k+t}{1-2\,|t|}\right)^2 \,\middle|\, k-1 \leqslant t \leqslant \frac{1-k}{3},\ |t| < \frac{1}{4}\right)\right\}, 1\right\}$$
$$= \min\left\{\left(\frac{k+t}{1-2\,|t|}\right)^2 \Bigg|_{t=0}, 1\right\} = k^2.$$

If $k \geqslant 1$, then $k+t \geqslant 1-2\,|t|$ and we have

$$\Lambda^0_{KLP} \geqslant \inf\left\{\frac{k+t}{1-2\,|t|}\,\middle|\,|t|<\frac{1}{4}\right\} = \left.\frac{k+t}{1-2\,|t|}\right|_{t=0} = k.$$

Thus we have (3.15) with $\leqslant$ replaced by $=$.

In the Kalf–Kumar case, $-\infty < s < \infty$, $|t| < 1/2$ and

$$\Lambda_{KK} = g_1(a(t)s, a(t)(k+t))^2.$$

An argument similar to that given for Λ_{KLP} for $k \geqslant \frac{1}{2}$ leads (3.11).

Now let us consider Λ_{AU}, where $-\infty < s < \infty$ and $|t| < 1$. The first and the second formulae of (3.12) are obvious. In the sequel we assume $k > 1$. Noting

$$\Lambda_{AU} = \frac{1}{2}\cdot\frac{1}{1-M^2}\,[K^2 + L + \sqrt{K^2(K^2+2L) + L^2M^2}\,]$$

$$= \frac{1}{4}\,\frac{1}{1-M^2}\,[\sqrt{K^2 + L\{1+\sqrt{1-M^2}\}} + \sqrt{K^2 + L\{1-\sqrt{1-M^2}\}}\,]^2,$$

where

$$L = 2\,|k+s+t|, \qquad K = |s|, \qquad M = |t|,$$

we put

$$\begin{cases} g_3(x, y, \alpha, \beta) = \frac{1}{2}\{\sqrt{A} + \sqrt{B}\}, \\ A = x^2 + 2\alpha\,|x+y|, \qquad B = x^2 + 2\beta\,|x+y|. \end{cases} \tag{3.16}$$

Then we have

$$\Lambda_{AU}(s, t; k) = \frac{1}{1-t^2}\,\{g_3(s, k+t, \alpha(t), \beta(t))\}^2, \qquad -\infty < s < \infty, \quad |t| < 1, \tag{3.17}$$

where

$$\alpha(t) = 1 + \sqrt{1-t^2}, \qquad \beta(t) = 1 - \sqrt{1-t^2}. \tag{3.18}$$

LEMMA 3.3. *Let*

$$y > 0, \qquad 2 \geqslant \alpha > 1 > \beta \geqslant 0, \qquad \textit{and} \qquad \alpha + \beta = 2. \tag{3.19}$$

We put $\gamma = \alpha\beta$. *When* $y > 1$, *we define* x_0 *as*

$$x_0 = \frac{1}{2(y-1)} [\gamma - \sqrt{\gamma\{\gamma + 4y(y-1)\}}]. \tag{3.20}$$

Then we have

$$\gamma(x_0 + y) = x_0^2(y-1) \qquad \textit{and} \qquad -y < x_0 \leqslant 0 \quad \textit{for} \ \ y > 1, \tag{3.21}$$

$$\frac{\partial g_3}{\partial x}(x_0, y, \alpha, \beta) = 0 \qquad \textit{if} \quad y > 1, \tag{3.22}$$

$$\begin{aligned} &\inf\{ g_3(x, y, \alpha, \beta) \mid -\infty < x < \infty\} \\ &= \begin{cases} g_3(-y, y, \alpha, \beta) = y & \textit{if} \quad 1 \geqslant y > 0, \\ g_3(x_0, y, \alpha, \beta) & \textit{if} \quad y > 1. \end{cases} \end{aligned} \tag{3.23}$$

Proof. A little calculation shows (3.21). In $x < -y$, we have

$$\frac{\partial g_3}{\partial x} = \frac{1}{2}\left[\frac{x-\alpha}{\sqrt{A}} + \frac{x-\beta}{\sqrt{B}}\right] < 0.$$

In $x > -y$, we have

$$\begin{cases} A - B = 2(\alpha-\beta)(x+y), \\ \beta A - \alpha B = -(\alpha-\beta)\, x^2, \\ AB = x^4 + 4(x+y)\, x^2 + 4\gamma(x+y)^2 \\ \qquad = \{x(x+2y)\}^2 + 4(x+y)\{\gamma(x+y) - x^2(y-1)\}, \end{cases} \tag{3.24}$$

and

$$\begin{aligned} \frac{\partial g_3}{\partial x} &= \frac{1}{2}\left[\frac{x+\alpha}{\sqrt{A}} + \frac{x+\beta}{\sqrt{B}}\right] \\ &= \frac{1}{2\sqrt{AB}} [x(\sqrt{A} + \sqrt{B}) + \alpha\sqrt{B} + \beta\sqrt{A}] \\ &= \frac{1}{2\sqrt{AB}} \cdot \frac{1}{\sqrt{A} - \sqrt{B}} [x(A-B) + \beta A - \alpha B + (\alpha-\beta)\sqrt{AB}] \\ &= \frac{\alpha-\beta}{2\sqrt{AB}} \cdot \frac{1}{\sqrt{A} - \sqrt{B}} \\ &\quad \times [x(x+2y) + \sqrt{\{x(x+2y)\}^2 + 4(x+y)\{\gamma(x+y) - x^2(y-1)\}}]. \end{aligned}$$

