Periodic pyramidal traveling fronts of bistable reaction–diffusion equations with time-periodic nonlinearity

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Abstract

This paper deals with the existence and stability of periodic pyramidal traveling fronts for reaction–diffusion equations with bistable time-periodic nonlinearity in \( \mathbb{R}^N \) with \( N \geq 3 \). It is well known that two-dimensional periodic traveling curved fronts exist and are stable. In this paper, by constructing various of supersolutions and subsolutions, we first show that there exist three-dimensional periodic pyramidal traveling fronts, and then we prove that such periodic pyramidal traveling fronts are asymptotically stable. Finally, we further prove that our existence result holds for \( \mathbb{R}^N \) with \( N \geq 4 \).

Keywords:
Pyramidal traveling fronts
Reaction–diffusion equations
Time periodic

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1. Introduction

This paper is concerned with the following reaction–diffusion equation

\[
\frac{\partial u(y,t)}{\partial t} = \Delta u(y,t) + f(u(y,t),t), \quad y \in \mathbb{R}^N, \quad t > 0.
\]  

(1.1)

Here, we assume that \( f \) satisfies the following hypotheses:

(H1) There exists \( T > 0 \) such that \( f(u,t) = f(u,t+T) \) for all \( (u,t) \in \mathbb{R}^2 \).

(H2) The period map \( P(\alpha) := \omega(\alpha, T) \) has exactly three fixed points \( \alpha^-, \alpha^0, \alpha^+ \) satisfying \( \alpha^- < \alpha^0 < \alpha^+ \), where \( \omega(\alpha, t) \) is the solution of

\[
\omega_t = f(\omega, t), \quad t \in \mathbb{R}, \quad \omega(\alpha, 0) = \alpha \in \mathbb{R}.
\]

Furthermore, they are non-degenerate and \( \alpha^\pm \) are stable, i.e.,

\[
\frac{d}{d\alpha} P(\alpha^\pm) < 1 < \frac{d}{d\alpha} P(\alpha^0).
\]

(H3) There exists \( \nu_0 > 0 \) such that \( \nu^+ + \nu^- + f_u(W^\pm(t), t) > \nu_0 \) for any \( t \in [0, T] \), where

\[
\nu^\pm := \frac{1}{T} \int_0^T f_u(W^\pm(\lambda), \lambda) d\lambda,
\]

and

\[
W^\pm(t) := \omega(\alpha^\pm, t), \quad W^0(t) := \omega(\alpha^0, t).
\]

A typical example of \( f \) satisfying (H1)–(H3) is the cubic potential

\[
f(u, t) = \left( 1 - u^2 \right) (2u - \rho(t)),
\]

where \( \rho(t) \in (-2, 2) \) is \( T \)-periodic. In fact, \( f \) is a particular case of the following more general example (see Alikakos et al. [1])

\[
f(u, t) = p(u)(-p'(u) - \rho(t)),
\]

where \( \rho \in C^1 \) and \( p \in C^3 \) satisfy \( \rho(\cdot + T) = \rho(\cdot) \), and \( p(\pm 1) = 0, \ p(\cdot) > 0 \) in \((-1, 1)\).

It is known from [1] that if \( f(u, t) \in C^{2,1}(\mathbb{R} \times \mathbb{R}) \) satisfies hypotheses (H1) and (H2), then there exists a unique solution pair \((c, U)\) to (1.1) in one-dimensional space satisfying

\[
\begin{cases}
U_t - cU_{\xi\xi} - U_{\xi} - f(U, t) = 0, & (\xi, t) \in \mathbb{R}^2, \\
U(\pm \infty, t) = \lim_{\xi \to \pm \infty} U(\xi, t) = W^\pm(t), & t \in \mathbb{R}, \\
U(\cdot, T) = U(\cdot, 0), & U(0, 0) = \alpha^0.
\end{cases}
\]

where the function \( U(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is the wave profile and the constant \( c \in \mathbb{R} \) is the speed. In addition, there hold the following properties:

(i) The profile \( U \) is monotone decreasing with respect to the moving coordinate for each \( t \). Namely, 

\[
U_{\xi}(\cdot, \cdot) < 0 \text{ in } \mathbb{R} \times \mathbb{R}.
\]
(ii) There exist positive constants $C_1$ and $\beta_1$ satisfying

$$\left| U(\pm \xi, t) - W(\pm t) \right| + \left| U_\xi(\pm \xi, t) \right| + \left| U_{\xi \xi}(\pm \xi, t) \right| \leq C_1 e^{-\beta_1 \xi}, \quad \xi \geq 0, \ t \in \mathbb{R}. \quad (1.2)$$

This means that $U$ the profile exponentially approaches its limits as $\xi \to \pm \infty$.

It is well known that $U(e \cdot y + ct, t)$ with $e = (e_1, e_2, \ldots, e_N)$ satisfying $\sum_{i=1}^{N} e_i^2 = 1$ is a planar traveling wave to (1.1) which has been well studied in one or higher-dimensional spaces, one can refer to [1,28,33,41–45] for time almost periodic or time periodic planar traveling wave solutions, and [2,3,17,18,29,31,34,38,48,50] for the autonomous case. For the related spatial discrete systems, we refer to [7–13].

Recently, the study on nonplanar traveling waves (multidimensional traveling waves) of autonomous reaction–diffusion equations has received increasing attention. For example, Ninomiya and Taniguchi [36,37] considered the following equation

$$\frac{\partial u(y, t)}{\partial t} = \Delta u(y, t) + f(u(y, t)), \quad y \in \mathbb{R}^N. \quad (1.3)$$

and proved that (1.3) with $N = 2$ admits a unique two-dimensional V-shaped curved front which is asymptotically stable, see also [4–6,16,19–25,39,26,5,27] for some related results. It needs to be pointed out that the authors in [22,23] also deals with cylindrically symmetric traveling fronts in arbitrary space dimensions. Taniguchi [46,47] further studied the existence, uniqueness and stability of three-dimensional pyramidal traveling fronts of (1.3) with $N = 3$. Very recently, Kurokawa and Taniguchi [30] considered the existence of $N$-dimensional traveling fronts of (1.3) with $N \geq 4$. Just as Kurokawa and Taniguchi [30] pointed out, the space dimension is crucial for the existence or nonexistence of multidimensional traveling fronts, see also Pino et al. [14,15] and Savin [40] for details.

Compared with the autonomous case, little attention has been paid to non-autonomous reaction–diffusion equations, even if the periodic problem (1.1). More recently, Wang and Wu [49] proved that there exists a two-dimensional periodic V-shaped traveling front to (1.1) with $N = 2$. Furthermore, they proved that such a traveling curved front is asymptotically stable. It is then natural to ask whether multidimensional periodic traveling fronts of (1.1) with $N \geq 3$ exist and are stable. Resolving this issue is the main contribution of our current study.

In this paper, we first consider the existence, uniqueness and stability of three-dimensional periodic pyramidal traveling fronts of (1.1) with $N = 3$, i.e., we consider the following equation

$$\frac{\partial u(x, y, z, t)}{\partial t} = \Delta u(x, y, z, t) + f(u(x, y, z, t), t), \quad (x, y, z) \in \mathbb{R}^3, \ t > 0. \quad (1.4)$$

Then we establish the existence of $N$-dimensional periodic pyramidal traveling fronts of (1.1) with $N \geq 4$. Our main method is the comparison principle coupled with the supersolution and subsolution technique. However, for the uniqueness and stability of $N$-dimensional periodic pyramidal traveling fronts of (1.4) with $N \geq 4$, they are difficulty and are left to be as interesting open problems.

Without loss of generality, we assume $c > 0$ (the case $c = 0$ is certainly a very interesting problem which needs to be considered separately. For details we refer to [19,6] and the references therein.) and the solutions travel towards $z$-direction. Set

$$u(x, y, z, t) = v(x, y, z - lt, t), \quad s = z - lt.$$  

For simplicity, we still denote $v(x, y, s, t)$ by $v(x, y, z, t)$. Substituting $v$ into (1.4), we have

$$L[v] := v_t - v_{xx} - v_{yy} - v_{zz} - lv_z - f(v, t) = 0, \quad (x, y, z) \in \mathbb{R}^3, \ t > 0,$$

$$v(x, y, z, 0) = v_0(x, y, z), \quad (x, y, z) \in \mathbb{R}^3. \quad (1.5)$$

Here, we always assume that $l > c$ holds true.
Let \( n \geq 3 \) be a given integer and set

\[
\tau := \frac{\sqrt{l^2 - c^2}}{c} > 0. \tag{1.6}
\]

Assume that \((A_j, B_j) \in \mathbb{R}^2\) satisfies

\[
A_j^2 + B_j^2 = 1 \quad \text{for all } j = 1, \ldots, n \tag{1.7}
\]

and

\[
A_j B_{j+1} - A_{j+1} B_j > 0, \quad 1 \leq j \leq n - 1,
A_n B_1 - A_1 B_n > 0. \tag{1.8}
\]

The vector \((-\tau A_j, -\tau B_j, 1)\) is the normal vector of a surface \(\{z = \tau (A_j x + B_j y)\}\). Additionally, we put

\[
h_j(x, y) := \tau (A_j x + B_j y),
\]

\[
h(x, y) = \max_{1 \leq j \leq n} h_j(x, y) = \tau \max_{1 \leq j \leq n} (A_j x + B_j y). \tag{1.9}
\]

Thus \(z = h(x, y)\) is a pyramid in \(\mathbb{R}^3\). Denoting

\[
\Omega_j = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = h_j(x, y)\},
\]

we get

\[
\mathbb{R}^2 = \bigcup_{j=1}^{n} \Omega_j.
\]

We note that condition (1.8) is to ensure that the location \(\Omega_1, \Omega_2, \ldots, \Omega_n\) is counterclockwise. We set

\[
E' := \bigcup_{j=1}^{n} \partial \Omega_j \subset \mathbb{R}^2.
\]

For \(j = 1, \ldots, n\), the lateral surfaces of the pyramid are given by

\[
S_j = \{(x, y, z) \in \mathbb{R}^3 \mid z = h_j(x, y), \ (x, y) \in \Omega_j\}.
\]

We define the edges \(\Gamma_j\) of the pyramid as

\[
\Gamma_j := \begin{cases} S_j \cap S_{j+1} & \text{if } 1 \leq j \leq n - 1, \\ S_n \cap S_1 & \text{if } j = n. \end{cases}
\]

Thus

\[
\Gamma := \bigcup_{j=1}^{n} \Gamma_j
\]
is the set of all edges. For each $\gamma > 0$, we define

$$D(\gamma) := \{(x, y, z) \in \mathbb{R}^3 \mid \text{dist}(x, y, z, \Gamma) > \gamma\}.$$ 

For every $(A_j, B_j)$ satisfying (1.7), it is easy to see that $U(\xi(z - h_j(x, y)), t)$ are planar traveling fronts to (1.5). Define

$$v^-(x, y, z, t) := U\left(\frac{c}{l}(z - h(x, y)), t\right) = \max_{1 \leq j \leq n} U\left(\frac{c}{l}(z - h_j(x, y)), t\right) = \max_{1 \leq j \leq n} U\left(\frac{c}{l}(z - \tau A_j x - \tau B_j y), t\right). \quad (1.10)$$

It is obvious that $v^-(x, y, z, t)$ is a subsolution to (1.5). In addition, we have $v^-(\cdot, \cdot, \cdot, \cdot + T) = v^-(\cdot, \cdot, \cdot, \cdot)$ in $\mathbb{R}^4$ and $v_z^-(x, y, z, t) < 0$ for $(x, y, z, t) \in \mathbb{R}^4$.

Our aim in this paper is to seek for the solution $V(x, y, z, t)$ with

$$\mathcal{L}[V] := V_t - V_{xx} - V_{yy} - V_{zz} - lV_z - f(V, t) = 0, \quad (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R},$$

$$V(\cdot, \cdot, \cdot, \cdot) = V(\cdot, \cdot, \cdot, \cdot + T), \quad (x, y, z) \in \mathbb{R}^3. \quad (1.11)$$

The main results of this paper are as follows.

**Theorem 1.1.** Assume that $l > c > 0$ holds, $h(x, y)$ is given by (1.9). Then, under the assumptions (H1)--(H3), there exists a solution $V(x, y, z, t)$ to (1.11)--(1.11) satisfying

$$U\left(\frac{c}{l}(z - h(x, y)), t\right) < V(x, y, z, t) < W^+(t), \quad (x, y, z, t) \in \mathbb{R}^3 \times [0, T]$$

and

$$\lim_{\gamma \to +\infty} \sup_{(x, y, z) \in D(\gamma), t \in [0, T]} \left|V(x, y, z, t) - U\left(\frac{c}{l}(z - h(x, y)), t\right)\right| = 0,$$

$$-V_z(x, y, z, t) > 0 \quad \text{for all} \; (x, y, z) \in \mathbb{R}^3. \quad (1.12)$$

**Theorem 1.2.** In addition to the assumptions in Theorem 1.1, if we also assume that

$$\lim_{\gamma \to +\infty} \sup_{(x, y, z) \in D(\gamma)} |v_0(x, y, z) - V(x, y, z, 0)| = 0 \quad (1.13)$$

holds, then the solution $v(x, y, z; v_0)$ to (1.5) satisfies

$$\lim_{t \to +\infty} \|v(\cdot, \cdot, \cdot, t) - V(\cdot, \cdot, \cdot, t)\|_{C(\mathbb{R}^3)} = 0,$$

or equivalently

$$\lim_{k \to +\infty} \|v(\cdot, \cdot, \cdot, \cdot + kT) - V(\cdot, \cdot, \cdot, \cdot)\|_{C(\mathbb{R}^3 \times [0, T])} = 0.$$

We note that the case $l < c < 0$ follows from a similar argument to [49, Theorem 1.2].
Remark 1.3. In fact, when \( f(u, t) = f(u) \), the conclusions of Theorems 1.1 and 1.2 have already been established in [46, Theorem 2] and [47, Theorem 2], respectively. More recently, Wang and Wu [49] have studied the existence and globally asymptotic stability of periodic traveling curved fronts for (1.4) in \( \mathbb{R}^2 \), which are planar ones in three-dimensional space. However, Theorems 1.1 and 1.2 show that the periodic pyramidal traveling front is asymptotically stable and nonplanar in \( \mathbb{R}^3 \).

