Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 909 (2016) 73–85

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Exact asymmetric Skyrmion in anisotropic ferromagnet and its helimagnetic application

Anjan Kundu

Theory Division, Saha Institute of Nuclear Physics, Calcutta, India

Received 12 October 2015; received in revised form 13 April 2016; accepted 27 April 2016

Available online 4 May 2016

Editor: Hubert Saleur

Abstract

Topological Skyrmions as intricate spin textures were observed experimentally in helimagnets on 2d plane. Theoretical foundation of such solitonic states to appear in pure ferromagnetic model, as exact solutions expressed through any analytic function, was made long ago by Belavin and Polyakov (BP). We propose an innovative generalization of the BP solution for an anisotropic ferromagnet, based on a physically motivated geometric (in-)equality, which takes the exact Skyrmion to a new class of functions beyond analyticity. The possibility of stabilizing such metastable states in helimagnets is discussed with the construction of individual Skyrmion, Skyrmion crystal and lattice with asymmetry, likely to be detected in precision experiments.

1. Introduction

Unusual spin textures, identified as magnetic Skyrmions of topological origin, were observed in helimagnetic crystals like Mn Si and $Fe_xCo_{1-x}Si$, in a series of recent experiments [1–3]. However, the possibility of such a phenomenon to occur on a two-dimensional (2d) plane in ferromagnetic model was predicted by Belavin and Polyakov 40 years back, in a pioneering work, where they found exact Skyrmions in a general framework with topological charge $Q = N$, linked to any analytic functions [4]. Problem with this beautiful theoretical result was, that due

E-mail address: anjan.kundu@saha.ac.in.

http://dx.doi.org/10.1016/j.nuclphysb.2016.04.043

to scale invariance of the solution it could give only metastable states, where for stabilizing the Skyrmions one needs to bring in coupling parameters through additional interactions. In a helimagnetic model a Dzyaloshinskii–Moriya (DM) spin-orbital interaction term is added to the spin–spin ferromagnetic interaction, where the DM coupling breaks the unwanted scale invariance and stabilizes the solitons through competing forces between the ferromagnetic and the effective DM interactions. Recent experimental observation of Skyrmions in magnetic models is therefore a landmark confirmation of an old basic theoretical result proposed in [4].

It is a bit surprising, that in the recent large collection of high-profile theoretical and experimental work dedicated to the magnetic Skyrmions in helimagnets [1–3,5,6], rarely the work of BP [4] is mentioned, which on the other hand is the theoretical basis for the topological Skyrmion in magnetic model in 2d. It is also rather unexpected, that no other extension or generalization of the BP solutions was proposed in 2d, over these long years of development in the subject. Our motivation here, is to bring in a novel contribution to the subject, by discovering a family of exact Skyrmions for an anisotropic extension of the ferromagnetic model, through a physically motivated geometric (in-)equality, which generalizes the BP result by going beyond the analytic functions and linking the Skyrmions to the contemporary $\bar{\partial}$ problem [8]. Such Skyrmions exhibit more inherent asymmetry and anisotropy, properties akin also to the helimagnetic crystals with noninversion symmetry and anisotropy exhibited in 2d. Therefore, a natural application of such asymmetric solitons to the helimagnetic model, through their stabilization under DM interaction might be a promising perspective to be verified in a precision experiment.

2. Ferromagnetic and helimagnetic models and BP solution

As evident from the experimental images, the Skyrmion magnetic pattern in helimagnets shows extended structure for individual Skyrmions in a 30 nm range, slowly varying over several lattice spacings [3]. Therefore, one can go for the continuum limit reducing the ferromagnetic Hamiltonian H_f from the Heisenberg spin model on a 2d lattice with nearest neighbor interactions as

$$H_{HS} = \frac{1}{2} \sum_{<\mathbf{j},\mathbf{j}'>} \mathbf{s_j s_{j'}} \longrightarrow \quad H_f = \frac{1}{2} \int d^2x (\nabla \mathbf{M})^2, \tag{1}$$

where $\mathbf{s_j} \to \mathbf{M}$, $\mathbf{s_j^2} \to \mathbf{M}^2 = 1$. For constructing a helimagnetic model a crucial addition of a spin-orbital interaction is needed to the ferromagnetic Hamiltonian (1) in the form of a DM term

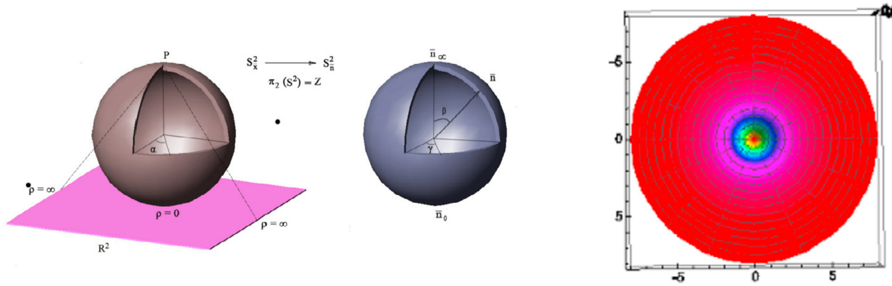
$$H_{dm} = \int d^2x (\mathbf{M} \cdot [\nabla \times \mathbf{M}]), \tag{2}$$

which was proposed phenomenologically way back in another landmark work [7].