Note that $\alpha > \beta$, $A > B$ and $x + 2y > 0$ by $y > 0$ and $x + y > 0$. It is obvious that (3.22) holds by (3.21), and

$$\frac{\partial g_3}{\partial x} \begin{cases} > 0 & \text{in} \quad x > 0, \\ \geqslant 0 & \text{in} \quad -y < x < 0, \gamma(x+y) - x^2(y-1) \geqslant 0, \\ < 0 & \text{in} \quad -y < x < 0, \gamma(x+y) - x^2(y-1) < 0. \end{cases}$$

In case $0 < y \leqslant 1$, we have $\gamma(x+y) - x^2(y-1) \geqslant 0$ for $-y < x < 0$. In case $y > 1$, note (3.21). In each case we have (3.23).

Define $\alpha = \alpha(t)$ and $\beta = \beta(t)$ by (3.18), $\gamma(t)$ as $\gamma(t) = \alpha(t)\beta(t) = t^2$ and $y(t)$ as $y(t) = k + t$. Then $y > 0$ since $k > 1$ and $|t| < 1$. When $y > 1$, we define $x_0(t)$ by (3.20) and put

$$g_4(t) = \frac{g_3(x_0(t), y(t), \alpha(t), \beta(t))}{\sqrt{1-t^2}}.$$

Then (3.17) and (3.23) show

$$\inf_{s \in \mathbf{R}} \Lambda_{AU}(s, t; k) = \begin{cases} \dfrac{(k+t)^2}{1-t^2} & \text{if} \quad k + t \leqslant 1, \\ g_4(t)^2 & \text{if} \quad k + t > 1. \end{cases} \tag{3.25}$$

Let us calculate $\inf\{(k+t)^2/(1-t^2) \,|\, k+t \leqslant 1, |t| < 1\}$. Since

$$\frac{d}{dt}\frac{(k+t)^2}{1-t^2} = \frac{2(k+t)(1+kt)}{(1-t^2)^2},$$

the infimum is attained at $t = -1/k$ if $-1 < -1/k \leqslant 1 - k$, that is, if $1 < k \leqslant (1+\sqrt{5})/2$, and at $t = 1-k$ if $-1 < 1-k < -1/k$, that is, if $(1+\sqrt{5})/2 < k < 2$. If $k \geqslant 2$, $k + t \leqslant 1$ and $|t| < 1$ do not hold simultaniously. Thus we have

$$\inf\left\{\frac{(k+t)^2}{1-t^2}\,\middle|\, k+t \leqslant 1, |t| < 1\right\}$$
$$= \begin{cases} \left.\dfrac{(k+t)^2}{1-t^2}\right|_{t=-1/k} = k^2 - 1 & \text{for} \quad 1 < k \leqslant \dfrac{1+\sqrt{5}}{2}, \\ \left.\dfrac{(k+t)^2}{1-t^2}\right|_{t=1-k} = \dfrac{1}{k(2-k)} & \text{for} \quad \dfrac{1+\sqrt{5}}{2} < k < 2. \end{cases} \tag{3.26}$$

Next, let us calculate $\inf\{g_4(t)^2 \,|\, k + t > 1, |t| < 1\}$. We assume $y = k + t > 1$ and $|t| < 1$. We will show

$$g_4'(t) > 0 \qquad \text{in} \quad 0 < t < 1 \tag{3.27}$$

and

$$g_4'(t)\begin{cases} >0 & \text{for} \quad 1<k<\dfrac{1+\sqrt{5}}{2}, \\ =0 & \text{for} \quad k=\dfrac{1+\sqrt{5}}{2}, \\ <0 & \text{for} \quad k>\dfrac{1+\sqrt{5}}{2} \end{cases} \tag{3.28}$$

in $0>t>\max\{-1, 1-k\}$. Then we have

$$\inf\{\, g_4(t)^2 \,|\, t>1-k, |t|<1\}$$
$$=\begin{cases} \lim\limits_{t\downarrow 1-k} g_4(t)^2=\dfrac{1}{k(2-k)} & \text{for} \quad 1<k\leqslant\dfrac{1+\sqrt{5}}{2}, \\ g_4(0)^2=k & \text{for} \quad k\geqslant\dfrac{1+\sqrt{5}}{2}. \end{cases}$$

Note that $k^2-1\leqslant 1/(k(2-k))$ for $1<k<2$ since $1/(k(2-k))-(k^2-1)=(k^2-k-1)^2/(k(2-k))$ and that $k\leqslant 1/(k(2-k))$ for $(1+\sqrt{5})/2\leqslant k<2$ since $1/(k(2-k))-k=(k-1)(k^2-k-1)/(k(2-k))$. The above formula with (3.25) and (3.26) yields (3.12).