The organization of this paper is as follows. In Section 2, we summarize some preliminaries. The existence of periodic pyramidal traveling fronts is proved in Section 3 and the uniqueness and globally asymptotic stability of such traveling fronts are shown in Section 4. Namely, we prove Theorems 1.1 and 1.2, respectively. The last section is devoted to the existence of \( N \)-dimensional periodic pyramidal traveling fronts with \( N \geq 4 \).

2. Preliminaries

In this section we will construct a mollified pyramid, which plays a key role in establishing the supersolution. One can see [46] for details.

Let \( \rho \in C^\infty[0, \infty) \) be a function with the following properties:

\[
\rho(r) > 0, \quad \rho(r) \leq 0 \quad \text{for } r \geq 0, \\
\rho(r) \equiv 1 \quad \text{if } r \geq 0 \text{ small enough,} \\
\rho(r) = e^{-r} \quad \text{if } r > 0 \text{ large enough.}
\]

Then \( \rho(r) \in C^\infty[0, \infty) \) satisfies

\[
2\pi \int_0^\infty r \rho(r) \, dr = 1. \tag{2.1}
\]

Then \( \rho \in C^\infty[\mathbb{R}^2] \) and satisfies \( \int_{\mathbb{R}^2} \rho = 1 \). For the pyramid \( z = h(x, y) \), we construct a mollified pyramid \( z = \varphi(x, y) \) by \( \varphi(x, y) := \rho * h \) with \( \varphi(x, y) \) defined as

\[
\varphi(x, y) = \int_{\mathbb{R}^2} \rho(x - x', y - y') h(x', y') \, dx' \, dy' = \int_{\mathbb{R}^2} \rho(x', y') h(x - x', y - y') \, dx' \, dy'. \tag{2.2}
\]

We set \((a_j, b_j) := \tau(A_j, B_j)\). Then \((a_j, b_j) \in \mathbb{R}^2\) satisfies

\[
\frac{l}{\sqrt{1 + a_j^2 + b_j^2}} = c \quad \text{for all } j = 1, \ldots, n.
\]

Set

\[
S(x, y) := \frac{l}{\sqrt{1 + \varphi_x^2(x, y) + \varphi_y^2(x, y)}} - c. \tag{2.3}
\]

Then we have the following lemmas which follow from [46].

Lemma 2.1. Let \( \varphi \) and \( S \) be defined in (2.2) and (2.3), respectively. Then we have

\[
\sup_{(x, y) \in \mathbb{R}^2} |D_{x}^{i_1} D_{y}^{i_2} \varphi(x, y)| < +\infty \quad \text{for all integers } i_1 \geq 0, i_2 \geq 0
\]
and

\[ h(x, y) < \varphi(x, y) \leq h(x, y) + 2 \pi \tau \int_0^\infty r^2 \bar{\rho}(r) \, dr, \quad (x, y) \in \mathbb{R}^2, \]

\[ |\nabla \varphi(x, y)| < \tau, \quad 0 < S(x, y) < l, \quad (x, y) \in \mathbb{R}^2. \]

In addition,

\[ \lim_{\lambda \to \infty} \sup \{ S(x, y) \mid (x, y) \in \mathbb{R}^2, \text{dist}(x, y, E') \geq \lambda \} = 0, \]

\[ \lim_{\lambda \to \infty} \sup \{ \varphi(x, y) - h(x, y) \mid (x, y) \in \mathbb{R}^2, \text{dist}(x, y, E') \geq \lambda \} = 0. \]

**Lemma 2.2.** There exist positive constants \( v_1, v_2 \) such that

\[ 0 < v_1 \leq \frac{\varphi(x, y) - h(x, y)}{S(x, y)} \leq v_2 \]

holds for all \((x, y) \in \mathbb{R}^2\).

**Proposition 2.3.** For all integers \( i_1 \geq 0, i_2 \geq 0 \) with \( 2 \leq i_1 + i_2 \leq 3 \), we have

\[ \sup_{(x, y) \in \mathbb{R}^2} \frac{|D_x^{i_1} D_y^{i_2} \varphi(x, y)|}{S(x, y)} < +\infty. \]

### 3. Existence of periodic pyramidal traveling fronts

In this section we show that there exists a periodic pyramidal traveling front of (1.4) by constructing a supersolution that is larger than \( v^- \). Hereafter, we always assume the planar traveling wave speed \( c > 0 \).

**Lemma 3.1.** There exist a positive constant \( \varepsilon^+_0 \) and a positive function \( \alpha^+_0(\varepsilon) \) such that, for \( 0 < \varepsilon \leq \varepsilon^+_0 \) and \( 0 < \alpha \leq \alpha^+_0(\varepsilon) \),

\[ v^+(x, y, z, t; \varepsilon, \alpha) := U \left( \frac{z - \frac{1}{\alpha} \varphi(\alpha x, \alpha y)}{\sqrt{1 + \frac{1}{\alpha^2}(\varphi_x^2(\alpha x, \alpha y) + \varphi_y^2(\alpha x, \alpha y))}}, t \right) + \varepsilon (a^+(t) + a^-(t)) S(\alpha x, \alpha y) \]

is a supersolution of (1.5) on \( t \in (-\infty, +\infty) \), where

\[ a^\pm(t) := \exp \left\{ \int_0^t W^\pm(\tau), \tau \right\}. \]

In addition,
Proof. Direct calculation shows $v^+(x, y, z; t; \varepsilon, \alpha) = v^+(\cdot, \cdot, \cdot + T; \varepsilon, \alpha) = v^+(\cdot, \cdot, \cdot; \varepsilon, \alpha)$. It only remains to show that $\mathcal{L}[v^+] \geq 0$ in $\mathbb{R}^3 \times [0, T]$. Let

$$\lim_{y \to \infty} \sup_{(x, y, z) \in D(y), t \in [0, T]} \left| v^+(x, y, z; t; \varepsilon, \alpha) - U \left( \frac{c}{T} (z - h(x, y)), t \right) \right| \leq (1 + a^*) \varepsilon, \quad (3.1)$$

$$U \left( \frac{c}{T} (z - h(x, y)), t \right) < v^+(x, y, z; t; \varepsilon, \alpha), \quad (3.2)$$

with $a^* := \max_{t \in [0, T]} (a^+(t) + a^-(t))$.

\[ \hat{\mu} = \frac{z - \frac{1}{\alpha} \varphi(\alpha x, \alpha y)}{\sqrt{1 + \frac{1}{\alpha^2} (\varphi^2_\xi(\alpha x, \alpha y) + \varphi^2_\eta(\alpha x, \alpha y))}} = \frac{1}{\alpha} \frac{\xi - \varphi(\xi, \eta)}{\sqrt{1 + \varphi^2_\xi(\xi, \eta) + \varphi^2_\eta(\xi, \eta)}}. \]

Then we have

\[ \hat{\mu}_x = \frac{-\varphi_\xi}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}}, \quad \hat{\mu}_{xx} = \alpha \hat{\mu} F_1(\xi, \eta), \quad \hat{\mu}_{xx} = \alpha G_{11}(\xi, \eta) + \alpha^2 \hat{\mu} H_{11}(\xi, \eta), \]

\[ \hat{\mu}_{xy} = \alpha G_{12}(\xi, \eta) + \alpha^2 \hat{\mu} H_{12}(\xi, \eta), \]

\[ \hat{\mu}_y = \frac{-\varphi_\eta}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}}, \quad \hat{\mu}_{yy} = \alpha G_{22}(\xi, \eta) + \alpha^2 \hat{\mu} H_{22}(\xi, \eta), \]

where

\[ F_1(\xi, \eta) := \sqrt{1 + \varphi^2_\xi + \varphi^2_\eta} \left( \frac{1}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \xi \right), \]

\[ F_2(\xi, \eta) := \sqrt{1 + \varphi^2_\xi + \varphi^2_\eta} \left( \frac{1}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \eta \right), \]

\[ G_{11}(\xi, \eta) := -\left( \frac{\varphi_\xi}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \xi \right) - \varphi_\xi \left( \frac{1}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \xi \right) \]

\[ = \frac{(-1 + \varphi^2_\xi - \varphi^2_\eta)\varphi_\xi + 2(\varphi^2_\xi + \varphi_\xi \varphi_\eta)\varphi_\eta}{(1 + \varphi^2_\xi + \varphi^2_\eta)^{\frac{3}{2}}}, \]

\[ G_{12}(\xi, \eta) := -\left( \frac{\varphi_\xi}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \xi \right) - \varphi_\eta \left( \frac{1}{\sqrt{1 + \varphi^2_\xi + \varphi^2_\eta}} \xi \right) \]

\[ = \frac{(-1 + \varphi^2_\xi + \varphi^2_\eta)\varphi_\xi + 2\varphi_\xi \varphi_\eta (\varphi_\xi + \varphi_\eta)}{(1 + \varphi^2_\xi + \varphi^2_\eta)^{\frac{3}{2}}}. \]
Substituting 

\[ G_{22}(\xi, \eta) := -\left( \frac{\varphi_{\eta}}{\sqrt{1 + \varphi_{\xi}^2 + \varphi_{\eta}^2}} \right) - \varphi_{\eta} \left( \frac{1}{\sqrt{1 + \varphi_{\xi}^2 + \varphi_{\eta}^2}} \right) \]

\[ = (-1 - \varphi_{\xi}^2 + \varphi_{\eta}^2)\varphi_{\xi\xi} + 2(\varphi_{\eta}^2 + \varphi_{\xi}\varphi_{\eta})\varphi_{\xi\eta}, \]

\[ H_{11}(\xi, \eta) := (F_1(\xi, \eta))_\xi + F_1(\xi, \eta)^2, \]

\[ H_{12}(\xi, \eta) := (F_1(\xi, \eta))_\eta + F_1(\xi, \eta)F_2(\xi, \eta), \]

and

\[ H_{22}(\xi, \eta) := (F_2(\xi, \eta))_\eta + F_2(\xi, \eta)^2. \]

For the above calculations, one can see [46].

Note that \( v^+(x, y, z, t; \varepsilon, \alpha) \) can be written as

\[ v^+(x, y, z, t; \varepsilon, \alpha) = U(\hat{\mu}, t) + (a^+(t) + a^-(t))\sigma(\xi, \eta). \]

Substituting \( v^+(x, y, z, t; \varepsilon, \alpha) \) into (1.5), we obtain

\[ \mathcal{L}[v^+(x, y, z, t; \varepsilon, \alpha)] = v^+_t - v^+_x - v^+_y - v^+_z - l v^+_z - f(v^+, t) \]

\[ = U_t - U_{xx} - U_{yy} - U_{zz} - lU_z - f(v^+, t) \]

\[ + \sigma \frac{d}{dt}(a^+(t) + a^-(t)) - (a^+(t) + a^-(t))\sigma_{xx} - (a^+(t) + a^-(t))\sigma_{yy} \]

\[ = U_t + (v^+ + v^- + f_u(W^+(t), t) + f_u(W^-(t), t))(a^+(t) + a^-(t))\sigma(\xi, \eta) \]

\[ - \alpha \frac{\partial U}{\partial \hat{\mu}} (G_{11}(\xi, \eta) + G_{22}(\xi, \eta)) - \alpha^2 \hat{\mu} \frac{\partial U}{\partial \hat{\mu}} (H_{11}(\xi, \eta) + H_{22}(\xi, \eta)) \]

\[ - \frac{\partial^2 U}{\partial \hat{\mu}^2} + 2\alpha \hat{\mu} \frac{\partial^2 U}{\partial \hat{\mu}^2} \frac{\varphi_{\xi}(\xi, \eta)F_1(\xi, \eta) + \varphi_{\eta}(\xi, \eta)F_2(\xi, \eta)}{\sqrt{1 + \varphi_{\xi}^2(\xi, \eta) + \varphi_{\eta}^2(\xi, \eta)}} \]

\[ - \alpha^2 \hat{\mu}^2 \frac{\partial^2 U}{\partial \hat{\mu}^2} (F_1^2(\xi, \eta) + F_2^2(\xi, \eta)) - \frac{l}{\sqrt{1 + \varphi_{\xi}^2(\xi, \eta) + \varphi_{\eta}^2(\xi, \eta)}} \frac{\partial U}{\partial \hat{\mu}} \]

\[ - \varepsilon \alpha^2 (a^+(t) + a^-(t))(S_{\xi\xi}(\xi, \eta) + S_{\eta\eta}(\xi, \eta)) \]

\[ - f(U(\hat{\mu}, t) + (a^+(t) + a^-(t))\sigma(\xi, \eta), t). \]