Note, that the DM Hamiltonian (2) exhibits an explicitly broken space inversion symmetry as well as anisotropy, due to the appearance of only first order derivatives and interaction of the 2d coordinate space with the internal spin space, rewriting (2) as

$$H_{dm} = \int d^2x \left(M^1 \partial_y M^3 + M^2 \partial_x M^3 + M^3 (\partial_x M^2 - \partial_y M^1) \right). \tag{3}$$

Therefore the basic helimagnetic model, for investigating the magnetic Skyrmions of contemporary interest, may be given simply by the model Hamiltonian $H_{heli} = J H_f + D\, H_{dm}$ where J is the spin exchange parameter, D is the spin-orbital coupling and the ferromagnetic H_f and the DM Hamiltonian H_{dm} may be given by (1) and (3), respectively.



a) b)

Fig. 1. Topological picture in 2D field model. a) *sphere to sphere* mapping with charge Q as the degree of this mapping. b) Symmetric Skyrmion in ferromagnetic model with topological charge $Q = -1$.

2.1. Topological charge

Here comes the most interesting topological concept, by noticing that the magnetic moment field $\mathbf{M}(\rho, \alpha)$ defined in a 2d space (ρ, α being the polar coordinates) takes values on a 2-*sphere* $\mathbf{M} \in S^2_M$. At the same time, for finite energy soliton solutions, $\mathbf{M}$ must go to a fixed vector at space infinities $\mathbf{M}_\infty(\rho \to \infty) \to (0.0.1)$. This in turn compactifies the 2D space $(\rho, \alpha) \in R^2$ also to a 2-*sphere* S^2_r (since space-infinities can be added as a fixed north pole to form a sphere), resulting to a $S^2_r \to S^2_M$ mapping (see Fig. 1a)

The *degree* of this mapping, which counts how many times Skyrmion field $\mathbf{M}$ wraps the target sphere S^2_M, when coordinate space S^2_r is covered once, may be defined as the topological charge Q. It is evident therefore, that topological charge is a geometrical property that can take only integer values $Q = N$ and since it is conserved topologically, Skyrmion with a fixed charge can not decay into a configuration with a different charge and hence acquires a topological stability.

2.2. BP Skyrmions

The original idea of a topological Skyrmion to appear in pure ferromagnetic model (1) was put forward in [4]. However due to scale invariance of such Skyrmions the states remain metastable and difficult to observe in experiments. Therefore for stabilizing the solitons some scale must be introduced, as done by additional $D\ H_{dm}$ coupling in a helimagnetic model.

For understanding therefore the original idea of [4] for constructing the Skyrmions, let us concentrate on ferromagnetic Hamiltonian H_f (1) alone and note that, since charge Q for a Skyrmion is conserved, if we ensure a lower bound for the Hamiltonian like $H_f \geq \text{const.}|Q|$, the *energy* can not decay into the vacuum, making the state stable and at the same time the configuration reaching the lowest bound (i.e. when *equality* is achieved) should minimize the energy, producing thus the needed soliton solution. To elaborate the picture let us express magnetic moment $\mathbf{M}$ through two spherical angles (β, γ) on S^2_M:

$$\beta(\rho, \alpha) \in [0, \pi], \gamma(\rho, \alpha) \in [0, 2\pi]$$

as

$$M^{\pm} = M^1 \pm i M^2 = \sin\beta e^{\pm i\gamma}, \ M^3 = \cos\beta \tag{4}$$

and rewrite for convenience ferromagnetic Hamiltonian (1) as

$$H_f = \int d^2x h_f(\beta,\alpha),\ h_f(\beta,\alpha) = \frac{1}{2}((\nabla\beta)^2 + \sin^2\beta(\nabla\gamma)^2) \tag{5}$$

and the topological charge as

$$Q = \frac{1}{4\pi}\int d^2x q(\beta,\alpha),\ \ q(\beta,\alpha) = \sin\beta[\nabla\beta\times\nabla\gamma]. \tag{6}$$

For further insight we introduce two field vectors

$$\mathbf{X} = \nabla\beta,\ \ \mathbf{Y} = \sin\beta\nabla\gamma \tag{7}$$

to rewrite

$$h_f(\beta,\alpha) = \frac{1}{2}(\mathbf{X}^2 + \mathbf{Y}^2),\ \ q(\beta,\alpha) = [\mathbf{X}\times\mathbf{Y}] \tag{8}$$

which by using an obvious inequality

$$\frac{1}{2}\left((X_1 - Y_2)^2 + (X_2 + Y_1)^2\right) \geq 0 \tag{9}$$

and regrouping the terms as

$$\frac{1}{2}(\mathbf{X}^2 + \mathbf{Y}^2) \geq (X_1Y_2 - X_2Y_1)$$

leads clearly to $h_f \geq |q|$, which by integrating both sides gives the required inequality $H_f \geq 4\pi|Q|$. For answering the next question, that is, when we can get the equality $H_f = 4\pi|Q|$, we focus again on (9) and see clearly that equality is achieved at the relations

$$X_1 = Y_2,\ \ X_2 = -Y_1, \tag{10}$$

which are simple first order equations, allowing exact Skyrmion solutions. Interestingly, mapping $\mathbf{M}$ to a complex field (stereographic projection): $\frac{M^+}{1+M^3} = f(z) = u + iv$ one can map the minimizing condition (10) to a similar relations $\partial_1 u = \partial_2 v$, $\partial_2 u = -\partial_1 v$, which is known to us very well as the Cauchy–Riemann (CR) condition for analyticity $\partial_{\bar{z}} f(z) = 0$ of a complex function $f(z)$. Therefore the BP result concludes, that any analytic function

$$f(z) = \frac{\prod_i (z - z_i)^{n_i}}{\prod_j (z - z_j)^{n_j}},\ N = \sum_i n_i - \sum_j n_j, \tag{11}$$

with z_i, z_j any arbitrary constant complex numbers, would be an exact Skyrmion solution of the ferromagnetic model with topological charge $Q = N$, where we have explicit scale invariance $z \to z_0 z$ of the solution.

