JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 5, 269-286 (1962)

The Limiting Speeds of Characteristics in Relaxation Hydrodynamics*

N. COBURN

Department of Mathematics, University of Michigan, Ann Arbor, Michigan

Submitted by C. L. Dolph

I. INTRODUCTION

The object of relaxation hydrodynamics is to study the motion of charged (infinitely conductive) compressible gases which are not in equilibrium. That is, energy is being added to or withdrawn from the system. As a result, the basic equations of the system consist of the usual equations of continuity, motion, and the Maxwell equations, but the usual single energy equation is replaced by two equations. One of these equations is the usual energy equation with an additional term involving a new dependent variable, the relaxation variable q. The additional equation states that the rate of change of q along a streamline, denoted by $\bar{q}$, is directly proportional to the rate of change of internal energy with respect to the relaxation variable, q. These equations have been studied in the linearized case by Stupochenko and Stakhanov [1].

In this paper, the nonlinear theory of relaxation hydrodynamics is discussed. Our purpose is to obtain the limiting speeds for characteristic waves in charged compressible relaxation hydrodynamics. Two speeds were shown to exist in the linearized theory by Stupochenko and Stakhanov [1] by determining the equation for the velocity vector in the nonmagnetic, compressible case. Our procedure is to formulate the general discontinuity relations for characteristics [2]. Although the original system of equations consists essentially of nine equations in nine dependent variables (the velocity vector; the magnetic field vector; the density; the entropy; and q, the relaxation variable), the set of discontinuity relations forms an *underdetermined system.* It is shown that this system can be completed by *assuming that the normal derivative of $\bar{q}$ along any discontinuity manifold is continuous.* Furthermore, it is shown that in the nonmagnetic case *this assumption suffices to determine the limiting speeds given by Stupochenko and Stakhanov* [1]. Again, it is shown

* This work was performed under a grant by the National Science Foundation (G-10103), administered by the Office of Research Administration of the University of Michigan.

that *in the nonlinear case (but not in the linear case) this assumption is consistent with the other equations of the system of discontinuity relations*. Both of the previously mentioned limiting speeds imply that the jumps of the derivatives of the various physical quantities approach zero; *another limiting speed exists for which the jumps of these derivatives approach infinity*. Finally, the limiting speeds for the magnetic case are determined. It is shown that the *magnetic field "splits" the non-magnetic limit speed into two magnetic limit speeds.*

II. The Basic Equations

In this section, we shall discuss the basic equations of relaxation hydrodynamics. These equations are: (1) the equations of motion of a charged (infinitely conductive), compressible fluid; (2) the equation of continuity; (3) the Maxwell relations; (4) the energy *relations*. Let ρ, S, p, e, $'T$, q be the density, specific entropy, pressure, specific internal energy, absolute temperature, relaxation parameter, respectively, and let v^j, H^j ($j = 1, 2, 3$) be the velocity and magnetic field vectors, respectively, in a general system of curvilinear coordinates x^j (with covariant derivative defined by ∇_j and time denoted by t). Finally, let K be a constant and η be the constant magnetic permeability.

First, we consider the thermodynamics of relaxation hydrodynamics. In this paper, we shall consider ρ, S, and q as the independent thermodynamic variables. Further, the first law of thermodynamics will be *assumed* to be the relation [1]

$$de = \frac{p}{\rho^2}\, d\rho + {'T} dS + \frac{\partial e}{\partial q}\, dq \tag{2.1}$$

By *assuming* that along a streamline, the relation [1]

$$\frac{de}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt} \tag{2.2}$$

is valid, as in the equilibrium flow of a compressible fluid, one obtains from (2.1) and (2.2)

$${'T}\frac{dS}{dt} = -\frac{\partial e}{\partial q}\frac{dq}{dt} \tag{2.3}$$

Further, it is *assumed* in relaxation hydrodynamics that [1]

$$\bar{q} = -K\frac{\partial e}{\partial q}, \qquad \bar{q} \equiv \frac{dq}{dt} \tag{2.4}$$

where K is some constant. Thus, (2.3) becomes by use of (2.4)

$$K'T\frac{dS}{dt} = \left(\frac{dq}{dt}\right)^2 \tag{2.5}$$

The relations (2.4), (2.5) are the two new relations of relaxation hydrodynamics which replace the energy relation

$$\frac{dS}{dt} = 0$$

of conventional hydrodynamics. Instead of working with (2.4), (2.5), we may work with (2.5) *and the relation obtained by replacing* $\partial e/\partial q$ of *(2.4) in (2.1), that is*

$$de = \frac{p}{\rho^2}\,d\rho + 'TdS - \frac{1}{K}\frac{dq}{dt}\,dq \tag{2.6}$$

Note, (2.6) implies (2.4).

Now, we consider some consequences of (2.1). Evidently, we can write

$$'T = \left.\frac{\partial e}{\partial S}\right)_{\rho,q} \tag{2.7}$$

$$p = \rho^2 \left.\frac{\partial e}{\partial \rho}\right)_{S,q} \tag{2.8}$$

By differentiation of (2.8) we obtain the relations

$$A \equiv \left.\frac{\partial p}{\partial \rho}\right)_{S,q} = 2\rho\frac{\partial e}{\partial \rho} + \rho^2\frac{\partial^2 e}{\partial \rho^2} \tag{2.9}$$

$$B \equiv \left.\frac{\partial p}{\partial S}\right)_{\rho,q} = \rho^2\frac{\partial^2 e}{\partial S\,\partial \rho} \tag{2.10}$$

$$C \equiv \left.\frac{\partial p}{\partial q}\right)_{S,\rho} = \rho^2\frac{\partial^2 e}{\partial \rho\,\partial q} \tag{2.11}$$