Let

\[ R(\xi, \eta, \hat{\mu}, t; \varepsilon, \alpha) := -\frac{\partial U}{\partial \hat{\mu}} (G_{11}(\xi, \eta) + G_{22}(\xi, \eta)) - \alpha \hat{\mu} \frac{\partial U}{\partial \hat{\mu}} (H_{11}(\xi, \eta) + H_{22}(\xi, \eta)) \]
which can be obtained by using (1.2), Lemma 2.1 and Proposition 2.3. Then we have

\[ L[U_t] = S = - S v_x(\xi, \eta) + S \eta(\xi, \eta) , \]

where

\[ S_{\xi\xi}(\xi, \eta) + S_{\eta\eta}(\xi, \eta) = \left( \frac{l}{1 + \varphi_{\xi}^2(\xi, \eta) + \varphi_{\eta}^2(\xi, \eta)} \right)_{\xi\xi} + \left( \frac{l}{1 + \varphi_{\xi}^2(\xi, \eta) + \varphi_{\eta}^2(\xi, \eta)} \right)_{\eta\eta} . \]

There exist a constant \( A > 0 \) such that

\[ \frac{|R(\xi, \eta, \hat{\mu}, t; \varepsilon, \alpha)|}{S(\xi, \eta)} < A \quad \text{for all } (\xi, \eta, \hat{\mu}) \in \mathbb{R}^3, \varepsilon \in (0, 1), \alpha \in (0, 1), t \in [0, T], \]

which can be obtained by using (1.2), Lemma 2.1 and Proposition 2.3. Then we have

\[
L[v^+(x, y, z; t; \varepsilon, \alpha)] = U_t + (v^+ + v^- + f_u(W^+(t), t) + f_u(W^-(t), t))(a^+(t) + a^-(t)) \sigma \\
- \frac{\partial^2 U}{\partial \mu^2} - \frac{l}{\sqrt{1 + \varphi_{\xi}^2(\xi, \eta) + \varphi_{\eta}^2(\xi, \eta)}} \frac{\partial U}{\partial \mu} \\
- f(U(\mu, t) + (a^+(t) + a^-(t)) \sigma, t) + \alpha R(\xi, \eta, \hat{\mu}, t; \varepsilon, \alpha) \\
= S(\xi, \eta) \frac{\partial U}{\partial \mu} + (v^+ + v^- + f_u(W^+(t), t) + f_u(W^-(t), t))(a^+(t) + a^-(t)) \sigma \\
- (a^+(t) + a^-(t)) \sigma(\xi, \eta) \int_0^1 f_u(U(\mu, t) + \theta(a^+(t) + a^-(t)) \sigma, t) d\theta + \alpha R(\xi, \eta, \hat{\mu}, t; \varepsilon, \alpha) \\
= S(\xi, \eta) \left( - \frac{\partial U}{\partial \mu} + \varepsilon(a^+(t) + a^-(t)) (v^+ + v^- + f_u(W^+(t), t) + f_u(W^-(t), t)) \\
- \varepsilon(a^+(t) + a^-(t)) \int_0^1 f_u(U(\mu, t) + \theta(a^+(t) + a^-(t)) \sigma(\xi, \eta, t) d\theta + \alpha \frac{\alpha R(\xi, \eta, \hat{\mu}, t; \varepsilon, \alpha)}{S(\xi, \eta)} \right) \\
\geq S(\xi, \eta) \left( - \frac{\partial U}{\partial \mu} + \varepsilon(a^+(t) + a^-(t)) (v^+ + v^- + f_u(W^+(t), t) + f_u(W^-(t), t)) \\
- \varepsilon(a^+(t) + a^-(t)) \int_0^1 f_u(U(\mu, t) + \theta(a^+(t) + a^-(t)) \sigma, t) d\theta - \alpha A \right). \]

Define

\[ \epsilon_0 := \frac{1}{2} \min_{t \in [0, T]} \{ W^+(t) - W^0(t), W^0(t) - W^-(t) \} . \]
Then there exists $\epsilon \in (0, \epsilon_0)$ such that

$$|f_u(W^\pm(t), t) - f_u(b(t), t)| \leq \frac{1}{2}\nu_0$$

for any $t \in [0, T]$ and $b(\cdot) \in C([0, T])$ with $\|b(\cdot) - W^\pm(\cdot)\|_{C([0, T])} \leq \epsilon$.

Since

$$\lim_{\hat{\mu} \to \pm\infty} U(\hat{\mu}, t) = W^\mp(t)$$

uniformly for $t \in [0, T]$, there exists $\hat{\mu}_0 > 0$ satisfying

$$|W^-(t) - U(\hat{\mu}, t)| \leq \frac{1}{4}\epsilon$$

for any $\hat{\mu} > \hat{\mu}_0, t \in [0, T]$ and

$$|W^+(t) - U(\hat{\mu}, t)| \leq \frac{1}{4}\epsilon$$

for any $\hat{\mu} < -\hat{\mu}_0, t \in [0, T]$.

Let $\epsilon^*_0 > 0$ be sufficiently small such that

$$\epsilon^*_0 \leq \frac{\epsilon}{4a^* l}.$$ 

Then for any $0 < \epsilon \leq \epsilon^*_0$, $|\hat{\mu}| > \hat{\mu}_0$ and $t \in [0, T]$, we obtain

$$I := \epsilon S(\xi, \eta)(a^+(t) + a^-(t))\left(\nu^+ + \nu^- + f_u(W^+(t), t) + f_u(W^-(t), t)\right)$$

$$- \epsilon S(\xi, \eta)(a^+(t) + a^-(t))\int_0^1 f_u(U(\hat{\mu}, t) + \theta(a^+(t) + a^-(t))\sigma(\xi, \eta, t)) d\theta$$

$$\geq \frac{1}{2}\nu_0\epsilon S(\xi, \eta)(a^+(t) + a^-(t)).$$

Consequently, we have

$$\mathcal{L}[\nu^+(x, y, z, t; \epsilon, \alpha)] \geq S(\xi, \eta)\left(-\frac{\partial U}{\partial \hat{\mu}} - \alpha A + \frac{1}{2}\nu_0\epsilon(a^+(t) + a^-(t))\right) \geq 0$$

for $|\hat{\mu}| > \hat{\mu}_0$ and $t \in [0, T]$, provided that

$$\alpha \leq \min\left\{1, \frac{\nu_0\epsilon a_+}{2A}, \frac{p}{A}\right\}, \tag{3.3}$$

where $a_+ := \min_{t \in [0, T]}(a^+(t) + a^-(t))$ and

$$p := \min_{|\hat{\xi}| \leq \hat{\nu}_0, t \in [0, T]} \left(-\frac{\partial U(\xi, t)}{\partial \hat{\xi}}\right) > 0.$$
For $|\hat{\mu}| \leq \hat{\mu}_0$ and $t \in [0, T]$, using (3.3) we have

$$\mathcal{L}[v^+(x, y, z, \xi, \eta; \epsilon, \alpha)] \geq S(\xi, \eta) \left( p - \alpha A + \frac{1}{2} v_0 \epsilon (a^+(t) + a^-(t)) \right) \geq 0.$$ 

In both cases we have proved that $v^+(x, y, z, \xi, \eta; \epsilon, \alpha)$ is a supersolution of (1.5).

Now, we show that (3.2) holds. By using a similar argument to [46], it suffice to show $v^+(x, y, z, \xi, \eta; \epsilon, \alpha) - U(z - a_j x - b_j y, t) > 0$. Temporarily, we denote $a_j, b_j$ by $a, b$ for simplicity, respectively. Let

$$\mu_1 := \frac{c}{l} (z - ax - by).$$

If $\hat{\mu} < \mu_1$, then (3.2) holds. Assume that $\hat{\mu} \geq \mu_1$. We have

$$\hat{\mu} - \mu_1 = \frac{z - \frac{1}{a} \psi(ax, \alpha y)}{\sqrt{1 + \frac{1}{a^2} (\varphi_\xi^2(ax, \alpha y) + \varphi_\eta^2(ax, \alpha y))}} - \frac{c}{l} (z - ax - by) \geq 0,$$

which is equivalent to

$$\left( \frac{l}{\sqrt{1 + \varphi_\xi^2(\xi, \eta) + \varphi_\eta^2(\xi, \eta)}} - c \right) (z - ax - by) \geq \frac{l \nu_1}{\alpha \sqrt{1 + \varphi_\xi^2(\xi, \eta) + \varphi_\eta^2(\xi, \eta)}}.$$

Combining this inequality with the definition of $S(\xi, \eta)$ and using Lemma 2.2, we have

$$z - ax - by \geq \frac{l \nu_1}{\alpha \sqrt{1 + \varphi_\xi^2(\xi, \eta) + \varphi_\eta^2(\xi, \eta)}} \geq \frac{c \nu_1}{\alpha}, \quad (3.4)$$

Utilizing Lemma 2.1, we get $\varphi(\xi, \eta) > a \xi + b \eta$.

Choosing $\alpha$ and $\epsilon$ such that

$$\frac{1}{l} \sup_{|\mu| \geq \frac{c \nu_1}{\alpha}, t \in [0, T]} \left| \mu U_\xi \left( \frac{c}{l} \mu, t \right) \right| < \frac{\epsilon a^*_\epsilon}{2}.$$

Hereafter, we use $U_\xi$ to denote the first order derivative of $U$ with respect to the first variable. It follows that

$$v^+ - U = U \left( \frac{z - \frac{1}{a} \psi(ax, \alpha y)}{\sqrt{1 + \varphi_\xi^2(ax, \alpha y) + \varphi_\eta^2(ax, \alpha y)}} , t \right) - U \left( \frac{c}{l} (z - ax - by), t \right) + (a^+(t) + a^-(t)) \sigma(\xi, \eta)$$

$$\geq U \left( \frac{z - ax - by}{\sqrt{1 + \varphi_\xi^2(\xi, \eta) + \varphi_\eta^2(\xi, \eta)}} , t \right) - U \left( \frac{c}{l} (z - ax - by), t \right) + (a^+(t) + a^-(t)) \sigma(\xi, \eta)$$

$$= \frac{(z - ax - by) S(\xi, \eta)}{l} \int_0^1 U_\xi \left( \frac{\theta}{\sqrt{1 + \varphi_\xi^2 + \varphi_\eta^2}} + \frac{c}{l} (1 - \theta) \right) (z - ax - by), t \right) d\theta$$

$$+ (a^+(t) + a^-(t)) \sigma(\xi, \eta)$$
\[ \geq S(\xi, \eta) \left( \varepsilon (a^+ + a^-) - \frac{1}{I} \sup_{|\mu| \geq \frac{c_{l\mu}}{a}} \left| \mu U_\xi \left( \frac{c}{I} \mu, t \right) \right| \right) \]
\[ \geq S(\xi, \eta) \left( \varepsilon a^* - \frac{1}{I} \sup_{|\mu| \geq \frac{c_{l\mu}}{a}} \left| \mu U_\xi \left( \frac{c}{I} \mu, t \right) \right| \right) > \frac{\varepsilon a^*}{2} S(\xi, \eta) > 0. \]

Actually, by (3.3), (3.4) and (1.2), we take
\[ \alpha_0^+ (\varepsilon) := \min \left\{ 1, \frac{\varepsilon a^*}{2A}, \frac{p}{A}, \beta_1 \frac{c^2 v_1}{1 + \ln \left( \frac{2C_1 c_{\varepsilon a^*}}{\varepsilon} \right)} \right\} \]
and
\[ \varepsilon_0^+ := \min \left\{ 1, \frac{\varepsilon}{4a^*l} \right\}. \]

Then (3.2) follows immediately for \((x, y, z) \in \mathbb{R}^3\) and \(t \in [0, T]\) if \(0 < \varepsilon \leq \varepsilon_0^+ > 0\) and \(0 \leq \alpha \leq \alpha_0^+(\varepsilon)\).

We next show that (3.1) holds. Since \(\varepsilon (a^+ + a^-) S \leq a^* \varepsilon\), we need only to show
\[ \left| \hat{\mu} (\mu, t) - U \left( \frac{c}{I} (z - h(x, y) + t) \right) \right| \leq \varepsilon. \]

Assume that this is not true. Then there exist two sequences \(\{\gamma_n\}_{n=1}^\infty\) and \(\{(x_n, y_n, z_n)\}_{n=1}^\infty\) such that
\[ \lim_{n \to \infty} \gamma_n = \infty, \quad (x_n, y_n, z_n) \in D(\gamma_n) \]
and
\[ \left| \hat{\mu} (\mu, t) - U \left( \frac{c}{I} (z_n - h(x_n, y_n), t) \right) \right| > \varepsilon. \]

Let \(\xi_n = \alpha x_n, \eta_n = \alpha y_n, \zeta_n = \alpha z_n\) and
\[ \hat{\mu}_n = \frac{1}{\alpha} \frac{\xi_n - \varphi (\xi_n, \eta_n)}{\sqrt{1 + \varphi_n^2 (\xi_n, \eta_n) + \varphi_\eta^2 (\xi_n, \eta_n)}} = \frac{z_n - h(x_n, y_n) - \frac{1}{\alpha} (\varphi (\xi_n, \eta_n) - h(\xi_n, \eta_n))}{\sqrt{1 + \varphi_n^2 (\xi_n, \eta_n) + \varphi_\eta^2 (\xi_n, \eta_n)}}. \]

If \(\lim_{n \to \infty} \text{dist}((\xi_n, \eta_n), E') = \infty\), by applying Lemma 2.1 we obtain \(\lim_{n \to \infty} |\varphi (\xi_n, \eta_n) - h(\xi_n, \eta_n)| = 0\) and \(\lim_{n \to \infty} S(\xi_n, \eta_n) = 0\). Recall \(E' := \bigcup_{j=1}^n \partial \Omega_n \subset \mathbb{R}^2\). Then
\[ \lim_{n \to \infty} \left| \hat{\mu}_n - \frac{c}{I} (z_n - h(x_n, y_n)) \right| = 0. \]

This contradicts (3.6). If \(\text{dist}((\xi_n, \eta_n), E')\) remains finite uniformly in \(n\), then (3.5) implies
\[ \lim_{n \to \infty} (z_n - h(x_n, y_n)) = \pm \infty \quad \text{and} \quad \lim_{n \to \infty} \hat{\mu}_n = \pm \infty, \]
which also contradicts (3.6). This completes the proof. \(\square\)
Next we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( v^- (x, y, z, t) \) be defined as in (1.10). Consider solutions of (1.5) given by \( v(x, y, z, t; v^-) \) and \( v(x, y, z, t; v^+) \). We have

\[
v^- (x, y, z, t) \leq v(x, y, z, t; v^-) \leq v(x, y, z, t; v^+) \leq v^+ (x, y, z, t; \varepsilon, \alpha)
\]

for \((x, y, z) \in \mathbb{R}^3 \) and \( t > 0 \).