The simplest exact solution obtained from the above general result as $f(z) = \frac{1}{z} = \frac{1}{\rho}e^{-i\alpha}$, leading to the Skyrmion solution for the magnetic moment in the ferromagnetic model (8) as

$$M^3 = \cos\beta = \frac{\rho^2 - 1}{\rho^2 + 1},\ \ M^3(\rho = 0) = -1,\ \ M^3(\rho = \infty) = +1,\ \gamma = -\alpha, \tag{12}$$

shows perfect circular symmetry (see Fig. 1b).

It is rather surprising, that even for the helimagnetic model exhibiting high level of asymmetry and anisotropy, the contemporary theoretical models look for only Skyrmions with circular symmetry. Our findings, as we show below, will be tuned towards asymmetric Skyrmion solutions.

3. Beyond BP Skyrmions

Our motivation is to look for more general solutions beyond symmetric Skyrmions, which might be detected in future precision experiments. We intend to find some way to extend the BP solutions. However since the BP procedure seems to be built on a specific construction using a particular inequality (9), it is difficult to get any clue for its generalization. Therefore, we turn for new ideas to some physically motivated inequalities of geometric nature with an universal appeal, called isoperimetric inequalities.

3.1. Isoperimetric inequalities

It is a common experience, that the soap bubbles, deformed at its starting always try to take the spherical shape in time. This is because the bubble surface S is lower bounded by its volume V and tends to reach the equilibrium point by minimizing its surface area, which is achieved for a sphere, at reaching the equality.

Interestingly, this universal concept remains valid also in 2d, where for any closed curve its perimeter P is bounded from below by the area enclosed: A, through the relation $P^2 \geq 4\pi A$. It is not difficult to check, that the equality is reached at the maximum symmetry, i.e. when the curve turns into a circle.

This picture becomes even more interesting at discrete symmetries, where for a polygon of n-sides the perimeter P_n and the enclosed area A_n are related through an inequality $P_n^2 \geq c_n A_n$, $c_n = (4n \tan \frac{\pi}{n})$. The equality is achieved, as expected, at the symmetric situation, when the polygon becomes a regular one with equal sides and equal angles (which is a nice check using simple geometric relations).

Particular cases of this (in-)equality with $n = 4, 3$ will proved to be important for us.

3.2. Inequalities for parallelogram and triangle

First let us consider a parallelogram generated by two vectors $\mathbf{X}$ and $\mathbf{Y}$, which should satisfy the inequality for n-polygon with $n = 4$: $P_4^2 \geq 16A_4$, since $c_4 = (4 \cdot 4 \tan \frac{\pi}{4}) = 16$. The inequality expressed through these vectors therefore takes the form (see Fig. 2a)

$$(2(|\mathbf{X}| + |\mathbf{Y}|))^2 \geq 16|[\mathbf{X} \times \mathbf{Y}]|,$$

where perimeter P_4 and area A_4 of the parallelogram are rewritten through the generating vectors. Using further a simple relation, known as Schwartz inequality: $\mathbf{X}^2 + \mathbf{Y}^2 \geq 2(\mathbf{X} \cdot \mathbf{Y})$ we derive an important relation

$$\frac{1}{2}(\mathbf{X}^2 + \mathbf{Y}^2) \geq |[\mathbf{X} \times \mathbf{Y}]|, \tag{13}$$

which surprisingly coincides with the ferromagnetic Hamiltonian and charge densities (8) if we represent our vectors as (7), yielding the same relation $h_f \geq q$ or its integrated version $H_f \geq 4\pi Q$. Thus we could derive the same lower bound of the ferromagnetic Hamiltonian H_f through topological charge Q, as obtained by BP [4] using a restricted inequality (9), from a general framework of universal inequalities for polygons through a geometric construction. Now, let us see how further we can go in retrieving the BP construction from the present geometric approach. As we should have guessed, the equality in (13) for the parallelogram would be achieved for its maximum symmetry, i.e. when it reduces to a square having (see Fig. 2a)

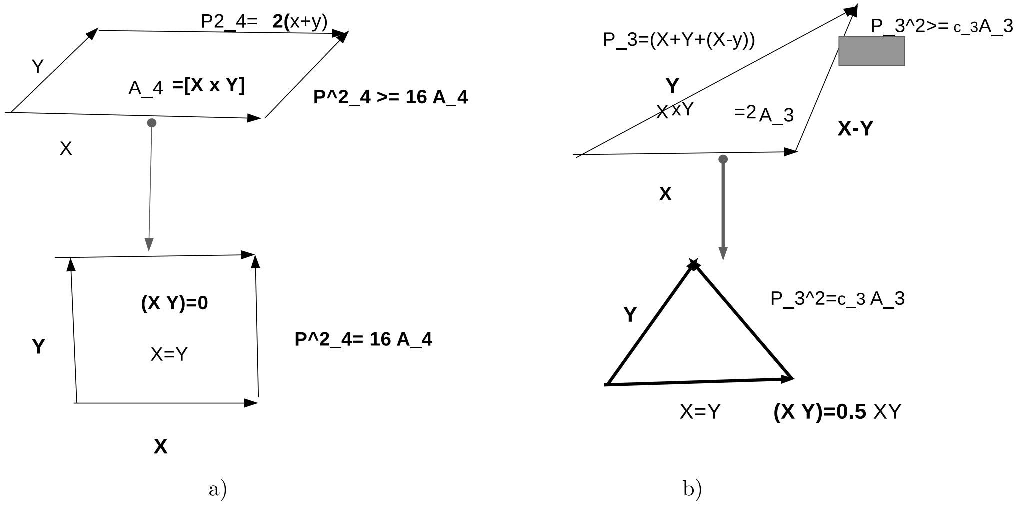


Fig. 2. Isoperimetric (in-)equalities. a) Parallelogram inequality reaching equality at *square* linked to $H_f \geq Q$ for ferromagnet and BP Skyrmion related to analytic functions. b) Triangle inequality reaching equality at *equilateral* triangle, linked to new anisotropic ferromagnet $\tilde{H}_f \geq Q$ and asymmetric Skyrmion related to $\bar{\partial}$-problem functions.