Further, by differentiation of (2.7) we find

$$D \equiv \left.\frac{\partial' T}{\partial \rho}\right)_{S,q} = \frac{\partial^2 e}{\partial \rho\,\partial S} \tag{2.12}$$

$$E \equiv \left.\frac{\partial' T}{\partial S}\right)_{\rho,q} = \frac{\partial^2 e}{\partial S^2} \tag{2.13}$$

$$F \equiv \left.\frac{\partial' T}{\partial q}\right)_{\rho,S} = \frac{\partial^2 e}{\partial q\,\partial S} \tag{2.14}$$

Finally, we define the coefficient G by

$$G \equiv \frac{\partial^2 e}{\partial q^2} \tag{2.15}$$

Instead of (2.4) or (2.6), one may use the relation obtained by differentiating (2.4) and using (2.11), (2.14), (2.15)

$$\frac{d\bar{q}}{dr} = -K\left(G\frac{dq}{dr} + F\frac{dS}{dr} + \frac{C}{\rho^2}\frac{d\rho}{dr}\right) \tag{2.16a}$$

where $\bar{q}$ is defined by the second equation of (2.4) or

$$\bar{q} \equiv \frac{dq}{dt} = \frac{\partial q}{\partial t} + v^j \nabla_j q \tag{2.16b}$$

and the directional derivative d/dr is

$$\frac{d}{dr} \equiv \frac{dt}{dr}\frac{\partial}{\partial t} + \frac{dx^j}{dr}\nabla_j \tag{2.16c}$$

The value of using (2.16a) rather than the first equation of (2.4) will become apparent when we discuss discontinuity theory.

Finally, we list the basic relations: (1) the equations of motion; (2) the equation of continuity; (3) the Maxwell equations; (4) the energy equations. These are [3]:

$$\rho\left(\frac{\partial v_j}{\partial t} + v^k \nabla_k v_j\right) + \nabla_j\left(p + \frac{\eta}{2}H^2\right) - \nabla_k(\eta H^k H_j) = 0 \tag{2.17}$$

$$\frac{\partial \rho}{\partial t} + \nabla_j(\rho v^j) = 0 \tag{2.18}$$

$$\frac{\partial H_j}{\partial t} + v^k \nabla_k H_j - H^k \nabla_k v_j + H_j \nabla_k v^k = 0 \tag{2.19}$$

$$\nabla_j H^j = 0 \tag{2.20}$$

$$K'T\left(\frac{\partial S}{\partial t} + v^j \nabla_j S\right) = \bar{q}^2 \tag{2.21}$$

$$\frac{\partial q}{\partial t} + v^j \nabla_j q \equiv \bar{q} = -K\frac{\partial e}{\partial q} \tag{2.22}$$

It should be noted that in the above equations

$$H^2 \equiv H_j H^j \tag{2.23}$$

III. The Cauchy Problem and Characteristic Manifolds

In order to discuss the theory of characteristic manifolds via the Cauchy problem, we consider a Euclidean four-space with coordinate variables x^j, t (cf. [4, p. 270]). We assume that the initial and characteristic manifolds, S_3, are three-dimensional of class C^2 with defining equation

$$\phi(x^j, t) = \text{constant} \tag{3.1}$$

Further, we shall use the notation, where g_{jk} is the metric tensor of Euclidean three-space

$$\phi_j \equiv \frac{\partial \phi}{\partial x^j}, \qquad \phi_0 \equiv \frac{\partial \phi}{\partial t}, \qquad \phi^j \equiv g^{jk}\phi_k, \qquad \Phi \equiv \phi^j \phi_j \tag{3.2a}$$

$$L \equiv \phi_0 + \phi_k v^k, \qquad '\Phi \equiv \phi^\alpha \phi_\alpha, \qquad K'T \equiv T \tag{3.2b}$$

If we denote the above four-vector by ϕ_α, where

$$\phi_\alpha \equiv (\phi_0, \phi_j),$$

then ϕ_α lies along the normal to S_3. Note, Latin indices have the range

$$j, k = 1, 2, 3;$$

Greek indices have the range

$$\alpha, \beta = 0, 1, 2, 3.$$

In our work, we shall introduce

$$\underset{a}{t_\alpha}\ (a = 1, 2, 3)$$

any three mutually orthogonal unit vectors tangent to S_3 at each point. Further, we define ∇_0, g_{00}, g_{0j} by

$$\nabla_0 \equiv \frac{\partial}{\partial t}$$

$$g_{00} \equiv 1, \qquad g_{0j} \equiv 0$$

To study the Cauchy problem, we assume that along any *known* initial S_3, the following are *known:* (1) v^j, H^j, ρ, S, q, and the tangential derivatives along S_3 of these quantities; (2) T, p, e, and all of their first partials with respect to ρ, S, q *except* $\partial e/\partial q$. That is, the last derivative when considered as a function of x^j, t is unknown along S_3. Hence, we may write

$$
\begin{aligned}
\nabla_\alpha v_j &= \phi_\alpha W_j + \sum_a \underset{a}{t_\alpha} \underset{a}{\bar{W}_j} \\
\nabla_\alpha H_j &= \phi_\alpha k_j + \sum_a \underset{a}{t_\alpha} \underset{a}{\bar{k}_j} \\
\nabla_\alpha \rho &= \phi_\alpha R + \sum_a \underset{a}{t_\alpha} \underset{a}{\bar{R}} \\
\nabla_\alpha S &= \phi_\alpha U + \sum_a \underset{a}{t_\alpha} \underset{a}{\bar{U}} \\
\nabla_\alpha q &= \phi_\alpha W + \sum_a \underset{a}{t_\alpha} \underset{a}{\bar{W}}
\end{aligned}
\tag{3.3}
$$