By the parabolic estimates, we know that there exists \( K > 0 \) such that the solutions \( v(x, y, z, t) \) of (1.5) with \( v_0 \in [\alpha^- - 1, \alpha^+ + 1] \) satisfy

\[
\|v(\cdot, \cdot, \cdot, \cdot)\|_{C^0(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [T, \infty))} < K \quad \text{and} \quad \|v(\cdot, \cdot, \cdot, \cdot)\|_{C^0(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [T, \infty))} < K,
\]

where the index \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) with \( \alpha_i \geq 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 3, \alpha_4 \leq 1, \) and \( \beta = (0, 0, 0, 2) \). Define

\[
V(x, y, z, t) := \lim_{k \to \infty} v(x, y, z, t + kT; v^-), \quad (x, y, z) \in \mathbb{R}^3, \quad t \in [0, T].
\]

Since \( v^- \) is a subsolution of (1.5) and satisfies \( v^- (x, y, z, t + T) = v^- (x, y, z, t) \) for all \((x, y, z) \in \mathbb{R}^3 \) and \( t \in [0, \infty) \), we get

\[
v(x, y, z, t + kT; v^-) \leq v(x, y, z, t + (k + 1)T; v^-), \quad (x, y, z) \in \mathbb{R}^3, \quad t \in [0, T].
\]

It follows that \( v(x, y, z, t + kT; v^-), \frac{\partial}{\partial t} v(x, y, z, t + kT; v^-), \frac{\partial^2}{\partial x^2} v(x, y, z, t + kT; v^-), \frac{\partial^2}{\partial y^2} v(x, y, z, t + kT; v^-) \) uniformly converge to \( V(x, y, z, t), \frac{\partial}{\partial t} V(x, y, z, t), \frac{\partial^2}{\partial x^2} V(x, y, z, t), \frac{\partial^2}{\partial y^2} V(x, y, z, t) \) as \( k \to \infty \) on the compact set of \((x, y, z, t) \in \mathbb{R}^3 \times [0, T] \), respectively. Since \( V(x, y, z, t) \leq v^+(x, y, z, t; \varepsilon, \alpha) \) for all \((x, y, z, t) \in \mathbb{R}^3 \times [0, T] \), by the arbitrariness of \( \varepsilon \) and \( \alpha \), we have

\[
\lim_{\gamma \to +\infty} \sup_{(x, y, z) \in D(\gamma), t \in [0, T]} |V(x, y, z, t) - v^- (x, y, z, t)| = 0.
\]

This completes the proof. \( \square \)

4. Uniqueness and stability

In the following, we first state the existence, uniqueness and stability results of a periodic V-form front in two-dimensional space. Then we characterize the periodic pyramidal traveling front as a combination of planar traveling fronts on the lateral surface and prove Theorem 1.2. Finally, we show that the three-dimensional periodic pyramidal traveling front is uniquely determined as a combination of two-dimensional V-form fronts on the edges.

Let \( \tilde{v}(\xi, \eta, t; \tilde{v}_0) \) be the solution of the following equation

\[
\tilde{v}_t - \tilde{v}_{\xi\xi} - \tilde{v}_{\eta\eta} - s\tilde{v}_{\eta} - f(\tilde{v}, t) = 0 \quad \text{for} \quad (\xi, \eta) \in \mathbb{R}^2, \quad t > 0,
\]

\[
\tilde{v}_0(\xi, \eta) = v(\xi, \eta, 0) \quad \text{for} \quad (\xi, \eta) \in \mathbb{R}^2.
\]
Theorem 4.1. (See [49, Theorem 1.1].) For any $s > c$, there exists a unique $\hat{v}(\xi, \eta, t; s)$ satisfying
\[
\hat{v}_t - \hat{v}_{\xi\xi} - \hat{v}_{\eta\eta} - s\hat{v}_\eta - f(\hat{v}, t) = 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, \ t \in \mathbb{R},
\]
and
\[
\hat{v}(\cdot, \cdot, \cdot; +T; s) = \hat{v}(\cdot, \cdot, \cdot; s) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, \ t \in \mathbb{R}.
\]
In addition,
\[
\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2} \left| \hat{v}(\xi, \eta) - U\left(\frac{c}{s} \left( \eta - \frac{\sqrt{s^2 - c^2}}{c} |\xi| \right), t \right) \right| = 0.
\]
One also has
\[
U\left(\frac{c}{s} \left( \eta - \frac{\sqrt{s^2 - c^2}}{c} |\xi| \right), t \right) < \hat{v}(\xi, \eta, t) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2,
\]
and
\[
\inf_{W^-(t) + \frac{\tau}{2} \leq \hat{v}(\xi, \eta) \leq W^+(t) - \frac{\tau}{2}} -\hat{v}_\eta(\xi, \eta) > 0 \quad \text{for } \epsilon \in (0, \epsilon_0).
\]
Furthermore, for any bounded initial function $\tilde{v}_0(\xi, \eta) \in C^0(\mathbb{R}^2)$ with
\[
\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2} \left| \tilde{v}_0(\xi, \eta) - U\left(\frac{c}{s} \left( \eta - \frac{\sqrt{s^2 - c^2}}{c} |\xi| \right), 0 \right) \right| = 0,
\]
there is
\[
\lim_{t \to \infty} \left\| \tilde{v}(\xi, \eta, t) - \hat{v}(\xi, \eta, t; s) \right\|_{C(\mathbb{R}^2)} = 0,
\]
or equivalently
\[
\lim_{k \to \infty} \left\| \tilde{v}(\xi, \eta, t + kT) - \hat{v}(\xi, \eta, t; s) \right\|_{C(\mathbb{R}^2 \times [0, T])} = 0.
\]
From (1.10) and Lemma 3.1, we obtain $v^-(x, y, z, t) < V(x, y, z, t) < v^+(x, y, z, t)$. Hereafter we set $x = (x, y, z) \in \mathbb{R}^3$. Since $\varphi(0, 0) > 0$, for any given $R > 0$, we have
\[
\lim_{\alpha \to 0} \inf_{|x| \leq R} v^+(x, t) \geq W^+(t). \quad (4.1)
\]
For any given $v(x, t) \in C^{2,1}(\mathbb{R}^3 \times (0, \infty))$ and $\epsilon \in (0, \epsilon_0)$ we define
\[
G(v; \epsilon \frac{\tau}{2}) := \inf_{W^-(t) + \frac{\tau}{2} \leq v(\xi, \eta) \leq W^+(t) - \frac{\tau}{2}} -v_z(x, t).
\]
Then for any $\epsilon \in (0, \epsilon_0)$ and $\alpha \in (0, \alpha_0^+(\epsilon))$, we have $G(v^+; \frac{\tau}{2}) > 0$ by utilizing $|\nabla \varphi| < \tau$ and (1.2). Similarly, we have $G(v^-; \frac{\tau}{2}) > 0$. 
Lemma 4.2. Assume that $v \in C^{2,1}(\mathbb{R}^3 \times (0, \infty))$ satisfies $G(v; \frac{x}{\rho^3}) > 0$. Choose a constant $\rho$ with

$$\rho > \frac{2K_0}{\beta G(v; \frac{x}{\rho^3})} \sup_{W^-(t) + \frac{x}{\rho^3} \leq W(t) - \frac{x}{\rho^3}, t \in [0,T]} |f(u, t)|$$

and a positive constant $\beta$ ($\beta < \frac{\lambda_0}{4}$) small enough. Let $\delta \in (0, \frac{\epsilon}{2K_0})$ be given. Then

$$w^+(x, t; v) := v(x, y, z - \rho \delta (1 - e^{-\beta t}), t) + \delta a(t)$$

is a supersolution of (1.5) if $\mathcal{L}[v] \geq 0$ in $\mathbb{R}^3 \times (0, \infty)$. Similarly,

$$w^-(x, t; v) := v(x, y, z + \rho \delta (1 - e^{-\beta t}), t) - \delta a(t)$$

is a subsolution of (1.5) if $\mathcal{L}[v] \leq 0$ in $\mathbb{R}^3 \times (0, \infty)$, where

$$a(t) = \exp \left\{ \left( v^+ + v^- - \frac{\nu_0}{4} \right) t + \int_0^t f(u(W^+(\tau), \tau)) d\tau + \int_0^t f(u(W^-(\tau), \tau)) d\tau \right\}$$

and

$$K_0 = \max_{t \in [0, T]} \exp \left\{ (v^+ + v^-) t + \int_0^t f(u(W^+(\tau), \tau)) d\tau + \int_0^t f(u(W^-(\tau), \tau)) d\tau \right\} \geq 1.$$

Proof. The proof is similar to that of [49, Lemma 3.4] and so we omit it. □

We denote the norm of $x = (x, y, z) \in \mathbb{R}^3$ by $|x| := \sqrt{x^2 + y^2 + z^2}$. Put

$$M_1 := \min \left\{ \inf_{x \in \mathbb{R}^3} v_0, -2 \max_{t \in [0, T]} \left\{ |W^+(t)|, |W^-(t)| \right\} - 1 - \|v_0\|_{L^\infty(\mathbb{R}^3)} \right\}.$$  \hspace{1cm} (4.4)

$$M_2 := \max \left\{ \sup_{x \in \mathbb{R}^3} v_0, 2 \max_{t \in [0, T]} \left\{ |W^+(t)|, |W^-(t)| \right\} + 1 + \|v_0\|_{L^\infty(\mathbb{R}^3)} \right\}.$$  \hspace{1cm} (4.5)

$$m := \max_{M_1 \leq u \leq M_2, t \in [0, T]} |f(u(t)|.$$  \hspace{1cm} (4.6)

For $x_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and $R > 0$, take

$$B(x_0; R) := \{ x \in \mathbb{R}^3 : |x - x_0| < R \}.$$  \hspace{1cm} (4.7)

For any subset $D \subset \mathbb{R}^3$ the characteristic function $\chi_D$ of $D$ is defined by

$$\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D. \end{cases}$$  \hspace{1cm} (4.8)

Let $F(x, t)$ be a given continuous function satisfying

$$\sup_{x \in \mathbb{R}^3, t > 0} |F(x, t)| \leq m.$$  \hspace{1cm} (4.9)
For a given initial value $u_0 \in L^\infty(\mathbb{R}^3)$, we consider the following linear equation

$$
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - l \frac{\partial}{\partial z} \right) u - F(x, t)u = 0, \quad x \in \mathbb{R}^3, \quad t > 0,
$$

$$
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3. \quad (4.6)
$$

The following lemma follows from [47, Lemma 3].

**Lemma 4.3.** Let $u(x, t)$ be a solution of (4.6). Then for all $t > 0$ we have

$$
\sup_{x \in \mathbb{R}^3} u(x, t) \leq e^{mt} \max \left\{ 0, \sup_{x \in \mathbb{R}^3} u_0(x) \right\},
$$

$$
e^{mt} \min \left\{ 0, \inf_{x \in \mathbb{R}^3} u_0(x) \right\} \leq \inf_{x \in \mathbb{R}^3} u(x, t),
$$

$$
\sup_{x \in \mathbb{R}^3} u(x, t) \leq e^{mt} \| u_0 \|_{L^\infty(\mathbb{R}^3)}.
$$

If $u_0 = 1 - \chi_{B(x_0, \sqrt{3}R)}$ for $x_0 \in \mathbb{R}^3$ and $R > 0$, we have

$$
|u(x_0, t)| \leq 3e^{mt} \operatorname{erfc}\left( \frac{R - lt}{\sqrt{4t}} \right),
$$

where the complementary error function $\operatorname{erfc}$ is given by

$$
\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.
$$

Furthermore, for any $\gamma > 0$ we have

$$
\sup_{x \in D(2\gamma)} u(x, t) \leq 3e^{mt} \operatorname{erfc}\left( \frac{\gamma - lt}{\sqrt{4t}} \right) \sup_{x \in D(\gamma)^c} u_0(x) + e^{mt} \sup_{x \in D(\gamma)} u_0(x).
$$

For each $j (1 \leq j \leq n)$ we consider a plane perpendicular to an edge $\Gamma_j = S_j \cap S_{j+1}$. Then the cross section of $z = \max\{h_j(x, y), h_{j+1}(x, y)\}$ in this plane has a periodic V-form front. Let $E_j$ be the two-dimensional periodic V-form front as in Theorem 4.1 corresponding to the cross section $z = \max\{h_j(x, y), h_{j+1}(x, y)\}$. We will write the precise definition of $E_j$ later.