$$|\mathbf{X}| = |\mathbf{Y}|, \quad \text{and} \quad (\mathbf{X} \cdot \mathbf{Y}) = 0, \quad \text{with} \quad \angle(\mathbf{X}, \mathbf{Y}) \equiv \theta_0 = 90^\circ. \tag{14}$$

It is easy to check this intriguing fact, that a geometric equality, obtained for a square (14) from an overall symmetry is exactly compatible with the self-duality condition (10) linked to the CR condition $\partial_{\bar{z}} f(z) = 0$, for an analytic function giving exact Skyrmion solutions in the ferromagnetic model [4]. The relation like (14) found here linked to the CR condition can give also an uncommon realization for the analyticity condition of a complex field $f = u + iv$ in the form

$$|\nabla u| = |\nabla v|, \quad (\nabla u \cdot \nabla u) = |\nabla u|^2 \cos 90^\circ = 0, \quad \angle(\nabla u, \nabla v) = 90^\circ. \tag{15}$$

4. Anisotropic ferromagnet and extension of BP result

Inspired by the above success in rederiving the exact BP Skyrmions for the ferromagnetic model, following a different geometric route of universal nature related to a parallelogram, we intend to focus now on a triangular inequality, in the hope for generalizing the BP result. Therefore, we focus on the above polygon relation and specialize for a new case: $n = 3$ leading to the inequality $P_3^2 \geq c_3 A_3$, $c_3 = (4 \cdot 3 \tan \frac{\pi}{3}) = 12\sqrt{3}$. The triangle generated by the above two vectors clearly defines the perimeter of the triangle as $P_3 = |\mathbf{X}| + |\mathbf{Y}| + |\mathbf{X} - \mathbf{Y}|$ while the area of the triangle may be given through $A_3 = \frac{1}{2}|[\mathbf{X} \times \mathbf{Y}]|$ expressing the triangular inequality explicitly as (see Fig. 2b)

$$(|\mathbf{X}| + |\mathbf{Y}| + |\mathbf{X} - \mathbf{Y}|)^2 \geq 6\sqrt{3}|[\mathbf{X} \times \mathbf{Y}]|. \tag{16}$$

Further working on the LHS and using repeatedly the Schwartz inequality one derives from (16) a new relation

$$\frac{1}{2}(\mathbf{X}^2 + \mathbf{Y}^2 - (\mathbf{X} \cdot \mathbf{Y})) \geq \frac{\sqrt{3}}{2}|[\mathbf{X} \times \mathbf{Y}]|, \tag{17}$$

which gives a physical interpretation for an extended ferromagnetic model. To see this we represent the vectors as (7) and comparing with the expression (8) used for the ferromagnetic model in the BP case, define a new anisotropic ferromagnetic model as

$$\tilde{H}_f = H_f + H_{anis},$$
$$H_{anis} = \frac{1}{2}\int d^2x \frac{1}{(1-M^{3^2})}\left(M^2\nabla M^1 - M^1\nabla M^2)\cdot\nabla M^3\right)$$
$$= -\frac{1}{2}\int d^2x \sin\beta(\nabla\beta\cdot\nabla\gamma), \tag{18}$$

where we have expressed the vectors $\mathbf{X}, \mathbf{Y}$ through the Magnetic moment vector $\mathbf{M}$ in deriving the anisotropic part H_{anis}. Interestingly, the lower bound for this new anisotropic ferromagnetic Hamiltonian $\tilde{H}_f$ through topological charge Q follows from the relation (17) as

$$\tilde{H}_f \geq 2\pi\sqrt{3}Q, \tag{19}$$

where $\tilde{H}_f$ is given by (18) with the standard ferromagnetic Hamiltonian as in (1) and the topological charge Q as in (6). We may note by passing, that comparing with the lower bound for the ferromagnetic model: $H_f \geq 4\pi|Q|$ as appears in the BP construction, the bound (19) we get here for the anisotropic extension is a bit lower.

The next important question is, can one find exact Skyrmion solutions minimizing the Hamiltonian of the new model (18). Since such a configuration should saturate the lower bound in (19) as $\tilde{H}_f = 2\pi\sqrt{3}Q$, we seek for the equality in the triangular relations (16) or (17) which, as per the general philosophy we have witnessed above, should be achieved at a maximal symmetry, i.e. for an equilateral triangle giving the condition (see Fig. 2b)

$$\mathbf{X}^2 = \mathbf{Y}^2, \;\; (\mathbf{X}\cdot\mathbf{Y}) = \frac{1}{2}|\mathbf{X}||\mathbf{Y}|, \;\; \theta_0 = 60^\circ, \tag{20}$$

which is consistent with the relation

$$X_1 = \cos\theta_0 Y_1 + \sin\theta_0 Y_2, \;\; X_2 = -\sin\theta_0 Y_1 + \cos\theta_0 Y_2, \;\; \theta_0 = 60^\circ \tag{21}$$