The quantities W_j, k_j, R, U, W are proportional to the *unknown* normal derivatives of v_j, H_j, ρ, S, q, respectively, and the barred quantities $\underset{a}{\bar{W}_j}$, $\underset{a}{\bar{k}_j}$, $\underset{a}{\bar{R}}$, $\underset{a}{\bar{U}}$, $\underset{a}{\bar{W}}$ are proportional to the *known* tangential derivatives of v_j, H_j, ρ, S, q, respectively. The Cauchy equations are obtained by substituting (3.3) into (2.17)-(2.22). We shall discuss the corresponding characteristic equations in detail. It should be noted that when the discontinuity relations form an underdetermined system (see (4.2), (4.3) for the nonmagnetic case and note k is "known" only to be *some* function of the unknown, $\phi(t, x^j) =$ constant) then the associated Cauchy initial value problem equations (which are nonhomogeneous forms of (4.2), (4.3), with $\phi(x^j, t)$ known in the nonmagnetic case) do not possess a unique solution.

We assume that a characteristic manifold, S_3, is determined by the following conditions: (1) v^j, H^j, the three independent thermodynamic variables ρ, S, q, and their tangential derivatives along S_3 are continuous and hence *only the normal derivatives along S_3 of v^j, H^j, ρ, S, q are discontinuous:* (2) T, p, e are continuous functions of ρ, S, q and hence of x^j, t; (3) the first partials of T, p with respect to ρ, S, q are continuous functions of x^j, t; (4) the first partials of e with respect to ρ and S are continuous functions of x^j, t *but the first partial of e with respect to q is a discontinuous function of x^j, t along S_3*; (5) all of the second derivatives of e with respect to ρ, S, q are continuous.

In view of the above assumptions and (3.2), (3.3), (2.16b), we may write along S_3

$$[\nabla_\alpha v_j] = \phi_\alpha V_j, \qquad [W_j] = V_j \tag{3.4}$$

$$[\nabla_\alpha H_j] = \phi_\alpha h_j, \qquad [k_j] = h_j \tag{3.5}$$

$$[\nabla_\alpha \rho] = \phi_\alpha P, \qquad [R] = P \tag{3.6}$$

$$[\nabla_\alpha S] = \phi_\alpha s, \qquad [U] = s \tag{3.7}$$

$$[\nabla_\alpha q] = \phi_\alpha Q, \qquad [W] = Q \tag{3.8}$$

$$[\bar{q}] = LQ \tag{3.9}$$

where the bracketts denote the "jump" of the enclosed quantity. Further, using the result that the jump of any tangential derivative of $\bar{q}$ is equal to that tangential derivative of the jump of $\bar{q}$ (Hadamard's theorem, [5]), we find from (3.9)

$$\frac{d}{dr} LQ = \left[\frac{d\bar{q}}{dr}\right] \tag{3.10}$$

where d/dr is a directional derivative in *any direction tangent* (say $\underset{a}{t}^{\alpha}$) to S_3. By forming the jump of (2.16a) and using (3.6)-(3.8), we see that (3.10) leads to [see (3.9)]

$$\underset{a}{t}^{\alpha}[\nabla_\alpha \bar{q}] = 0 \tag{3.11a}$$

$$LQ = k = [\bar{q}] \tag{3.11b}$$

where *k is constant over any S_3*. Note, (3.11b) implies that if ϕ_α is replaced by any vector along the normal (say $'\phi_\alpha$) then Q must be replaced by $'Q$ where $'\phi_\alpha v^\alpha{}'Q = k$. To obtain the jump of the normal derivative of $\bar{q}$ along S_3, we form the jump of (2.16a) when d/dr represents a directional derivative along this normal. From (3.6)-(3.8) and (2.16a) we find

$$\phi^\alpha[\nabla_\alpha \bar{q}] = -K'\Phi\left(GQ + Fs + \frac{CP}{\rho^2}\right) \tag{3.12}$$

The relation (3.12) and the first equation of (3.11) are equivalent to

$$[\nabla_\alpha \bar{q}] = -\phi_\alpha K\left(GQ + Fs + \frac{CP}{\rho^2}\right) \tag{3.13}$$

Finally, we note by use of the chain rule and (2.9)-(2.11), (3.6)-(3.8)

$$[\nabla_\alpha p] = \phi_\alpha(AP + Bs + CQ) \tag{3.14}$$

The relations (3.4)-(3.9), (3.11), (3.13), (3.14) are the basic jump relations for characteristic S_3.

To obtain the discontinuity relations, we form the jumps of (2.17)-(2.21). From (3.2), (3.4)-(3.8), (3.11b), and (3.14) we obtain (cf. [2, p. 384, eqs. (8.18), (8.19)])

$$\rho L V_j + \phi_j(AP + Bs + CQ + \eta H^k h_k) - \eta\phi_k H^k h_j = 0 \tag{3.15}$$

$$PL + \rho\phi_j V^j = 0 \tag{3.16}$$

$$Lh_j - \phi_k H^k V_j + H_j \phi_k V^k = 0 \tag{3.17}$$

$$\phi_j h^j = 0 \tag{3.18}$$

$$TLs = k^2 + 2k\bar{q}_1 \tag{3.19}$$

$$LQ = k \tag{3.20}$$

where $\bar{q}_1$ is the value of $\bar{q}$ on one side of S_3. First, we note that (3.18) follows from (3.17) by forming the scalar product of (3.17) with ϕ^j. Further, by eliminating Q, s from (3.15) by using (3.19) and (3.20), the equation (3.15) becomes

$$\rho L V_j + \phi_j \left\{ AP + \frac{B}{TL}(k^2 + 2k\bar{q}_1) + \frac{C}{L} k + \eta H^k h_k \right\} - \eta \phi_k H^k h_j = 0 \tag{3.21}$$