Define

$$
p_j := A_j B_{j+1} - A_{j+1} B_j > 0,
$$

$$
q_j := \sqrt{(A_{j+1} - A_j)^2 + (B_{j+1} - B_j)^2} > 0.
$$

Let

$$
A_{n+1} := A_1, \quad B_{n+1} := B_1.
$$
Then we have
\[ p_n := A_n B_1 - A_1 B_n > 0, \quad q_n := \sqrt{(A_1 - A_n)^2 + (B_1 - B_n)^2} > 0. \]

The direction of \( \Gamma_j \) is given by
\[ \nu_j \times \nu_{j+1} = \frac{1}{\sqrt{\tau^2 p_j^2 + q_j^2}} \left( \begin{array}{c} B_{j+1} - B_j \\ A_j - A_{j+1} \\ \tau (A_j B_{j+1} - A_{j+1} B_j) \end{array} \right), \]
and the traveling direction of a two-dimensional V-form wave \( E_j \) is given by
\[ \frac{v_{j+1} - v_j}{|v_{j+1} - v_j|} \times v_j \times v_{j+1} = \frac{1}{q_j} \left( \begin{array}{c} A_j - A_{j+1} \\ B_{j+1} - B_j \\ 0 \end{array} \right) \times \frac{1}{\sqrt{\tau^2 p_j^2 + q_j^2}} \left( \begin{array}{c} B_{j+1} - B_j \\ A_j - A_{j+1} \\ \tau (A_j B_{j+1} - A_{j+1} B_j) \end{array} \right) \]
\[ = \frac{1}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} \left( \begin{array}{c} \tau (B_{j+1} - B_j) p_j \\ \tau (A_j - A_{j+1}) p_j \\ q_j^2 \end{array} \right). \]

Let \( s_j \) be the speed of \( E_j \) and \( 2\theta_j \) (\( 0 < \theta_j < \pi/2 \)) be the angle between \( S_j \) and \( S_{j+1} \). Then we get
\[ s_j \sin \theta_j = c, \quad \sin \theta_j = \frac{\sqrt{\tau^2 p_j^2 + q_j^2}}{q_j \sqrt{1 + \tau^2}} \]
and
\[ s_j = \frac{l q_j}{\sqrt{\tau^2 p_j^2 + q_j^2}}. \]

The speed of \( E_j \) toward the z-axis equals
\[ \sqrt{\frac{\tau^2 p_j^2 + q_j^2}{q_j}} s_j = c \sqrt{1 + \tau^2} = l, \]
which coincides with the speed of \( V \). Let
\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_j \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = R_j^T \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \]
where \( R_j^T \) is the transposed matrix of \( R_j \). Here we take
\[ R_j = \begin{pmatrix} \frac{A_j - A_{j+1}}{q_j} & \frac{\tau (B_j - B_{j+1}) p_j}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} & \frac{B_j - B_{j+1}}{\sqrt{\tau^2 p_j^2 + q_j^2}} \\ \frac{B_{j+1} - B_j}{q_j} & \frac{\tau (A_{j+1} - A_j) p_j}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} & \frac{A_{j+1} - A_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} \\ 0 & \frac{\tau p_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} & -\frac{\tau p_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} \end{pmatrix}. \]
$R^j_T = \begin{pmatrix}
\frac{A_j - A_{j+1}}{q_j} & \frac{B_j - B_{j+1}}{q_j} & 0 \\
\frac{\tau (B_j - B_{j+1}) p_j}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} & \frac{\tau (A_{j+1} - A_j) p_j}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} & \frac{q_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} \\
\frac{B_j - B_{j+1}}{\sqrt{\tau^2 p_j^2 + q_j^2}} & \frac{A_{j+1} - A_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} & -\frac{\tau p_j}{\sqrt{\tau^2 p_j^2 + q_j^2}}
\end{pmatrix}.

Define $E_j$ as

$$E_j(x, y, z, t) := \hat{v} \left( \frac{(A_j - A_{j+1}) x + (B_j - B_{j+1}) y}{q_j}, \frac{\tau (B_j - B_{j+1}) p_j x + \tau (A_{j+1} - A_j) p_j y + q_j^2 z}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}}, t; \frac{lq_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} \right).$$

Direct calculations show that

$$\mathcal{L}[E_j] = \hat{v}_t - \hat{v}_{\xi \xi} - \hat{v}_{\eta \eta} - s_j \hat{v}_\eta - f (\hat{v}, t) = 0 \quad \text{for all } (\xi, \eta, t) \in \mathbb{R}^2 \times [0, T].$$

Hence $E_j(x, t)$ satisfies (1.5) for each $j$ ($1 \leq j \leq n$). We call $E_j$ a planar periodic V-form front corresponding to an edge $\Gamma_j$.

Set

$$Q_j := \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \Gamma) = \text{dist}(x, \Gamma_j) \}, \quad 1 \leq j \leq n.$$

Then we have

$$\mathbb{R}^3 = \bigcup_{j=1}^n Q_j.$$

Define

$$\hat{E}(x, t) := \max_{1 \leq j \leq n} E_j(x, t).$$

We have that $\hat{E}(x, t)$ is strictly monotone decreasing in $z$, because $E_j(x, t)$ is strictly monotone decreasing in $z$. In addition, $\hat{E}(x, t)$ has the following properties.

**Lemma 4.4.** The function $\hat{E}(x, t)$ satisfies

$$v^- (x, t) < \hat{E}(x, t) < V (x, t), \quad x \in \mathbb{R}^3, t \in [0, T]$$

and

$$\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0, T]} \left| \hat{E}(x, t) - v^- (x, t) \right| = 0. \quad (4.7)$$
Proof. By Theorem 4.1 we have

\[ \max \left\{ U \left( \frac{c}{t} (z - h_j(x, y)), t \right), U \left( \frac{c}{t} (z - h_{j+1}(x, y)), t \right) \right\} < E_j(x, t). \]

So

\[ v^-(x, t) = U \left( \frac{c}{t} (z - h(x, y)), t \right) < \hat{E}(x, t). \]

We consider the left-hand side and the right-hand side as the initial value of (1.5), respectively. Namely, we consider the solution \( v(x, t + kT; U(c\cdot (z - h_j(x, y)), 0)) \) and \( v(x, t + kT; E_j(x, 0)) \) of (1.5). Letting \( k \to \infty \), we have

\[ E_j(x, t) < V(x, t) \quad \text{for all} \quad x \in \mathbb{R}^3, t \in [0, T]. \]

Consequently, the comparison principle implies

\[ \hat{E}(x, t) < V(x, t) \quad \text{for all} \quad x \in \mathbb{R}^3, t \in [0, T]. \]

Without loss of generality, as \( \gamma \to \infty \) we can assume that either

\[ |z - h(x, y)| \to \infty \]

or

\[ \sup |z - h(x, y)| < \infty, \quad \text{dist}(x, \Gamma) \to \infty, \quad x \in Q_j \text{ for some } j. \]

For the former case \( \hat{E}(x, t) \) and \( v^-(x, t) \) converge to \( W^+(t) \) and thus the equality stated above holds true. For the latter case \( E_j(x, t) \) converges to \( W^-(t) \) for any \( i \neq j \) and we have \( |E_j(x, t) - v^-(x, t)| \to 0. \) Thus, we get (4.7). \( \Box \)

Now, we consider the following ordinary differential equations

\[ \frac{d}{dt} w(t) = f(w(t), t), \quad w(0) = M_1. \]

Similarly, we define \( \bar{w}(t) \) by

\[ \frac{d}{dt} \bar{w}(t) = f(\bar{w}(t), t), \quad \bar{w}(0) = M_2. \]

By (4.4) and (4.5) we know that \( M_1 < v_0 < M_2. \) Hence

\[ w(t) \leq v(x, t; v_0) \leq \bar{w}(t) \]

follows from the comparison principle. By the hypotheses (H2) we know that the Poincaré map \( P(\alpha) \) is monotonic and has only three fixed points with \( \alpha^\pm \) being stable, thus \( P(\alpha) > \alpha \) for all \( \alpha < \alpha^- \) and \( P(\alpha) < \alpha \) for all \( \alpha > \alpha^+ \) (see also [1]). It follows that
Proposition 4.5. Assume that \( v_0 \) satisfies (1.13). For any given \( \varepsilon_1 > 0 \), we can choose \( k_* \in \mathbb{N} \) large enough such that

\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0, T]} |v(x, t + kT; v_0) - \hat{E}(x, t)| < \varepsilon_1 \quad \text{for any fixed } k \geq k_*,
\]

and

\[
\lim_{k \to \infty} w(t + kT) = W^-(t) \quad \text{uniformly for } t \in [0, T].
\]

Hence for all \( \delta \in (0, \frac{\varepsilon}{2k_0}) \), \( x \in \mathbb{R}^3 \) and \( t \in [0, T] \), there exists a number \( k_1 \) satisfying

\[
W^-(t + kT) - \delta \leq w(t + kT) \leq v(x, t + kT; v_0) \leq \sup_{t \geq 0} (v(x, t + kT)) \leq W^+(t + kT) + \delta, \quad k \geq k_1.
\]

By Lemma 4.3, we get

\[
\sup_{x \in D(2\gamma)} |v(x, t + kT; v_0) - V(x, t)| \leq 3e^{m(t+kT)} \frac{\gamma - l(t+kT)}{\sqrt{4(t+kT)}} \sup_{D(\gamma)} |v_0(x) - V(x, 0)| + e^{m(t+kT)} \sup_{D(\gamma)} |v_0(x) - V(x, 0)|
\]

for \( t \in [0, T] \). Then we obtain

\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0,T]} |v(x, t + kT; v_0) - V(x, t)| = 0 \quad \text{for any fixed } k \in \mathbb{N}.
\]

Notice that (4.9) implies

\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} |v_0(x) - v^-(x, 0)| = 0.
\]

Let \( v^+(x, t) \) be as in Lemma 3.1. We define

\[
V^+(x, t) := \lim_{k \to \infty} v(x, t + kT; v^+) \quad \text{for any } x \in \mathbb{R}^3 \text{ and } t \in [0, T].
\]

Since \( v^+(x, t) \) is a supersolution of (1.5) and satisfies \( v^+(x, t + T) = v^+(x, t) \), we have that \( v(x, t + kT; v^+) \geq v(x, t + (k + 1)T; v^+) \) for any \( x \in \mathbb{R}^3 \), \( t \in [0, T] \) and \( k \in \mathbb{N} \). Then proceeding the similar argument as to \( V(x, t) \), we obtain that \( V^+(x, t) \) is \( C^2 \) in \( x \) and \( C^1 \) in \( t \), and satisfies (1.5) and \( V^*(x, t) = V^+(x, t + T) \) for all \( x \in \mathbb{R}^3 \), \( t \in \mathbb{R} \). It is clear that

\[
V(x, t) \leq V^*(x, t), \quad (x, t) \in \mathbb{R}^3 \times [0, T].
\]

Now we state some lemmas, which play key roles in proving the asymptotic stability.
In addition we have

\[
\liminf_{k \to \infty} \inf_{x \in \mathbb{R}^3, t \in [0, T]} (v(x, t + k\tau; v_0) - \hat{E}(x, t)) \geq 0.
\]

**Proof.** Without loss of generality we assume \(0 < \epsilon_1 < 1/2\). Set

\[
I_j := \Omega_j \cap \Omega_{j+1} = \left\{ \left( \frac{A_j + A_{j+1}}{B_j + B_{j+1}} \right) \left| s \geq 0 \right\}, \quad 1 \leq j \leq n - 1,
\]

\[
I_n := \Omega_n \cap \Omega_1 = \left\{ \left( \frac{A_n + A_1}{B_n + B_1} \right) \left| s \geq 0 \right\}.
\]

Then \(I_j\) is the projection of \(\Gamma_j\) onto the \(x-y\) plane and \(\bigcup_{j=1}^n I_j\) is the projection of \(\Gamma\) onto the \(x-y\) plane.

Without loss of generality we assume \(x \in Q_j\) for some \(1 \leq j \leq n\) as \(|x| \to \infty\). Since \((\partial/\partial x)^2 + (\partial/\partial y)^2\) is invariant under rotations on the \(x-y\) plane, we can assume \(\Omega_j \cap \Omega_{j+1} = \{(0, y, 0) \mid y \geq 0\}\) and

\[
(A_j, B_j) = (A, B), \quad (A_{j+1}, B_{j+1}) = (-A, B),
\]

where \(A > 0\), \(B > 0\) and \(A^2 + B^2 = 1\). Two planes \(S_{j+1}\) and \(S_j\) are \(z = \tau(\pm Ax \pm By)\) and \(z = \tau(Ax + By)\), respectively. The common line \(\Gamma_j\) of them is \(x = 0, z = \tau By\). The projection of \(Q_j\) onto the \(x-y\) plane is given by \(\{y \geq a|x|, x \geq 0\} \cup \{y \geq b|x|, x \leq 0\}\) for some \(a > 0\) and \(b > 0\). Hereafter we denote \(Q_j\) by \(Q\) for simplicity.