and clearly is a generalization of (10). The relation (21) or equivalently the equilateral condition (20) minimizes the model Hamiltonian $\tilde{H}_f$ and therefore take us beyond the BP Skyrmion. At the same time, the minimization condition (21) maps under the stereographic projection to a complex function generalizing an analytic function and hence satisfying an intriguing extension of the CR-condition:

$$F(z,\bar{z}) = u + iv, \;\; \partial_{\bar{z}}F(z,\bar{z}) \neq 0$$
$$\partial_1 u = \cos\theta_0\partial_1 v + \sin\theta_0\partial_2 v, \;\; \partial_2 u = -\sin\theta_0\partial_1 v + \cos\theta_0\partial_2 v, \tag{22}$$

with $\theta_0 = \frac{\pi}{3}$ in the present case. This is like a nontrivial rotation in the 2d gradient space $\nabla u = \hat{R}_{\theta_0}\nabla v$, where rotational angle $\theta_0 = \frac{\pi}{2}$ recovers the standard CR relations, while $\theta_0 = \frac{\pi}{3}$ corresponds to our case, giving new exact Skyrmion solutions through a class of complex functions beyond usual analyticity.

In general for finding the topological solitons in such magnetic models for the magnetic moments

$$M^3 = \cos\beta(\rho,\alpha), \; M^{\pm} = \sin\beta e^{\pm i\gamma(\rho,\alpha)}$$

one has to solve the governing Euler-equations for the angles β, γ by minimizing the model Hamiltonian. However, these equations are highly nonlinear coupled PDEs and difficult to solve in general. It is therefore a pleasant rescue in the present case, that the configuration satisfying the energy minimizing condition (21), which can be expressed explicitly as

$$\partial_\rho \beta = \frac{1}{2} \sin\beta (\partial_\rho \gamma + \frac{\sqrt{3}}{\rho} \partial_\alpha \gamma), \quad \frac{1}{\rho} \partial_\alpha \beta = \frac{1}{2} \sin\beta (-\sqrt{3} \partial_\rho \gamma + \frac{1}{\rho} \partial_\alpha \gamma), \tag{23}$$

corresponds automatically to the solutions of the Euler equations derived from our Hamiltonian $\tilde{H}_f$ (18). Thus we can solve the governing Euler equations of the model, without really solving them directly, but by solving only much simpler first-order equations (23), which fortunately, allows exact Skyrmion solutions in a general form.

4.1. General Skyrmion solution in anisotropic ferromagnet through generalized analytic function

Recall that the general BP Skyrmions for the ferromagnetic model (5) can be given by any analytic function (11), which may be expressed by

$$f(z) = |f|(\cos\theta_f + i \sin\theta_f), \text{ with } \partial_{\bar{z}} f(z) = 0. \tag{24}$$

On the other hand, for our anisotropic ferromagnetic model (18) the solutions for the energy minimizing condition (23) consistent with the *generalized* CR relations (22), may be given by exact Skyrmions linked to any complex function F of the form

$$\begin{aligned} &F(z, \bar{z}) = -\bar{c}_1 f(z) + c_2 \bar{f}(z), \\ &c_1 = i - e^{-i\theta_0}, \; c_2 = i + e^{-i\theta_0}, \text{ with } \theta_0 = 60^\circ, \end{aligned} \tag{25}$$

showing $\partial_{\bar{z}} F(z, \bar{z}) = c_2 \partial_{\bar{z}} \bar{f}(z) \neq 0$, where $f(z)$ is any analytic function in the form (11) and $\bar{f}(z)$ is its complex conjugate. Note, that since in (25) one gets $c_1 = 2i$, $c_2 = 0$, for $\theta_0 = 90^0$, function F reduces to an analytic function $2if(z)$, as expected for a BP Skyrmion.

It is rather surprising, that a restricted class of complex functions, which goes beyond the analyticity with $\partial_{\bar{z}} F \neq 0$, appears here as exact Skyrmion solutions with high degree of asymmetry. To analyze the property of such functions, which satisfy a special condition $(c_1 \partial_{\bar{z}} F + c_2 \partial_{\bar{z}} \bar{F}) = 0$ together with its complex conjugate, and to see how they differ from the $\bar{\partial}$ problem known in the literature [8], we represent the function in the allowed integral form [8]

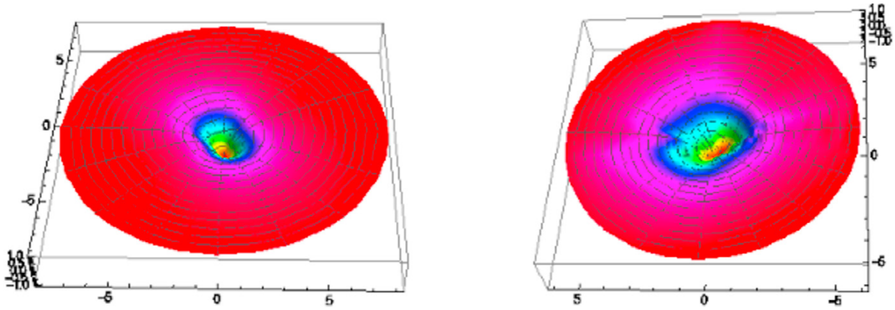
$$\begin{aligned} F(z, \bar{z}) &= -\bar{c}_1 f(z) - \frac{1}{2\pi i} \int\int_\Omega \frac{\partial_{\bar{\zeta}} F(\zeta, \bar{\zeta})}{(\zeta - z)} d\zeta \wedge d\bar{\zeta} \\ &= -\bar{c}_1 f(z) - c_2 \frac{1}{2\pi i} \int d\bar{\zeta} \partial_{\bar{\zeta}} \bar{f}(\zeta) \int d\zeta \frac{1}{(\zeta - z)} = -\bar{c}_1 f(z) + c_2 \bar{f}(z), \end{aligned} \tag{26}$$

which recovers the solution through $\bar{\partial}$ like problem, though it differs a bit from the usual $\bar{\partial}$ assumption of $\partial_{\bar{z}} F = f(z)$ [8].