Now let us show (3.27) and (3.28). Now,

$$(1-t^2)\; g_4'(t)=\frac{t\cdot g_3}{\sqrt{1-t^2}}+\sqrt{1-t^2}\,\frac{d}{dt}\, g_3(x_0(t), y(t), \alpha(t), \beta(t)).$$

$$\frac{dg_3}{dt}=\frac{\partial g_3}{\partial x}\frac{dx_0}{dt}+\frac{\partial g_3}{\partial y}\frac{dy}{dt}+\frac{\partial g_3}{\partial \alpha}\frac{d\alpha}{dt}+\frac{\partial g_3}{\partial \beta}\frac{d\beta}{dt}.$$

Using (3.22) and

$$\frac{\partial g_3}{\partial y}=\frac{1}{2}\left[\frac{\alpha}{\sqrt{A}}+\frac{\beta}{\sqrt{B}}\right], \qquad \frac{dy}{dt}=1,$$

$$\frac{\partial g_3}{\partial \alpha}=\frac{1}{2}\cdot\frac{x_0+y}{\sqrt{A}}, \qquad \frac{d\alpha}{dt}=\frac{-t}{\sqrt{1-t^2}},$$

$$\frac{\partial g_3}{\partial \beta}=\frac{1}{2}\cdot\frac{x_0+y}{\sqrt{B}}, \qquad \frac{d\beta}{dt}=\frac{t}{\sqrt{1-t^2}},$$

we have

$$(1-t^2)\sqrt{AB}\,g_4'(t)=\frac{1}{2}\left[\frac{t\sqrt{AB}}{\sqrt{1-t^2}}\{\sqrt{A}+\sqrt{B}\}+t(x_0+y)\{\sqrt{A}-\sqrt{B}\}\right.$$
$$\left.+\sqrt{1-t^2}\{\alpha\sqrt{B}+\beta\sqrt{A}\}\right],$$

from which (3.27) is obvious. In order to show (3.28), in the sequel we assume $k>1$ and $0>t>\max\{-1,1-k\}$. Then we have $y=k+1>1$ and $\gamma=\alpha\beta=t^2>0$. By (3.20) and (3.21) we have $-y<x_0<0$. By (3.24), (3.21), $x_0<0$, and $x_0+2y>x_0+y>0$ we have

$$\sqrt{AB}=\sqrt{x_0^2(x_0+2y)^2}=-x_0(x_0+2y).$$

Using (3.24) we have

$$\begin{aligned}
&(1-t^2)\sqrt{AB}(\sqrt{A}-\sqrt{B})\,g_4'(t)\\
&=\frac{1}{2}\left[\frac{t}{\sqrt{1-t^2}}\sqrt{AB}(A-B)+t(x_0+y)(A+B-2\sqrt{AB})\right.\\
&\quad\left.+\sqrt{1-t^2}\{(\alpha-\beta)\sqrt{AB}+(\beta A-\alpha B)\}\right]\\
&=2(x_0+y)[t(x_0+y)-x_0(1+ty-t^2)]\\
&=x_0(x_0+y)[t+\sqrt{t^2+4y(y-1)}-2(1+ty-t^2)]\\
&=x_0(x_0+y)[\sqrt{(2y-1)^2-(1-t^2)}-t(2y-1)-2(1-t^2)]\\
&=\frac{x_0(x_0+y)(1-t^2)\{(2y-1-2t)^2-5\}}{\sqrt{(2y-1)^2-(1-t^2)}+t(2y-1)+2(1-t^2)}\\
&=\frac{4x_0(x_0+y)(1-t^2)(k^2-k-1)}{\sqrt{(2y-1)^2-(1-t^2)}+t(2y-1)+2(1-t^2)},
\end{aligned}$$

where in the third equality we have used

$$2t(x_0+y)=\frac{2}{t}x_0^2(y-1)=x_0\{t+\sqrt{t^2+4y(y-1)}\},$$

which follows from (3.20), (3.21), $\gamma=t^2$ and $t<0$, and in the last equality we have used $y=k+t$. The sign of the numerator of the above formula is oposite to the sign of $k^2 - k - 1$. The denominator of the above formula is positive since

$$\sqrt{(2y-1)^2 - (1-t^2)} + t(2y-1)$$
$$= \sqrt{\{t(2y-1)\}^2 + 4y(y-1)(1-t^2)} + t(2y-1) > 0$$

by $y > 1$ and $-1 < t < 0$. Thus we have (3.28).

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