Actually, (4.10) implies

\[
\lim_{y \to \infty} \sup_{x \in D(\gamma)} \left| v_0(x) - U \left( \frac{c}{1}(z - \tau By - \tau A|x|), 0 \right) \right| = 0.
\]

The unit normal vector of the common line \(\Gamma_j\) is given by \(\frac{1}{\sqrt{1 + \tau^2 B^2}}(0, -\tau B, 1)\), \(2\theta_j\) is the angle between \(S_j\) and \(S_{j+1}\) \((0 < \theta_j < \pi/2)\). We denote \(\theta_j\) by \(\theta\) for simplicity. Then we get

\[
\frac{\sqrt{1 + \tau^2 B^2}}{\sqrt{1 + \tau^2}} = \sin \theta.
\]

The change of variables is as follows:

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & -\frac{\tau B}{\sqrt{1 + \tau^2 B^2}} \\
  0 & \frac{1}{\sqrt{1 + \tau^2 B^2}} & -\frac{\tau B}{\sqrt{1 + \tau^2 B^2}} \\
  0 & \frac{1}{\sqrt{1 + \tau^2 B^2}} & -\frac{\tau B}{\sqrt{1 + \tau^2 B^2}}
\end{pmatrix}
\begin{pmatrix}
  \xi \\
  \eta \\
  \zeta
\end{pmatrix}.
\]

Then we have

\[
\begin{pmatrix}
  \xi \\
  \eta \\
  \zeta
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & \frac{\tau B}{\sqrt{1 + \tau^2 B^2}} \\
  0 & \frac{1}{\sqrt{1 + \tau^2 B^2}} & -\frac{\tau B}{\sqrt{1 + \tau^2 B^2}} \\
  0 & \frac{1}{\sqrt{1 + \tau^2 B^2}} & -\frac{\tau B}{\sqrt{1 + \tau^2 B^2}}
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}.
\]
For this change of variables we get

\[ U\left( \frac{c}{s_j} \left( \eta - \frac{\sqrt{s_j^2 - c^2}}{c} |\xi| \right), t \right) = U\left( \frac{c}{t} \left( z - \tau By - \tau A|x| \right), t \right), \]

where \( s_j = \frac{1}{\sqrt{1 + \tau^2 B^2}} \). Hereafter we denote \( s_j \) and \( E_j \) by \( s \) and \( E \) for simplicity, respectively. Now

\[ E(x, t) = \hat{v}(\xi, \eta, t; s) = \hat{v}(x, \frac{z - \tau By}{\sqrt{1 + \tau^2 B^2}}, t; s) \]

is a solution of (1.5). Let \( \tilde{W}(\xi, \eta, t) = \tilde{W}(\xi, \eta, t; \tilde{W}_0) \) be the solution of

\[ \begin{align*}
\tilde{W}_t - \tilde{W}_{\xi\xi} - \tilde{W}_{\eta\eta} - s\tilde{W}_{\eta} - f(\tilde{W}, t) &= 0, \quad (\xi, \eta) \in \mathbb{R}^2, \quad t > 0, \\
\tilde{W}(\xi, \eta, 0) &= \tilde{W}_0(\xi, \eta), \quad (\xi, \eta) \in \mathbb{R}^2.
\end{align*} \tag{4.12} \]

Taking

\[ W_0(x, y, z) = \tilde{W}_0 \left( x, \frac{z - \tau By}{\sqrt{1 + \tau^2 B^2}}, 0 \right), \]

we have \( W(x, y, z, t; W_0) = \tilde{W}(\xi, \eta, t; \tilde{W}_0) \) satisfying

\[ \begin{align*}
\mathcal{L}[W] &= 0, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \\
W(x, 0) &= W_0(x), \quad (x, y, z) \in \mathbb{R}^3.
\end{align*} \tag{4.13} \]

Notice that

\[ U\left( \frac{c}{s} \left( \eta - \frac{\sqrt{s^2 - c^2}}{c} |\xi| \right), t \right) = U\left( \frac{c}{t} \left( z - \tau By - \tau A|x| \right), t \right) \leq U\left( \frac{c}{t} (z - h(x, y)), t \right). \]

By the strong comparison principle we get

\[ E(x, t) < V(x, t), \quad x \in \mathbb{R}^3 \text{ and } t \in \mathbb{R}. \]

Since this inequality holds true for all edges, we have

\[ \hat{E}(x, t) < V(x, t), \quad x \in \mathbb{R}^3 \text{ and } t \in \mathbb{R}. \]

By utilizing (4.7) and (4.10), we get

\[ \lim_{{\gamma \to \infty}} \sup_{{x \in D(\gamma) \cap Q}} |v_0(x) - E(x, 0)| = 0. \]
We choose a function \( g(\cdot) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}) \) with
\[
\begin{align*}
g(\gamma) &= \sup_{x \in D(\gamma) \cap Q} |v_0(x) - E(x, 0)| \quad \text{for } \gamma \geq 1, \\
\sup_{x \in D(\gamma) \cap Q} |v_0(x) - E(x, 0)| &\leq g(\gamma) \leq \min \{ |W^+(t)|, |W^-(t)| \} + 1 + \|v_0\|_{L^\infty(\mathbb{R}^2)}, \\
0 &< \gamma < 1, \\
g'(\gamma) < 0, \quad 0 < \gamma < 1, \\
g(\gamma) &\equiv g(-\gamma) \quad \text{for } \gamma \in \mathbb{R}.
\end{align*}
\]
It is obvious that \( g'(\gamma) \) is monotone non-increasing in \( \gamma > 0 \) and satisfies \( \lim_{\gamma \to \infty} g(\gamma) = 0 \). Since
\[
\dist(x, \Gamma) = \dist((x, y, z), \Gamma) = \sqrt{1 + \tau^2 B^2 x^2 + (z - \tau B y)^2}
\]
for \( x \in Q \), we have
\[
|v_0(x) - E(x, 0)| \leq g(|x|) = g\left(\sqrt{1 + \tau^2 B^2 x^2 + (z - \tau B y)^2}\right) \quad \text{for } x \in Q.
\]
We study (4.12) for \( \tilde{W}_0^\pm(\xi, \eta) := \tilde{v}(\xi, \eta, 0; s) \pm g\left(\sqrt{\xi^2 + \eta^2}\right) \), which is equivalent to study (4.13) for
\[
W_0^\pm(x) := E(x, 0) \pm g\left(\sqrt{x^2 + \frac{1}{1 + \tau^2 B^2} (z - \tau B y)^2}\right),
\]
respectively. It is obvious that
\[
\lim_{\gamma \to \infty} \sup_{R \to \infty, \xi^2 + \eta^2 > R^2} |\tilde{W}_0^\pm(\xi, \eta) - \tilde{v}(\xi, \eta, 0; s)| = 0.
\]
For \( s = \frac{1}{\sqrt{1 + \tau^2 B^2}} \), applying Theorem 4.1 we have
\[
\lim_{k \to \infty} \| \tilde{W}(\xi, \eta, t + kT; \tilde{W}_0^\pm) - \tilde{v}(\xi, \eta, t; s) \|_{C(\mathbb{R}^2 \times [0, T])} = 0.
\]
Thus
\[
\lim_{k \to \infty} \| W(x, t + kT; W_0^\pm) - E(x, t) \|_{C(\mathbb{R}^3 \times [0, T])} = 0.
\]
Taking \( k_j' \in \mathbb{N} \) large enough such that
\[
\sup_{k \geq k_j'} \| W(\cdot, \cdot + kT; W_0^\pm) - E(\cdot, \cdot) \|_{C(\mathbb{R}^3 \times [0, T])} < \frac{\epsilon_1}{2}.
\] (4.15)
Let $\nu(x, t)$ be a solution of (1.5) and set

$$\tilde{\nu}^\pm(x, t) = \nu(x, t) - W(x, t; W_0^\pm).$$

Then $\tilde{\nu}^\pm$ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - l \frac{\partial}{\partial z} + \int_0^1 f_\theta(\theta \nu(x, t) + (1 - \theta)W(x, t; W_0^\pm)) d\theta \tilde{\nu}^\pm(x, t) = 0, \\
&x \in \mathbb{R}^3, \ t > 0,
\end{aligned}$$

$$\tilde{\nu}^\pm(x, 0) = \nu_0(x) - E(x, 0) \mp g\left(\sqrt{x^2 + \frac{1}{1 + r^2B^2}(z - \tau By)^2}\right), \ x \in \mathbb{R}^3,$$

respectively. In particular, from (4.14) we get

$$\tilde{\nu}^+(x, 0) \leq 0, \quad \tilde{\nu}^-(x, 0) \geq 0 \quad \text{if } x \in Q.$$

Let $\hat{\nu}^\pm(x, t)$ be defined by

$$\begin{aligned}
&\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - l \frac{\partial}{\partial z} + \int_0^1 f_\theta(\theta \nu(x, t) + (1 - \theta)W(x, t; W_0^\pm)) d\theta \hat{\nu}^\pm(x, t) = 0, \\
&x \in \mathbb{R}^3, \ t > 0,
\end{aligned}$$

$$\hat{\nu}^\pm(x, 0) = \left(\nu_0(x) - E(x, 0) \pm g\left(\sqrt{x^2 + \frac{1}{1 + r^2B^2}(z - \tau By)^2}\right)\right)(1 - \chi_Q(x)), \ x \in \mathbb{R}^3.$$

By the comparison principle we obtain

$$\tilde{\nu}^+(x, t) \leq \hat{\nu}^+(x, t), \quad -\hat{\nu}^-(x, t) \leq \tilde{\nu}^-(x, t). \quad (4.16)$$

Then we have

$$2M_1 - 1 \leq \left(\nu_0(x) - E(x, 0) \pm g\left(\sqrt{x^2 + \frac{1}{1 + r^2B^2}(z - \tau By)^2}\right)\right)(1 - \chi_Q(x)) \leq 2M_2 + 1.$$ 

Applying Lemma 4.3 to $\hat{\nu}^\pm(x, t)$, for $t > 0$ we have

$$0 \leq \hat{\nu}^\pm(x, t) \leq 3(2M_2 + 1)e^{mt}\text{erfc}\left(\frac{R - lt}{\sqrt{4t}}\right) \quad \text{if } x \in Q \quad \text{and} \quad \sqrt{3R} < \text{dist}(x, \partial Q).$$

It follows that

$$\lim_{R \to \infty} \sup_{x \in Q, \text{dist}(x, \partial Q) \geq R, t \in [0, T]} \hat{\nu}^\pm(x, t + kT) = 0$$
for any fixed $k \in \mathbb{N}^+$. Applying this equality, (4.15) and (4.16) to $v(x, t + kT, v_0) = \tilde{v}(x, t + kT) + W(x, t + kT, W_0)$, we can take a constant $r_j > 0$ large enough such that

$$
\sup_{x \in Q, \text{dist}(x, \partial Q) \geq r_j, t \in [0, T]} |v(x, t + kT, v_0) - E(x, t)| < \varepsilon_1.
$$

Thus we have obtained the estimates on $Q$, namely, on $Q_j$ for some $j$. We set

$$k_* := \max\{k'_1, \ldots, k'_n\}.
$$

Applying the argument stated above for each $j$ ($1 \leq j \leq n$), we have

$$
\max_{1 \leq j \leq n} \sup_{x \in Q, \text{dist}(x, \partial Q) \geq r_j(t), t \in [0, T]} |v(x, t + kT, v_0) - E(x, t)| < \varepsilon_1
$$

for any $k \geq k_*$. 

(4.17)

with $\hat{r} := \max\{r_1, r_2, \ldots, r_n\}$. Applying Lemma 4.3 to $v(x, t + kT, v_0) - V(x, t)$ we have

$$
\sup_{x \in D(2\gamma), t \in [0, T]} |v(x, t + kT, v_0) - V(x, t)|
$$

$$
\leq 3e^{m(t + kT)} \text{erfc} \left( \frac{\gamma - l(t + kT)}{\sqrt{4(t + kT)}} \right) \sup_{D(\gamma)} |v_0(x) - V(x, 0)| + e^{m(t + kT)} \sup_{D(\gamma)} |v_0(x) - V(x, 0)|.
$$

From the definitions of $\Gamma$ and $Q_j$ we get

$$
\lim_{R \to \infty} \inf_{|x| \geq R, \text{dist}(x, \partial Q_j) \leq \hat{r}} = \infty \quad \text{for all } 1 \leq j \leq n.
$$

Thus we obtain

$$
\lim_{R \to \infty} \sup_{|x| \geq R, \text{dist}(x, \partial Q_j) \leq \hat{r}, t \in [0, T]} |v(x, t + kT, v_0) - \hat{E}(x, t)| < \varepsilon_1 \quad \text{for all } 1 \leq j \leq n.
$$

By this estimate and (4.17), we have

$$
\sup_{|x| \geq R, t \in [0, T]} |v(x, t + kT, v_0) - \hat{E}(x, t)| < \varepsilon_1 \quad \text{for any fixed } k \geq k_*.
$$

We next estimate $v(x, t + kT, v_0)$ from below. We set $v_0(x) = \tilde{v}_0(\xi, \eta, \zeta)$. Define

$$
\hat{p}(\xi, \eta) := \inf_{\xi \in \mathbb{R}} \min \left\{ 0, \tilde{v}_0(\xi, \eta, \zeta) - \tilde{v}(\xi, \eta, t; s) \right\} \leq 0.
$$

From (4.11), $0 < \varepsilon < 1$ and the assumptions of $v_0$ we have

$$
\lim_{R \to \infty} \sup_{|\xi|^2 + |\eta|^2 > R^2} |\hat{p}(\xi, \eta)| = 0.
$$

We choose a positive value function $\hat{q}(r) \in C^\infty[0, \infty)$ satisfying $\lim_{r \to \infty} \hat{q}(r) = 0$ and

$$
\sup_{|\xi|^2 + |\eta|^2 > r^2} |\hat{p}(\xi, \eta)| \leq \hat{q}(r) \quad \text{for all } r \geq 0.
$$
We set \( W_0 \) as in (4.13) and
\[
\tilde{W}_0(\xi, \eta) = \hat{v}(\xi, \eta, 0; s) + \tilde{q}(\sqrt{\xi^2 + \eta^2}).
\]
Then we have
\[
\tilde{W}_0(\xi, \eta) \leq \hat{v}_0(\xi, \eta, \zeta),
\]
and thus
\[
\tilde{W}(x; W_0) \leq v(x; t; v_0) \quad \text{for all } x \in \mathbb{R}^3 \text{ and } t > 0,
\]
\[
W(x; W_0) \leq v(x; t; v_0) \quad \text{for all } x \in \mathbb{R}^3 \text{ and } t > 0.
\]