Thus, the *system of discontinuity relations consists of (3.16), (3.17), (3.19)-(3.21). For specified k, the* equations (3.16), (3.17), (3.21) *are a system of seven linear algebraic equations in the seven unknowns*

$$V_j, h_j, P$$

Since there is no simple scheme for directly specifying k as some function of the unknown $\phi(x^j, t)$ of (3.1), we shall use an extended form of condition (5) on characteristic manifolds. That is, in addition to all second partials of e with respect to ρ, S, q being continuous [see (2.4) and Section VI], we *assume*

$$[\nabla_\alpha \bar{q}] = 0 \tag{3.22}$$

Thus, from (3.13), (3.22), we find

$$GQ + Fs + \frac{CP}{\rho^2} = 0 \tag{3.23}$$

The equations (3.16), (3.17), (3.19)-(3.21), (3.23) constitute a system of ten nonlinear homogeneous algebraic conditions for the ten unknowns

$$V_j, h_j, P, k, Q, s$$

By using (3.20), k can be eliminated from the above system. However, the question still remains whether LQ can be constant along any characteristic manifold S_3 when (3.23) is valid. We shall show that in the *nonlinear theory* a partial differential equation [see (4.7), (5.9)] always exists involving $k = LQ$ and ϕ_j, so that by specifying k as a function of ϕ, the condition that k is constant over any S_3, ϕ = constant, is satisfied.

Finally, we shall introduce the concept of *weak reactions* in a manner similar to that of *weak shocks*. That is, we have used the expansion (see 3.19)

$$[\bar{q}^2] = [\bar{q}]^2 + 2[\bar{q}]\,\bar{q}_1 \tag{3.24}$$

When the term $[\bar{q}]^2$ is negligible in comparison to $[\bar{q}]$ then

$$[\bar{q}^2] = 2\bar{q}_1[\bar{q}] \tag{3.25}$$

is valid and the *reaction will be called weak*. Evidently, for weak reactions, the only modification of the discontinuity relations occurs in (3.19), (3.21): *the term containing k^2 must be omitted from the relations*. The discontinuity relations form a system of linear homogeneous algebraic equations. But, *no scheme exists for satisfying the condition, k = constant, over any characteristic manifold S_3.*

IV. The Nonmagnetic Case

In order to understand the difference between the customary compressible and relaxation compressible hydrodynamics, we shall discuss the nonmagnetic case. Here,

$$h_j = H_j = 0 \tag{4.1}$$

and (3.16), (3.21) become

$$PL + \rho\phi_j V^j = 0 \tag{4.2}$$

$$\rho L V_j + \phi_j \left\{ AP + \frac{B}{TL}(k^2 + 2k\bar{q}_1) + \frac{C}{L}k \right\} = 0 \tag{4.3}$$

If we form the scalar product of (4.3) with ϕ^j and then eliminate $\phi^j V_j$ from the resulting equation by use of (4.2), we obtain

$$LP(A\Phi - L^2) + \Phi\left\{ Ck + \frac{B}{T}(k^2 + 2k\bar{q}_1) \right\} = 0 \tag{4.4}$$

By eliminating k from (4.4), we obtain, using (3.20),

$$TP(A\Phi - L^2) + \Phi\{TCQ + B(LQ^2 + 2Q\bar{q}_1)\} = 0 \tag{4.5}$$

Further, if we eliminate s in (3.23) by using (3.19), (3.20), we find

$$T\rho^2 GQ + F\rho^2(LQ^2 + 2Q\bar{q}_1) + TCP = 0 \tag{4.6}$$

Finally, if we eliminate P from (4.5) (when $A\Phi - L^2 \neq 0$) by using (4.6), we obtain the *degenerate quadratic equation in Q*

$$Q^2L\{F\rho^2L^2 + \Phi(BC - \rho^2FA)\}$$
$$+ Q\{T(L^2G\rho^2 + C^2\Phi - A\Phi\rho^2G) + 2\bar{q}_1(\rho^2FL^2 - \rho^2FA\Phi + BC\Phi)\} = 0 \tag{4.7}$$

Now, we consider the limiting wave speeds. First, we note that

$$c = L\Phi^{-1/2} \tag{4.8}$$

is the speed of wave propagation [cf. (4, p. 285)]. From (4.7), we see that *three limiting speeds exist*, corresponding to the relations

$$A\Phi - L^2 = 0 \tag{4.9}$$

$$F\rho^2L^2 + \Phi(BC - \rho^2FA) = 0 \tag{4.10}$$

$$G\rho^2L^2 + \Phi(C^2 - \rho^2AG) = 0 \tag{4.11}$$

Case 1. $A\Phi - L^2 = 0$. From (4.5), we see that $k = LQ$ takes on either of the two following limit values

$$Q = 0 \tag{4.12a}$$

$$BLQ + 2B\bar{q}_1 + TC = 0 \tag{4.12b}$$

In the case (4.12a), it follows from (4.6), (4.3), (3.19), and (3.20) that P, s, V_j take on the limit value zero; in the case (4.12b), it follows from the above noted relations that in the limit s, Q, P, V_j do not vanish. Thus, (4.12b) leads to convential or equilibrium compressible fluid dynamics. From (4.8), (4.9), we see that for case 1, the value of c^2 is