We can therefore find the exact Skyrmion solutions for the present model in a general form for the fields β, γ as

$$\tan\frac{\beta}{2} e^{i\gamma} = F \equiv |f|(\sin(\theta_f + \theta_0) + i \sin\theta_f), \; \theta_0 = 60^\circ \tag{27}$$

given explicitly as



a) b)

Fig. 3. Asymmetric Skyrmion for M^3 in magnetic models. a) Exact asymmetric Skyrmion in anisotropic ferromagnetic model with $Q = -1$. b) Asymmetric Skyrmion in Helimagnetic model, with soliton a) deformed under perturbation of DM coupling.

$$\beta = 2\tan^{-1}(|f|\delta_f), \; \delta_f^2 = \sin^2(\theta_f + 60^\circ) + \sin^2\theta_f), \tag{28}$$

$$\sin\gamma = \frac{1}{\delta_f}\sin\theta_f, \quad \cos\gamma = \frac{1}{\delta_f}\sin(\theta_f + 60^\circ). \tag{29}$$

Note that with the analytic function f(z) (24) given as (11) the localized Skyrmion solution for our anisotropic ferromagnetic model can be given in the general form (29) with topological charge $Q = \pm N$.

For understanding the structure of such topological solitons let us focus more closely on the simplest case with $Q = -1$, obtained as a reduction of (29) with $|f| = \frac{1}{\rho}$, and $\theta_f = -\alpha$ to construct the explicit exact Skyrmion (see Fig. 3a for graphical representation)

$$M^3 = \cos\beta = \frac{\rho^2 - \delta_f^2(\alpha)}{\rho^2 + \delta_f^2(\alpha)}, \quad \delta_f^2(\alpha) = \sin^2\alpha + \sin^2(\alpha - \theta_0)$$
$$\cos\gamma = -\frac{1}{\delta_f}\sin(\alpha - \theta_0), \; \theta_0 = 60^\circ, \; Q = -1. \tag{30}$$

It is important to note that even in the simplest case the exact Skyrmion solution does not exhibit circular symmetry since $\beta = \beta(\rho, \alpha)$ (Fig. 2a), whereas at $\theta_0 = 90^\circ$, we get $\delta_f = 1$ and hence the asymmetric Skyrmion reduces to the BP case with circular symmetry: $\beta = \beta(\rho)$, $\gamma = -\alpha$. Note that the Skyrmions in both these cases are localized structures with the asymptotic: $M^3|_{\rho=0} = -1$, $M^3|_{\rho=\infty} = 1$.

The simplest asymmetric topological soliton constructed here, is perhaps the only known spherically nonsymmetric exact solution found in any solvable model, in two and higher dimensions. We hope therefore, that the present novel approach could be applicable to other known models for discovering new asymmetric solutions.

5. Asymmetric Skyrmions in Helimagnetic model

Our next intention is to apply our result for seeking asymmetric Skyrmions for the helimagnetic model, with the hope that they could be detected in future precision experiments. Note that in spite of the discovery of exact Skyrme solitons with $Q = \pm N$ in a general form (25), for anisotropic ferromagnetic model (18), one should pay attention that such solutions are again scale invariant as the BP Skyrmions and hence difficult to observe in real experiments due to their metastability. Therefore for physically observed configurations such Skyrmions need to be stabilized by coupling to other interactions, which can break the explicit scale invariance of the system. This was realized in the known helimagnetic model [6], where a DM interaction (3) was considered in addition to the ferromagnetic Hamiltonian (5). We shall adopt a similar strategy in stabilizing the asymmetric Skyrmion solutions of our anisotropic ferromagnetic model (18) by adding the same DM interaction (3) to it. Therefore, we propose to look for the Skyrmion spin pattern in an anisotropic helimagnetic model, theoretically, by taking the helimagnetic model as

$$H_{anheli} = \frac{J}{2}(H_f + H_{anis}) + DH_{dm}, \tag{31}$$

where the ferromagnetic Hamiltonian H_f is given in the standard form (1), H_{anis} is its anisotropic extension as in (18) and H_{dm} is the DM interaction term (3). Since helimagnetic materials show a high degree of asymmetry, noninversion symmetry and additionally anisotropy, when projected on a 2d plane, it might be reasonable to consider a helimagnetic model like (31) showing explicit asymmetry and anisotropy.

Recall, that the DM interactions in magnetic crystals in general might have antisymmetric as well as symmetric contributions under spin exchange, intensity of which would depend on the crystal symmetry [9]. Notice that, the H_{dm} term containing a single derivative is antisymmetric under space inversion, representing the antisymmetric part of the DM interaction, while H_{anis} part involving two gradients is symmetric under space inversion and might appear in magnetic crystals with higher symmetry.