Theorem 4.1 implies
\[
\lim_{k \to \infty} \sup_{x \in \mathbb{R}^3, t \in [0,T]} |W(x, t + kT; W_0) - E(x, t)| = 0.
\]

For any given \( \varepsilon' > 0 \) there exists \( k'_j \in \mathbb{N} \) such that
\[
E(x, t) - \varepsilon' \leq v(x, t + kT; v_0) \quad \text{for all } x \in \mathbb{R}^3, t \in [0, T] \text{ and } k > k'_j.
\]
Applying the above argument to all \( j \) we can obtain
\[
\hat{E}(x, t) - \varepsilon' \leq v(x, t + kT; v_0) \quad \text{for all } x \in \mathbb{R}^3, k > \max_{1 \leq j \leq n}\{k'_1, k'_2, \ldots, k'_n\}.
\]
The proof is complete. \( \square \)

**Lemma 4.6.** Let \( V \) be defined as in Theorem 1.1. Then it satisfies
\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0,T]} |V(x, t) - \hat{E}(x, t)| = 0.
\]
\[
\lim_{R \to \infty} \sup_{|x-h(x,y)| \geq R, t \in [0,T]} |V^*_z(x, t)| = 0. \tag{4.18}
\]

In addition, for any \( \delta \in (0, \frac{\varepsilon}{2K_0}) \), we have \( G(V; \delta) > 0 \). Similarly, \( V^* \) satisfies
\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0,T]} |V^*(x, t) - \hat{E}(x, t)| = 0. \tag{4.19}
\]

**Proof.** By taking \( v_0 = V \) and \( v_0 = V^* \) in Proposition 4.5, respectively, we obtain that (4.18) and (4.19) hold.

Now we show that \( G(V; \delta) > 0 \) holds. Since we have \( V_z < 0 \) in \( \mathbb{R}^3 \), \( -V_z \) has a positive minimum on any compact subset of \( \mathbb{R}^3 \). Thus we need only to study \( V_z(x, t) \) as \( |x| \to \infty \). Assume that \( x_i = (x_i, y_i, z_i) \) satisfies \( \lim_{t \to \infty} |x_i| = \infty \) and \( W^-(t) + \frac{\varepsilon}{2K_0} \leq V(x, t) \leq W^+(t) - \frac{\varepsilon}{2K_0} \) for all \( t \in [0, T] \). It suffices to prove \( \liminf_{t \to \infty} t \in [0,T] - V_z(x_i, t) > 0 \). By using (1.12) and the definition of \( v^- \), we have \( \limsup_{t \to \infty} \text{dist}(x_i; \Gamma) < \infty \). Without loss of generality we can assume
\[
\lim_{t \to \infty} |x_i| = \infty, \quad \limsup_{t \to \infty} \text{dist}(x; \Gamma_j) < \infty \quad \text{for some } 1 \leq j \leq n.
\]
Then we obtain
\[ \lim_{i \to \infty} \text{dist}(x_i; \Gamma_m) = \infty \quad \text{for all } m \neq j. \]

It follows from (4.18) that
\[ \lim_{R \to \infty} \sup_{|x_i| \geq R, t \in [0, T]} |V(x_i, t) - E_j(x_i, t)| = 0. \]

Namely,
\[ \lim_{i \to \infty} \sup_{|x_i| \in B(x_i; 2), t \in [0, T]} |V(x_i, t) - E_j(x_i, t)| = 0. \]

By the interpolation \( \| \cdot \|_{C^1} \leq \sqrt{\| \cdot \|_{C^0} \cdot \| \cdot \|_{C^2}} \), we have
\[ \left\| \frac{\partial}{\partial z} V(x_i, t) - D_3 E_j(x_i, t) \right\|_{C^0(B(x_i; 2) \times [0, T])} \to 0, \]

where
\[ D_3 E_j(x_i, t) := \frac{q_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} D_2 \hat{v}(x_i, t) \left( \frac{(A_j - A_{j+1})x + (B_j - B_{j+1})y}{q_j}, \right. \]
\[ \left. \frac{\tau (B_j - B_{j+1})p_j x + (A_j + 1 - A_j)p_j y + q_j^2 z}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}}, t; s_j \right), \]

with \( s_j = \frac{b_j}{\sqrt{\tau^2 p_j^2 + q_j^2}} \), \( D_2 := \frac{\partial}{\partial y} \) and \( D_3 := \frac{\partial}{\partial z} \). Combining this estimate and Theorem 4.1 we obtain
\[ \lim_{i \to \infty} \inf_{t \in [0, T]} V_z(x_i, t) > 0. \]

Namely, \( G(V; \delta) > 0 \).

The assumption \( z - h(x, y) \to \infty \) implies \( \text{dist}(x, \Gamma) \to \infty \). Since
\[ \lim_{R \to +\infty} \sup_{|z-h(x,y)| \geq R, t \in [0, T]} \left| V(x, y, z, t) - U \left( \frac{c}{1}(z-h(x, y)), t \right) \right| \to 0 \]

and
\[ \lim_{R \to \infty} \sup_{|z-h(x,y)| \geq R, t \in [0, T]} \left| U_z \left( \frac{c}{1}(z-h(x, y)), t \right) \right| \to 0, \]

the interpolation \( \| \cdot \|_{C^1} \leq \sqrt{\| \cdot \|_{C^0} \cdot \| \cdot \|_{C^2}} \) implies
\[ \lim_{R \to \infty} \sup_{|z-h(x,y)| \geq R, t \in [0, T]} |V_z| = 0. \]

This completes the proof. \( \Box \)
\textbf{Lemma 4.7.} Fix any $\delta \in (0, \frac{\epsilon}{2K_0})$. We have $G(E_j; \delta) > 0$ for each $j$ ($1 \leq j \leq n$). For any $x \in \mathbb{R}^3$ with $\alpha^- + \delta \leq \max_{1 \leq j \leq n} E_j(x, 0) \leq \alpha^+ - \delta$, we have

$$\sup_{0 < \lambda < \lambda_0} \frac{\hat{E}(x, y, z + \lambda, 0) - \hat{E}(x, 0)}{\lambda} \leq - \min_{1 \leq j \leq n} G\left(E_j; \frac{\delta}{2}\right) < 0,$$

where $\lambda_0$ is a positive constant depending on $\delta_0$ and is independent of $(x, y, z)$.

\textbf{Proof.} $G(E_j; \delta) > 0$ is an immediate consequence of Theorem 4.1 and the definition of $E_j$. The remaining part follows from [47, Lemma 6]. \qed

\textbf{Lemma 4.8.} For $x \in \mathbb{R}^3$ and $t \in [0, T]$, $V^*(x, t) \equiv V(x, t)$.

\textbf{Proof.} Assume the contrary. Namely, $V^*(x, 0) \neq V(x, 0)$ for some $x$. Then we have $V(x, 0) < V^*(x, 0)$ in $\mathbb{R}^3$ by the strong maximum principle. In view of (4.18) and (4.19), for $\delta \in (0, \frac{\epsilon}{2K_0})$ and sufficiently large $\lambda > 0$ we have

$$V^*(x, 0) \leq V(x, y, z - \lambda, 0) + \delta.$$

Due to Lemma 4.2 we know that

$$V(x, y, z - \lambda - \rho \delta \left(1 - e^{-\beta t}\right), t) + \delta a(t)$$

is a supersolution of (1.5) on $t \geq 0$, where $\rho$ and $a(t)$ are defined in (4.2) and (4.3), respectively. Thus we have

$$V^*(x, t) \leq V(x, y, z - \lambda - \rho \delta \left(1 - e^{-\beta (t + kT)}\right), t + kT) + \delta a(t + kT)$$

for $x \in \mathbb{R}^3$, $t \in [0, T]$ and $k \in \mathbb{N}$. Sending $k \to \infty$ we get

$$V^*(x, t) \leq V(x, y, z - \lambda - \rho \delta, t) \quad \text{for } x = (x, y, z) \in \mathbb{R}^3 \text{ and } t \in [0, T].$$

Define

$$\Lambda := \inf\{\lambda \in \mathbb{R} | V^*(x, 0) \leq V(x, y, z - \lambda, 0)\}.$$

We have $\Lambda \geq 0$ and

$$V^*(x, 0) \leq V(x, y, z - \Lambda, 0) \quad \text{for } x = (x, y, z) \in \mathbb{R}^3.$$

The assumption $V^*(x, 0) \neq V(x, 0)$ yields $\Lambda > 0$. By Lemma 4.6 we can take $R_* > 0$ sufficiently large satisfying

$$2\rho \sup_{|z - h(x, y)| \geq R_* - \rho \frac{\epsilon}{2K_0}} |V^*(x, y, z - \Lambda, 0)| < 1.$$

Define

$$D_1 := \{x = (x, y, z) \in \mathbb{R}^3 | |z - h(x, y)| \leq R_*\}.$$
Thus the strong maximum principle implies

$$V^*(x, 0) < V(x, y, z - 2\rho\epsilon_1, 0) \text{ for } x = (x, y, z) \in \mathbb{R}^3.$$  

We choose a constant $\epsilon_1 > 0$ sufficiently small satisfying

$$0 < \epsilon_1 < \min\left\{\frac{\epsilon}{4K_0}, \frac{\Lambda}{4\rho}\right\}.$$  

Utilizing Lemma 4.7, for $(x, y, z) \in D_1$ we get

$$\hat{E}(x, y, z - \frac{\Lambda}{2}, 0) - \hat{E}(x, y, z - \frac{\Lambda}{4}, 0) = -\hat{E}(x, y, z - \frac{\Lambda}{2} + \frac{\Lambda}{4}, 0) + \hat{E}(x, y, z - \frac{\Lambda}{2}, 0) \geq \min\left\{\lambda_0, \frac{\Lambda}{4}\right\} \min_{1 \leq j \leq n} G\left(\frac{E_j}{\frac{1}{2}\delta_0}\right) > 0,$$

where

$$\delta_0 := \min\left\{\frac{\epsilon}{4K_0}, \alpha^+ - \max_{1 \leq j \leq n, 0 \leq \psi \leq 1} \sup_{|z - h(x, y)| \leq R_*} E_j\left(x, y, z + \frac{\Lambda}{2}\psi, 0\right), \max_{1 \leq j \leq n, 0 \leq \psi \leq 1} \sup_{|z - h(x, y)| \leq R_*} E_j\left(x, y, z + \frac{\Lambda}{2}\psi, 0\right) - \alpha^-\right\} \in (0, 1),$$

and $\lambda_0$ is defined in Lemma 4.7 associated with $\delta_0$. Thus we get

$$\inf_{(x, y, z) \in D_1} \left(\hat{E}(x, y, z - \Lambda + 2\rho\epsilon_1, 0) - \hat{E}(x, y, z, 0)\right) > \min\left\{\lambda_0, \frac{\Lambda}{4}\right\} \min_{1 \leq j \leq n} G\left(\frac{E_j}{\frac{1}{2}\delta_0}\right) > 0.$$  

If $x \in D_1$ and $|x|$ is large enough, say $|x| \geq R_0$ for some $R_0 > 0$, applying Lemma 4.6, we get

$$V^*(x, 0) < V\left(x, y, z - \frac{\Lambda}{2}, 0\right) \leq V(x, y, z - \Lambda + 2\rho\epsilon_1, 0).$$  

Since $D_1 \cap B(0; R_0)$ is a compact set in $\mathbb{R}^3$, we have

$$V^*(x, 0) < V(x, y, z - \Lambda + 2\rho\epsilon_1, 0) \text{ in } D_1 \cap B(0; R_0)$$

for sufficiently small $\epsilon_1$. Thus we obtain

$$V^*(x, 0) < V(x, y, z - \Lambda + 2\rho\epsilon_1, 0) \text{ in } D_1.$$  

In $\mathbb{R}^3 \setminus D_1$, we have
Utilizing (4.1), (4.8) and (4.11), we take
\[ V(x, y, z - \Lambda + 2\rho \epsilon_1, 0) - V(x, y, z - \Lambda, 0) \]
\[ = 2\rho \epsilon_1 \int_0^1 V_z(x, y, z - \Lambda + 2\theta \rho \epsilon_1, 0) d\theta \geq -\epsilon_1. \]

Combining both cases together, we have
\[ V^*(x, 0) < V(x, y, z - \Lambda + 2\rho \epsilon_1, 0) + \epsilon_1 \text{ in } \mathbb{R}^3. \]

Define
\[ v^{++}(x, t) := V(x, y, z - \Lambda + 2\rho \epsilon_1 - \rho \epsilon_1(1 - e^{-\beta t}), t) + \epsilon_1 a(t) \]
for \( x \in \mathbb{R}^3 \) and \( t \geq 0 \). By Lemma 4.2 we know that \( v^{++}(x, t) \) is a supersolution of (1.5). Thus we obtain
\[ V^*(x, t) = V^*(x, t + kT) \leq v^{++}(x, t + kT) \]
\[ = V(x, y, z - \Lambda + 2\rho \epsilon_1 - \rho \epsilon_1(1 - e^{-\beta(t+kT)}), t + kT) + \epsilon_1 a(t + kT) \]
for \( x \in \mathbb{R}^3, t \in [0, T] \) and \( k \in \mathbb{N} \). Letting \( k \to \infty \) yields
\[ V^*(x, t) \leq V(x, y, z - \Lambda + \rho \epsilon_1, t) \text{ for all } x \in \mathbb{R}^3 \text{ and } t \in [0, T]. \]
This contradicts the definition of \( \Lambda \). Thus \( \Lambda = 0 \) follows and we proved \( V^*(x, t) \equiv V(x, t) \). The proof is complete. \( \square \)

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \delta \in (0, \frac{\epsilon}{4K_0(1+a_T)}) \) be arbitrarily given. We take \( \epsilon_1 \in (0, \min\{\epsilon^+_0, \frac{\epsilon}{4K_0(1+a_T)}\}) \).