$$'c^2 = A \tag{4.12c}$$

That is, *this wave propagates at the sonic speed.*

Case 2. $F\rho^2L^2 + \Phi(BC - \rho^2FA) = 0$. Since (4.11) cannot be valid (as the characteristics are three-dimensional), it follows from (4.7) that Q must approach infinity as a limiting value. Hence, from (4.6), (3.19), (4.3), it follows that P, s, V_j approach infinity. By solving (4.10) for $L/\Phi^{1/2}$, we find the limiting value of c^2 by means of (2.9)-(2.15)

$$''c^2 = \frac{FA - BC\rho^{-2}}{F} = \frac{\dfrac{\partial p}{\partial \rho}\dfrac{\partial^2 e}{\partial q\,\partial S} - \dfrac{\partial p}{\partial S}\dfrac{\partial^2 e}{\partial \rho\,\partial q}}{\dfrac{\partial^2 e}{\partial q\,\partial S}} \tag{4.13}$$

Case 3. $\rho^2 GL^2 + \Phi(C^2 - \rho^2 AG) = 0$. Finally, if $\bar{q}_1$ vanishes and (4.11) is valid, then (4.7) implies that Q approaches zero, as a limiting value. Hence, from (4.6), (3.19), (4.3), it follows that P, s, V_j approach zero. By solving (4.11) for $L/\Phi^{1/2}$ we find, from (2.9)-(2.15),

$$'''c^2 = \frac{AG - \rho^{-2}C^2}{G} = \frac{\dfrac{\partial p}{\partial \rho}\dfrac{\partial^2 e}{\partial q^2} - \dfrac{\partial p}{\partial q}\dfrac{\partial^2 e}{\partial \rho\, \partial q}}{\dfrac{\partial^2 e}{\partial q^2}} \tag{4.14}$$

In the case of weak reactions for $\bar{q}_1 = 0$, it is easily shown that $s = 0$ (see 3.19) and $'c$, $'''c$ are the only limiting speeds. Both of these speeds have been obtained by Stupochenko and Stakhanov [1] in the linearized case, when entropy changes are negligible.

V. The Magnetic Case

Our problem is to find the magnetic equivalent of (4.5). If we eliminate $\phi_k V^k$ in (3.17) by use of (3.16), we obtain

$$\rho L h_j = \rho \phi_k H^k V_j + PLH_j \tag{5.1}$$

Now, we eliminate h_j in (3.15) by use of (5.1) and obtain

$$\begin{aligned} \rho L V_j + (AP + Bs + CQ)\,\phi_j + \eta\Big(\frac{P}{\rho} H^2 + \frac{H^p \phi_p H^k V_k}{L}\Big)\phi_j \\ - \eta \frac{P}{\rho} H^k \phi_k H_j - \frac{\eta}{L}(H^k\phi_k)^2 V_j = 0 \end{aligned} \tag{5.2}$$

If we form the scalar product of (5.2) with ϕ^j and eliminate $\phi^j V_j$ by use of (3.16), we find

$$-PL^2 + \Big(AP + Bs + CQ + \frac{\eta}{\rho} PH^2\Big)\Phi + \frac{\eta}{L} H^j H^k \phi_j V_k \Phi = 0 \tag{5.3}$$

To determine $H^k V_k$ of (5.3), we form the scalar product of (5.2) with H^j and find

$$\rho L H^k V_k = -(AP + Bs + CQ) H^j \phi_j \tag{5.4}$$

If we eliminate $H^k V_k$ in the third term of (5.3) by use of (5.4), and if we express s in terms of Q by use of (3.19) and (3.20) in the resulting equation, we obtain the magnetic equivalent of (4.5)

$$MTP + NQ^2 + RQ = 0 \tag{5.5}$$

where $'K = H^j\phi_j$ and

$$M \equiv A\Phi\left(L^2 - \frac{\eta}{\rho}\,'K^2\right) + L^2\left(\frac{\eta}{\rho}H^2\Phi - L^2\right) \tag{5.6}$$

$$N \equiv BL\Phi\left(L^2 - \frac{\eta}{\rho}\,'K^2\right) \tag{5.7}$$

$$R \equiv (2\bar{q}_1 B + CT)\frac{N}{BL} \tag{5.8}$$

Now, we study the wave speeds. When $M = 0$, the wave speeds are those of conventional (or equilibrium) magneto-hydrodynamics [3]. If we exclude this case, then we can solve (5.5) for P and eliminate P in (4.6). We obtain the *degenerate quadratic for the magnetic case*

$$Q^2(NC - \rho^2 MFL) + Q\{RC - \rho^2 M(2\bar{q}_1 F + TG)\} = 0 \tag{5.9}$$

Thus, the limiting speeds for the magnetic case correspond to

$$M = 0 \tag{5.10}$$

$$NC - \rho^2 MFL = 0 \tag{5.11}$$

$$RC - \rho^2 M(2\bar{q}_1 F + TG) = 0 \tag{5.12}$$

Case 1. $M = 0$. As we have noted, $M = 0$ furnishes the limiting speeds of conventional magneto-hydrodynamics. From (5.5), we see that in this case the limiting values of Q are: $Q = 0$ or $Q = -R/N$. In the first case (i.e. $Q = 0$), it follows from (3.19), (3.20), (3.23) that $P = s = 0$, and hence from (5.4)

$$H^k V_k = 0 \tag{5.13}$$

Further, from (3.16), (3.17) we see that

$$\phi_j V^j = 0 \tag{5.14}$$

$$Lh_j = \phi_k H^k V_j \tag{5.15}$$

From (3.15), (5.15), we find that (noting that $'K = H^j\phi_j$)

$$\left(\rho L - \frac{\eta' K^2}{L}\right) V_j = -\eta H^k h_k \phi_j \tag{5.16}$$