For finding the spin texture in this helimagnetic model (31) described by the Skyrmion solution, one has to solve the associated Euler equations for the two angle variables $\beta(\rho, \alpha)$, and $\gamma(\rho, \alpha)$, derived from the model Hamiltonian. The corresponding energy minimizing Euler equations in the static case for our helimagnetic model (31) may be given by

$$J\left(\nabla^2\beta - \frac{1}{2}\sin 2\beta(\nabla\gamma)^2 - \frac{1}{2}\cos\beta(\nabla\beta\cdot\nabla\gamma)\right) + 2D\, M_\beta(\beta,\gamma) = 0, \tag{32}$$

for angle $\beta(\rho, \alpha)$ and

$$\begin{aligned} J & \left(\sin 2\beta(\nabla\beta\cdot\nabla\gamma) + \sin^2\beta(\nabla^2\gamma) - \frac{1}{2}(\cos\beta(\nabla\beta)^2 + \sin\beta\nabla^2\beta\right) \\ & + 2D\, M_\gamma(\beta,\gamma) = 0, \end{aligned} \tag{33}$$

with respect to angle $\gamma(\rho, \alpha)$, where the additional DM terms are

$$M_\beta(\beta,\gamma) \equiv \sin^2\beta(\partial_\rho\gamma\cos(\gamma-\alpha) + \frac{1}{\rho}\partial_\alpha\gamma\sin(\gamma-\alpha)) \tag{34}$$

and

$$M_\gamma(\beta,\gamma) \equiv -\sin^2\beta(\partial_\rho\beta\cos(\gamma-\alpha) + \frac{1}{\rho}\partial_\alpha\beta\sin(\gamma-\alpha). \tag{35}$$

It is crucial to observe, that under a scale transformation: $\rho \to \lambda\, \rho$ the exchange terms and the DM terms behave differently, since all exchange terms contain gradients in second order, while the terms in the DM interaction have them only in the first order. As a result under the above scale transformation only the DM terms (34), (35) acquire a multiplicative factor λ while the exchange terms remain scale invariant. Therefore, in the absence of DM interaction the scaling factor λ remains arbitrary including a vanishing value, making the ferromagnetic Skyrmion to be metastable. However, with the additional DM terms as in our case,, with the factor λ explicitly appearing in the Euler equations, it gets fixed through the coupling parameters J, D of the model acquiring a nonvanishing value. Therefore the Skyrmion solutions to the above Euler equations minimizing the helimagnetic Hamiltonian (31) is likely to be a stable solution, similar to those in the known helimagnetic model, solved numerically and observed experimentally [3].

These equations however are highly nonlinear coupled partial differential equations in two variables and do not allow in general separation of variables due to high degree of asymmetry and therefore difficult to handle in general. It is also evident that the exact solvability of the ferromagnetic part (18) of the model is lost now, due to the additional DM interaction with $D \neq 0$. Therefore, for constructing Skyrmion solutions for the anisotropic helimagnetic model our strategy would be to treat the system perturbatively by considering $\epsilon = D/J$, the relative coupling between the DM and the ferromagnetic exchange interaction, as small parameter. The perturbative solution therefore may be given by expanding around the arbitrary exact solutions of our ferromagnetic model:

$$\beta(\rho,\alpha) = \beta_0(\rho,\alpha) + \epsilon\, \beta_1(\rho,\alpha),\ \gamma(\rho,\alpha) = \gamma_0(\rho,\alpha) + \epsilon\, \gamma_1(\rho,\alpha), \tag{36}$$

in the first order of approximation, where $\beta_0(\rho,\alpha)$, $\gamma_0(\rho,\alpha)$ are the unperturbed solutions induced by the anisotropic ferromagnetic Hamiltonian (18) alone, while $\beta_1(\rho,\alpha)$, $\gamma_1(\rho,\alpha)$ are the deformations suffered, when the DM interaction (3) is switched on, perturbatively. The parameter D also serves as the scaling parameter, breaking the scale invariance of the unperturbed Skyrmions and thus providing the required stability to the soliton solutions. Since, we have derived already the set of general solutions β_0, γ_0 in the analytic form (25), we may extract the deforming solutions β_1, γ_1 by solving the corresponding Euler equations for the helimagnetic model (31) through the perturbative expansion (36). This would reduce the equations to a linear set of coupled partial differential equations, which can be solved numerically. Adding these perturbing solutions to the Skyrmions β_0, γ_0 found above for the anisotropic ferromagnetic model, we can finally obtain the Skyrmions for the helimagnetic model in the form (36).

For demonstrating our approach we extract an individual Skyrmions for the Helimagnetic model (31) taking the unperturbed solitons as the simplest solution (30) shown in Fig. 3a. The corresponding perturbative solution for the Helimagnetic model solved numerically (see the Supplementary material for the Mathematica 10 code) is shown in Fig. 3b. It is interesting to notice, that the circular asymmetry of the exact Skyrmion of the anisotropic ferromagnetic model has been much enhanced in the helimagnetic model. Such asymmetric solutions never predicted earlier for the helimagnetic models, would encourage to look for them in future experiments.

Note that one can adopt the above approach without including the H_{anis} interaction term in (31), which has been considered recently by us [10] for finding more general Skyrme solutions in helimagnetic model. However our present aim is to propose a helimagnetic model with inherently asymmetric Skyrmion solutions which can not be reduced to a symmetric one and might carry experimental relevance. Our basic solutions are also novel exact Skyrmions without circular symmetry (25). Note that exact Skyrmion solution with unit topological charge without spherical symmetry has never been found earlier.

It is also a matter of concern, that though experimentally the images of Skyrmion lattice and Skyrmion crystals in the hexagonal form have been observed vividly [3], no satisfactory theoretical proposal seem to has been put forward, where such crystalline structures could be derived as a solution from the governing Euler equations of the helimagnetic model, without neglecting the nonlinear interactions between the individual Skyrme molecules, forming the crystal. Note, that our approach as outlined above, could give one of such solutions for the Skyrmion crystals or Skyrmion lattice in a helimagnetic model, by perturbing a ferromagnetic Skyrmion crystal or lattice with anisotropy.