Utilizing (4.1), (4.8) and (4.11), we take \( \alpha \in (0, \alpha^+_0(\epsilon_1)) \) and get
\[ v(x, t + k_1 T; v_0) \leq v^{++}(x, t) + \delta \text{ for all } x \in \mathbb{R}^3 \text{ and } t \in [0, T]. \]
For each \( x \in \mathbb{R}^3 \) and \( t \in [0, T] \), we have \( v(x, t + kT; v^-) \) and \( v(x, t + kT; v^+) \) are monotone increasing and monotone decreasing in \( t > 0 \), respectively. Let \( m > 1 \) be an arbitrarily given number. Taking \( k_2 \leq k_1 \) and \( r_2 > 0 \) sufficiently large, if \( |x| \geq r_2 \) and \( k \geq k_2 \), we get
\[ V(x, t) - \frac{\delta}{m} \leq v(x, t + kT; v^-) \leq V(x, t), \]
\[ V(x, t) \leq v(x, t + kT; v^+) \leq V(x, t) + \frac{\delta}{m} \]
for \( t \in [0, T] \). On the other hand, in \( B(0; r_2) \) the interior estimates (see [32]) yield
\[ \lim_{k \to \infty} \sup_{x \in B(0; r_2), t \in [0, T]} \left| v(x, t + kT; v^-) - V(x, t) \right| = 0, \]
\[ \lim_{k \to \infty} \sup_{x \in B(0; r_2), t \in [0, T]} \left| v(x, t + kT; v^+) - V(x, t) \right| = 0. \]
Since $m > 1$ can be taken arbitrarily large, we get

$$
\lim_{k \to \infty} \| v(x, t + kT; v^-) - V(x, t) \|_{L^\infty(\mathbb{R}^3 \times [0, T])} = 0.
$$

$$
\lim_{k \to \infty} \| v(x, t + kT; v^+) - V(x, t) \|_{L^\infty(\mathbb{R}^3 \times [0, T])} = 0.
$$

Thus for any $\delta > 0$ we take $\hat{k} > 0$ large enough such that

$$
V(x, t) - \delta < v(x, t + kT; v^-) \leq v(x, t + kT; v^+) < V(x, t) + \hat{\delta} \quad \text{for } x \in \mathbb{R}^3, k > \hat{k} \text{ and } t \in [0, T].
$$

By Lemma 4.3 we get

$$
V(x, t) - \delta < v(x, t + (k + \hat{k})T; v_0) < V(x, t) + \hat{\delta} \quad \text{for } x \in \mathbb{R}^3, t \in [0, T] \text{ and } k \in \mathbb{N},
$$

if

$$
v^-(x, t) - \hat{\delta}e^{-m(kT)} < v(x, t + kT; v_0) < v^+(x, t) + \hat{\delta}e^{-m(kT)} \quad \text{for some } k \in \mathbb{N}. \quad (4.20)
$$

Thus to prove the theorem, it suffices to show that (4.20) holds for any small $\hat{\delta}e^{-m(kT)}$.

We first study the upper estimate. From the definition of $v^+$ and $\varepsilon_1 < \frac{\varepsilon}{4K_0(1+a_\ast)}$, we have $G(\varepsilon_1; \frac{\varepsilon}{2K_0}) > 0$. Let $\rho$ be as in Lemma 4.2, then we know that

$$
v^{++}(x, t) = v^+(x, y, z - \rho \delta(1 - e^{-\beta(t+kT)}), t + kT) + \delta a(t + kT)
$$

and

$$
v^{--}(x, t) = v^-(x, y, z + \rho \delta(1 - e^{-\beta(t+kT)}), t + kT) - \delta a(t + kT)
$$

are a supersolution and a subsolution of (1.5), respectively. Sending $k \to \infty$ in the right-hand side we have

$$
\limsup_{k \to \infty} \sup_{x \in \mathbb{R}^3, t \in [0, T]} (v(x, t + kT; v_0) - v^+(x, y, z + \rho \delta), t) \leq 0,
$$

$$
\liminf_{k \to \infty} \inf_{x \in \mathbb{R}^3, t \in [0, T]} (v(x, t + kT; v_0) - v^-(x, y, z + \rho \delta), t) \geq 0.
$$

So far we have proved that (4.20) holds true, since the lower estimate is an immediate consequence of Proposition 4.5 and Lemma 4.4.

Due to the arbitrariness of $\hat{\delta}$, we have completed the proof. \(\square\)

**Corollary 4.9.** Let $V$ be the three-dimensional periodic pyramidal traveling front associated with the pyramid $z = h(x, y)$. If (1.5) has a solution $v$ with

$$
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0, T]} |v(x, t) - \hat{E}(x, t)| = 0,
$$

then we have $v \equiv V$. 


It is known from Corollary 4.9 that a three-dimensional periodic pyramidal traveling front is uniquely determined as a combination of two-dimensional V-form fronts.

5. Existence of periodic pyramidal traveling fronts in $\mathbb{R}^N$ with $N \geq 4$

In this section, we study the existence of periodic $N$-dimensional nonplanar traveling waves to (1.1) with $N \geq 4$. Without confusion, we use the same notations as in previous sections. Without loss of generality, we denote $y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$ and $y' = (y_1, y_2, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$. In addition, we assume $c > 0$ and the solutions travel towards $y_N$-direction. Let

$$u(y, t) = v(y', y_N - lt, t), \quad s = y_N - lt.$$  

For simplicity, we denote $v(y', s, t)$ by $v(y', y_N, t)$. Substituting $v$ into (1.4), we have

$$L[v] := v_t - \Delta v - l \frac{\partial v}{\partial y_N} - f(v, t) = 0, \quad y \in \mathbb{R}^N, \quad t > 0,$$

$$v(y, 0) = v_0(y), \quad y \in \mathbb{R}^N. \quad (5.1)$$

Let $n \geq 3$ be a given integer and $\tau > 0$ be given by (1.6). Let $\{A_j\}_{j=1}^n \in \mathbb{R}^N$ be a set of unit vectors such that $A_j \neq A_i$, if $i \neq j$. Then $A_j = (A_{1,j}, A_{2,j}, \ldots, A_{N-1,j})$ satisfies

$$|A_j| = \sum_{i=1}^{N-1} A_{i,j}^2 = 1 \quad (5.2)$$

for $j = 1, 2, \ldots, n$. Thus $(-\tau A_j, 1) \in \mathbb{R}^N$ is a normal vector of $\{y \in \mathbb{R}^N | y_N = \tau (A_j, y')\}$, where $(A_j, y')$ is defined by

$$(A_j, y') := \sum_{i=1}^{N-1} A_{i,j} y_i.$$  

Put

$$h_j(y') := \tau (A_j, y'),$$
$$h(y') := \max_{1 \leq j \leq n} h_j(y') = \tau \max_{1 \leq j \leq n} (A_j, y'). \quad (5.3)$$

Then $\{y \in \mathbb{R}^N | y_N = h(y')\}$ is a pyramid in $\mathbb{R}^N$. We define $\Omega_j, S_j, \Gamma_j, D(y')$ as in Section 2 by replacing $(x, y)$ and $(x, y, z)$ with $y'$ and $y$, respectively. We denote the boundary of $\Omega_j$ by $\partial \Omega_j$. For every $A_j$ satisfying (5.2), (5.1) has a planar front solution $U(\frac{c}{l}(y_N - h_j(y')), t)$. Define

$$v^-(y, t) := U\left(\frac{c}{l}(y_N - h(y')), t\right) = \max_{1 \leq j \leq n} U\left(\frac{c}{l}(y_N - h_j(y')), t\right). \quad (5.4)$$

It is obvious that $v^-(y, t)$ is a subsolution of (5.1).
Let \( \tilde{\rho}(r) \in C^\infty[0, \infty) \) be a function defined as

\[
\tilde{\rho}(r) > 0, \quad \tilde{\rho}'(r) \leq 0 \quad \text{for } r \geq 0, \\
\tilde{\rho}(r) \equiv 1 \quad \text{if } r > 0 \text{ small enough}, \\
\tilde{\rho}(r) = e^{-r} \quad \text{if } r > 0 \text{ large enough, say } r > R_0,
\]

\[
\int_{\mathbb{R}^{N-1}} \tilde{\rho}(|y'|) \, dy' = 1.
\]

Without loss of generality we assume \( R_0 > 1 \). Thus we have

\[
\int_{\mathbb{R}^{N-1}} \tilde{\rho}(|y'|) \, dy' = \frac{(N - 1)\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \int_0^\infty r^{N-2} \tilde{\rho}(r) \, dr,
\]

where \( \Gamma \) is the Gamma function. Setting \( \rho(y') = \tilde{\rho}(|y'|) \), we have

\[
\int_{\mathbb{R}^{N-1}} \rho(y') \, dy' = 1.
\]

In addition, for any nonnegative integers \( j_1, \ldots, j_{N-1} \) satisfying \( 0 \leq \sum_{p=1}^{N-1} j_p \leq 3 \), there exists a positive constant \( M_1 \) such that

\[
|D_{j_1}^1 \cdots D_{j_{N-1}}^{j_{N-1}} \rho(y')| \leq M_1 \rho(y') \quad \text{for all } y' \in \mathbb{R}^{N-1}.
\] 

We put \( \varphi := \rho \ast h \) which implies the convolution of \( \rho \) and \( h \) defined by

\[
\varphi(y') := \int_{\mathbb{R}^{N-1}} \rho(y'') h(y' - y'') \, dy''
\]

\[
= \int_{\mathbb{R}^{N-1}} \rho(y' - y'') h(y'') \, dy'' \quad \text{for all } y', y'' \in \mathbb{R}^{N-1}.
\]

Thus we obtain a mollified pyramid \( y_N = \varphi(y') \) associated with a pyramid \( y_N = h(y') \). Define

\[
S(y') := \frac{l}{\sqrt{1 + |\nabla \varphi(y')|^2}} - c,
\]

where \( \nabla \varphi(y') = (\partial \varphi/\partial y_1, \ldots, \partial \varphi/\partial y_{N-1}) \). By a similar argument to that of [30], we can obtain a series of lemmas as in Section 2. Proceed the same procedure as in the previous sections, we obtain the following lemma and theorem.

**Lemma 5.1.** There exist a positive constant \( \varepsilon_0^+ \) and a positive function \( \alpha_0^+(\varepsilon) \) such that, for \( 0 < \varepsilon \leq \varepsilon_0^+ \) and \( 0 < \alpha \leq \alpha_0^+(\varepsilon) \),

\[
v^+(y; \varepsilon, \alpha) := U\left(\frac{y_N - \frac{1}{\alpha} \varphi(\alpha y')}{\sqrt{1 + |\nabla \varphi(\alpha y')|^2}}, t\right) + \varepsilon a(t) S(\alpha y')
\]
is a supersolution of (5.1) on $t \in (-\infty, +\infty)$, where

$$a(t) := 2 \exp \left\{ \frac{1}{2} \left( v^+ t + v^- t + \int_0^t f_u(W^+(\tau), \tau) d\tau + \int_0^t f_u(W^-(\tau), \tau) d\tau \right) \right\}.$$  

In addition,

$$\lim_{\gamma \to \infty} \sup_{y \in D(\gamma), t \in [0, T]} |v^+(y, t; \epsilon, \alpha) - v^-(y, t)| \leq (1 + a^*) \epsilon, \quad (5.8)$$

$$v^-(y, t) < v^+(y, t; \epsilon, \alpha), \quad (5.9)$$

with $a^* := \max_{t \in [0, T]} a(t)$.

Our aim in this section is to seek for the solution $V(y, t)$ with

$$\mathcal{L}[V] := V_t - \Delta V - l \frac{\partial V}{\partial y_N} - f(V, t) = 0, \quad y \in \mathbb{R}^N, \quad t \in [0, T], \quad (5.10)$$

$$V(\cdot, \cdot) = V(\cdot, \cdot + T), \quad y \in \mathbb{R}^N. \quad (5.11)$$

**Theorem 5.2.** Assume $l > c > 0$, and let $h(y')$ and $v^-(y, t)$ be given by (5.3) and (5.4), respectively. In addition, we also assume that the hypotheses (H1)–(H3) hold. Then there exists a solution $V(y, t)$ of (5.10) and (5.11) satisfying

$$U\left(\frac{c}{l} (y_N - h(y')), t\right) < V(y, t) < W^+(t), \quad (y, t) \in \mathbb{R}^N \times [0, T]$$

and

$$\lim_{\gamma \to \infty} \sup_{y \in D(\gamma), t \in [0, T]} |V(y, t) - v^-(y, t)| = 0,$$

$$\frac{\partial V}{\partial y_N}(y, t) < 0 \quad \text{for all} \quad y \in \mathbb{R}^N. \quad (5.12)$$

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**References**