Forming the scalar product of (5.16) with V^j and using (5.14), we see that *either V_j vanishes or*

$$L^2 = \frac{\eta}{\rho}\,'K^2 \tag{5.17}$$

By substituting (5.6) into (5.10), and then eliminating L^2 by use of (5.17) in the resulting equation, we find that H_j *lies along* ϕ_j. Thus, the case $Q = 0$ implies: (1) $P = s = 0$, $V_j = 0$, and, by (3.17), $h_j = 0$; or (2) $P = s = 0$, the field ϕ_j lies along H^j, and from (5.14), (3.17), h_j, V_j are both tangent to $\phi =$ constant. From (5.17), we see that this last wave moves with the Alfvén speed [3]. The case $Q = -R/N$ leads to the general case of equilibrium magneto-hydrodynamics.

Case 2. $NC - \rho^2 MFL = 0$. Here, the limiting speeds are obtained from (5.11). Substituting (5.6), (5.7) into (5.11), we obtain the following equation for c^2 by using (4.8), after dividing by $\Phi^{5/2}$

$$Fc^4 + c^2\left(\frac{CB}{\rho^2} - FA - \frac{\eta}{\rho}FH^2\right) + \frac{\eta}{\rho}\left(FA - \frac{CB}{\rho^2}\right)\bar{K}^2 = 0 \tag{5.18}$$

Note, the quantity

$$\bar{K} = \frac{'K}{\Phi^{1/2}} \tag{5.19}$$

is the projection of H^j on the space normalized unit normal to $\phi =$ constant. For either of the *limiting speeds determined by (5.18),* P, Q, s, V_j, h_j *approach infinity*, as is easily verified by use of (5.5), (3.16), (3.17), (3.19), (3.20). Further, by comparing (4.13), (5.18), we see that the *single limiting speed of the nonmagnetic case is "split" into two magnetic limiting speeds whose squares, when summed, are greater than the nonmagnetic limit speed squared by* [*see (2.23)*].

$$\frac{\eta}{\rho}H^2 \tag{5.20}$$

It should be noted that in case $H_j = 0$, then $\bar{K} = H = 0$ and (5.18) reduces to (4.13).

Case 3. $RC - \rho^2 M(2\bar{q}_1 F + TG) = 0$. In this last case, the limiting speeds are given by (5.12). Substituting (5.6), (5.8) into (5.12) and dividing the resulting equation by Φ^2, we obtain, by using (4.8),

$$\begin{aligned}&(2\bar{q}_1 F + TG)\,c^4\\ &+\left\{-2\bar{q}_1\left(-\frac{BC}{\rho^2} + FA + \frac{\eta}{\rho}H^2F\right) - TG\left(A + \frac{\eta}{\rho}H^2\right) + \frac{C^2T}{\rho^2}\right\}c^2\\ &+\left\{-2\bar{q}_1\frac{\eta}{\rho}\bar{K}^2\left(\frac{BC}{\rho^2} - AF\right) - \frac{\eta}{\rho}\bar{K}^2T\left(\frac{C^2}{\rho^2} - AG\right)\right\} = 0\end{aligned} \tag{5.21}$$

For the case when $\bar{q}_1 = 0$, (5.21) becomes

$$Gc^4 + \left(-AG + \frac{C^2}{\rho^2} - \frac{\eta}{\rho}H^2G\right)c^2 - \frac{\eta}{\rho}\bar{K}^2\left(-AG + \frac{C^2}{\rho^2}\right) = 0 \tag{5.22}$$

For either of the limiting speeds determined by (5.21) or (5.22), P, Q, s, V_j, h_j *approach zero.* This follows from (5.5), (3.16), (3.17), (3.19), (3.20). Further, by comparing (4.14), (5.22), we see that for $\bar{q}_1 = 0$, the *single limiting speed of the non-magnetic case is "split" into two magnetic limiting speeds such that the sum of their squares is greater than the square of the nonmagnetic limiting speed by (5.20).* Since (5.18), (5.22) contain $\bar{K}$, it would be more precise to talk about *classes of speed.*

VI. Concluding Remarks

Three questions remain to be answered. These are:

(1) Since the original equations (2.17)-(2.22) consist of nine equations (note, $\nabla_j H^j = 0$ follows from the other equations and properly given initial data [3]) in nine dependent variables, why is the discontinuity system for the jumps or their ratios underdetermined (and hence no unique solution exists for the associated Cauchy problem)?

(2) What is the significance of the assumption ($\nabla_\alpha \bar{q}$ is continuous along a characteristic S_3) which leads to a determined discontinuity system?

(3) What is the significance of each of the limiting speeds?