Note, that hexagonal Skyrmion crystals in helimagnets are formed from $N = 7$ individual Skyrme molecules and the Skyrmion crystals in tern serve as the building blocks of the Skyrmion lattice, which are formed through repeated entries of the Skyrmion crystals by periodic shifting of their centers. Therefore for finding first the Skyrmion crystal solution for our helimagnetic model, we could proceed with the building of a hexagonal Skyrmion crystal for our anisotropic ferromagnetic model, which can be constructed from the general Skyrmion solutions in the analytic form (27), where the function $F(z, \bar{z})$ (25) expressed through analytic function $f(z)$ (24) may be chosen as a Skyrme soliton in a hexagonal lattice form with equal sides a_s, explicitly as

$$f^{-1}(z) = \prod_{j=1}^{7} (z - z_j) = -a_s^6 \rho e^{i\alpha} + \rho^7 e^{i7\alpha}, \tag{37}$$

having topological charge $Q = -7$ with 7 centers chosen equidistantly by properly adjusting the constant parameters z_j. For finding the corresponding Skyrmion crystal solution for the helimagnetic model the above crystal solution could be subjected to DM-interaction perturbatively.

For extending the solution for building the Skyrmion lattice, solution (37) can be repeated n-times with arbitrary n and by shifting the parameters z_j, which would again be an exact solution, since the n-product of solution (37) is another solution, due to general form of the analytic function (11). Such Skyrmion lattice subsequently can be treated perturbatively for DM-interaction to get finally the required Skyrme lattice solution for the anisotropic helimagnetic model (31).

The derivation and construction of the Skyrme crystal and Skyrme lattice solutions for the helimagnetic model outlined above, will not be presented here in detail, since our basic aim is to put forward our novel idea and to show how it works.

6. Concluding remarks

The recent experimental observation of Skyrmions in helimagnetic materials on a 2d plane, is a triumph of the theoretical prediction of such exact topological solitons in a ferromagnetic model proposed 40 years ago by Belavin and Polyakov (BP) [4], though due to the metastable states they could not be detected in the original model. On the other hand, during all these years, there was hardly any proposal which could generalize the idea of BP for constructing novel exact solitons of topological origin in other 2d models, which consequently, could be tested experimentally. We have presented here such a new idea based on some geometric inequalities of universal nature and found new exact topological solitons with intrinsic asymmetry in an anisotropic ferromagnetic model. This result generalizes the well known Cauchy–Riemann condition as a new minimization condition for the model Hamiltonian and extends the BP Skyrmions to a novel class of exact asymmetric Skyrmions. Such Skyrmion solutions are linked to a new class of *generalized* functions related to the $\bar{\partial}$ problem and might have far reaching consequences in terms of generalized symmetries in conformal field theory and related algebraic structures [11]. On the other hand, our result could be applied to the helimagnetic models of contemporary interest, for finding individual Skyrmions as well as Skyrmion crystals and lattice exhibiting asymmetry and anisotropy, as a natural solutions of the governing equations, which hopefully could be verified in precision experiments.

Therefore the expected impact of the present proposal is twofold. As a theoretical result, it goes beyond the original idea of BP and constructs exact Skyrmions with intrinsic asymmetry, giving an intriguing generalization of the class of analytic functions. As an application, our approach constructs solutions for Skyrmions, Skyrmion crystals and Skyrmion lattice with asymmetry, bearing experimental interests in helimagnets.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.nuclphysb.2016.04.043.

References

[1] M. Lee, et al., Phys. Rev. Lett. 102 (186601) (2009) 1;
A. Neubauer, et al., Phys. Rev. Lett. 102 (186602) (2009) 1.
[2] S. Mühlbauer, et al., Science 323 (2009) 915;
W. Münzer, et al., Phys. Rev. B 81 (2010) 041203(R).
[3] X.Z. Yu, et al., Nature 465 (2010) 901.
[4] A.A. Belavin, A.M. Polyakov, JETP Lett. 22 (1975) 245.
[5] C. Pfleiderer, S.R. Julian, G.G. Lonzarich, Nature 414 (2001) 427;
C. Pfleiderer, et al., Nature 427 (2004) 227;
B. Binz, A. Vishwanath, Physica B 403 (2008) 1336.
[6] U.K. Rößler, A.N. Bogdanov, C. Pfleiderer, Nature 442 (2006) 797;
U.K. Rößler, A.A. Leonov, A.N. Bogdanov, Chiral Skyrmionic matter in noncentrosymmetric magnets, arXiv:1009.4849, 2010;
U.K. Rößler, A.A. Leonov, A.N. Bogdanov, Skyrmion textures in chiral magnets, arXiv:0907.3651, 2009;
A.B. Butenko, A.A. Leonov, U.K. Rößler, A.N. Bogdanov, Phys. Rev. B 82 (2010) 052402.
[7] L.D. Landau, E.M. Lifshitz, Statistical Physics, Pergamon, New York, 1977.
[8] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, Proc. R. Soc. Lond. A 453 (1999) 182;
A.S. Fokas, B. Pelloni, A Transform Method for Evolution PDEs on the Interval, IMA J. Appl. Math. 75 (2005) 564–587;
B.G. Konopelchenko, On dbar-problem and integrable equations, arXiv:nlin/0002049, 2000.
[9] T. Moriya, Phys. Rev. 120 (1960) 91.
[10] A. Kundu, Asymmetric Skyrmion lattice in helimagnets, arXiv:1510.05767, 2015.
[11] P. Ginsparg, Applied conformal field theory, arXiv:hep-th/9108028, 1991.