The first question is easily answered. If $\partial e/\partial q$ is a *known* function of ρ, S, q, and hence of x^k, t along the initial S_3 then the system (2.17)-(2.22) is determined in the sense that the Cauchy problem has a unique solution. Similarly, if $\partial e/\partial q$ is a *continuous* function of x^k, t along any characteristic manifold then the ratio of every pair of discontinuities is determined. However, in this case, the characteristic manifolds coincide with those of equilibrium compressible nonmagnetic (and magnetic) hydrodynamics. In our work, we eliminated $\partial e/\partial q$ by differentiating the first equation of (2.4). But, the resulting equation (2.16a) contains the derivative $d\bar{q}/dr$. For the Cauchy (or discontinuity) problem, the tangential derivatives

$$\underset{a}{t^\alpha}\nabla_\alpha \rho, \qquad \underset{a}{t^\alpha}\nabla_\alpha S, \qquad \underset{a}{t^\alpha}\nabla_\alpha q$$

are known (continuous). Thus, (2.16a) can be used to determine the tangential derivative, $\underset{a}{t^\alpha}\nabla_\alpha \bar{q}$. However, (2.16a) relates the normal derivative $\phi^\alpha \nabla_\alpha \bar{q}$ to the unknown (discontinuous) normal derivatives

$$\phi^\alpha \nabla_\alpha \rho, \qquad \phi^\alpha \nabla_\alpha S, \qquad \phi^\alpha \nabla_\alpha q$$

Thus, one of the energy equations introduces a new equation and a new possible unknown (or possible discontinuity). The *assumption in our work is that* $\phi^\alpha \nabla_\alpha \bar{q}$ *is known (continuous)* and this completes our system.

In order to answer the second question (what is the significance of the assumption $\nabla_\alpha \bar{q}$ is continuous?), we can use the general theory developed by Thomas [6]. We shall use a slightly modified form of this theory which is more suitable to our special problem. From the definition (2.16b), we find

$$\nabla_\alpha \bar{q} = \nabla_\alpha \nabla_0 q + (\nabla_\alpha v^j)(\nabla_j q) + v^j \nabla_\alpha \nabla_j q \tag{6.1}$$

Let us expand $[\nabla_\alpha \nabla_\beta q]$ in terms of the four-tuple of *unit* orthogonal vectors, $\underset{a}{t}^\alpha$ ($a = 1, 2, 3$) of Section III and $'\phi_\alpha$ where (see (3.2b))

$$'\phi_\alpha = \frac{\phi_\alpha}{'\Phi^{1/2}} \tag{6.2}$$

We obtain

$$[\nabla_\alpha \nabla_\beta q] = \sum_{a,b} \underset{ab}{A}\, \underset{a}{t}_\alpha\, \underset{b}{t}_\beta + \sum_a \underset{a}{C}\, \underset{a}{t}_\alpha\, '\phi_\beta + B\, '\phi_\alpha\, '\phi_\beta + \sum_a \underset{a}{D}\, '\phi_\alpha\, \underset{a}{t}_\beta$$

where

$$\begin{aligned}
\underset{ab}{A} &= \underset{a}{t}^\alpha\, \underset{b}{t}^\beta [\nabla_\alpha \nabla_\beta q] \\
\underset{a}{C} &= \underset{a}{t}^\alpha\, '\phi^\beta [\nabla_\alpha \nabla_\beta q] \\
B &= '\phi^\alpha\, '\phi^\beta [\nabla_\alpha \nabla_\beta q] \\
\underset{a}{D} &= '\phi^\alpha\, \underset{a}{t}^\beta [\nabla_\alpha \nabla_\beta q]
\end{aligned} \tag{6.4}$$

Hadamard's theorem [5] states that if "right and left hand derivatives" exist, then the jump of any tangential derivative of any quantity (scalar, etc.) is equal to that tangential derivative of the jump of this quantity. Applying this theorem to $\nabla_\beta q$, we find, by using (3.8), (6.2),

$$\underset{a}{t}^\alpha [\nabla_\alpha \nabla_\beta q] = \underset{a}{t}^\alpha \nabla_\alpha ('\phi_\beta\, 'Q) \tag{6.5}$$

where $'Q = Q\, '\Phi^{1/2}$. From a well known result in differential geometry (cf. [7, p. 246, eq. (5.9)]), it follows that

$$\nabla_\alpha\, '\phi_\beta = h_{\alpha\beta} + '\phi_\alpha u_\beta \tag{6.6}$$

where $h_{\alpha\beta}$ is the known (or continuous) symmetric second fundamental tensor of S_3, which lies in S_3 (i.e., $'\phi^\alpha h_{\alpha\beta} = 0$), and u_β is the known (or continuous) curvature vector of the $'\phi_\alpha$ congruence. Substituting (6.6) into (6.5), we find, by forming scalar products of (6.5) with $'\underset{b}{t}^\beta$ and $'\phi^\beta$ and using (6.4),

$$\underset{ab}{A} = \underset{a}{t}^\alpha\, \underset{b}{t}^\beta [\nabla_\alpha \nabla_\beta q] = 'Q h_{\alpha\beta}\, \underset{a}{t}^\alpha\, \underset{b}{t}^\beta \tag{6.7}$$

$$\underset{a}{C} = \underset{a}{t}^\alpha\, '\phi^\beta [\nabla_\alpha \nabla_\beta q] = \underset{a}{t}^\alpha \nabla_\alpha\, 'Q \tag{6.8}$$

Further, since in Euclidean space with coordinates x^j, t, the operator relation $\nabla_\alpha \nabla_\beta = \nabla_\beta \nabla_\alpha$ is valid, it follows from (6.4) that

$$\underset{a}{C} = \underset{a}{D} \tag{6.9}$$

Finally, we note a well known relation (cf. [8, p. 96])

$$\sum_a \underset{a}{t}^\lambda \underset{a}{t}_\alpha = \delta^\lambda_\alpha - '\phi^\lambda\, '\phi_\alpha \tag{6.10}$$

Substituting (6.7)-(6.9) into (6.3), we find, by using (6.10) and the fact that $'\phi^\alpha$ lies in the null domain of $h_{\lambda\mu}$,

$$[\nabla_\alpha \nabla_\beta q] = 'Q h_{\alpha\beta} + '\phi_\alpha \nabla_\beta\, 'Q + '\phi_\beta \nabla_\alpha\, 'Q + (B - 2'\phi^\lambda \nabla_\lambda\, 'Q)\, '\phi_\alpha '\phi_\beta \tag{6.11}$$

Forming the jump of (6.1), we find by using (3.4), (3.8), and (6.11) and expanding the product jump $[(\nabla_\alpha v^j)(\nabla_j q)]$ (cf. [2, p. 384])

$$\begin{aligned}[\nabla_\alpha \bar{q}] = {}& 'Q(h_{0\alpha} + v^j h_{j\alpha}) + 'L \nabla_\alpha\, 'Q + '\phi_\alpha v^j \nabla_j\, 'Q \\ & + (B - 2'\phi^\lambda \nabla_\lambda\, 'Q)\, 'L\, '\phi_\alpha + (\nabla_j q)_1\, 'V^j\, '\phi_\alpha \\ & + (\nabla_\alpha v^j)_1\, 'Q\, '\phi_j + ('\phi_j 'V^j)\, 'Q\, '\phi_\alpha + '\phi_\alpha \nabla_0\, 'Q \end{aligned} \tag{6.12}$$

where $'V^j = V^j{}'\Phi^{1/2}$, and $'L = '\phi_0 + '\phi_j v^j$ and the subscript indicates the value of $\nabla_j q$, $\nabla_\alpha v^j$ on one side of S_3. First, it should be noted from (3.11a) that $\underset{a}{t}^\alpha[\nabla_\alpha \bar{q}]$ must vanish identically. This can be verified by use of (6.6) and (3.11b). Thus, the nonvanishing of (6.12) leads to only one condition

$$\begin{aligned}'L(B - '\phi^\alpha \nabla_\alpha\, 'Q) + v^j \nabla_j\, 'Q + \nabla_0\, 'Q + 'Q\, '\phi_j\, 'V^j & \\ + '\phi_j\, '\phi^\alpha (\nabla_\alpha v^j)_1\, 'Q + 'V^j (\nabla_j q)_1 = \phi^\alpha [\nabla_\alpha \bar{q}] & \end{aligned} \tag{6.13}$$

Equation (6.13) determines B (see (6.3)) when $'Q$, $'V^j$ etc. are known. Note, if $\nabla_\alpha 'Q$ is discontinuous, then one must replace $\nabla_\alpha 'Q$ in (6.13) by $(\nabla_\alpha 'Q)_1$. We note that if $'Q$, $'V^j$ vanish, but $\phi^\alpha[\nabla_\alpha \bar{q}]$ *does not vanish*, then B *does not vanish* and by (6.11), $[\nabla_\alpha \nabla_\beta q]$ *does not vanish.* Thus, if v^j, q, and their first partials with respect to x^j, t are continuous [these are the only quantities, except for $\partial e/\partial q$, that occur in the basic relations (2.17)-(2.23)] then the *second derivatives of* q with respect to x^j, t will be continuous if and only if *the first normal derivative of* $\bar{q}$ or $\partial e/\partial q$ is continuous with respect to x^j, t. Thus, the *assumption* that $[\nabla_\alpha \bar{q}]\, \phi^\alpha$ *vanishes is a condition that will assure the continuity of all second derivatives of* q *when* q, v^j *and their first partials are continuous.* Note that the *second partials of* q enter the basic equations when (2.22) is replaced by (2.16a).

Our final question was: what is the significance of the limiting speeds?

In equilibrium hydrodynamics, the *characteristic manifolds are independent of the jumps* V_j, h_j, P, Q, s [see (3.4)-(3.9)]. However, in relaxation hydrodynamics, the characteristic manifolds depend on Q [see (4.7), (5.9)], except for the special cases of equilibrium hydrodynamics (see case 1 for the nonmagnetic and magnetic cases). The relations (4.7), (5.9) are of the form

$$\bar{A}Q^2 + \bar{B}Q = 0 \tag{6.14}$$

Hence, *for every finite value of* Q *except* $Q = 0$, the relation (6.14) determines a class of manifolds along which Q has this finite value. From (6.14), we see that for $Q \neq 0$,

$$\bar{A}Q + \bar{B} = 0 \tag{6.15}$$

But as $\bar{B}$ approaches zero, Q also approaches zero. Again, we note $\bar{B} = 0$, $\bar{A} = 0$ lead to two-dimensional manifolds. Similarly, from (6.15), we see that as $\bar{A}$ approaches zero, Q approaches infinity.

Appendix

By comparing the relations for the limiting speeds (4.12c), (4.13), (4.14) of the nonmagnetic case with the corresponding limiting speeds of the magnetic case [see (5.6) with $M = 0$, $L^2/\Phi = c^2$, (5.18), (5.22)], we see that if the nonmagnetic limiting speed is determined by

$$c^2 = E^2 \tag{1}$$

then the corresponding magnetic limiting speeds are the solutions of the biquadratic

$$c^4 - \left(E^2 + \frac{\eta}{\beta} H^2\right) c^2 + \frac{\eta}{\rho} (\bar{K}E)^2 = 0, \tag{2}$$

where H is the magnitude of the magnetic field [see (2.23)] and $\bar{K}$ is the component of the magnetic field which lies along the normal to the characteristic wave [see (5.19)]. Further, we note that the *same relation for* $k = LQ$, where k is constant along a characteristic wave and Q is determined by (3.8) or (3.9), *occurs in the corresponding nonmagnetic and magnetic cases.* Thus, case 1 (equilibrium hydrodynamics) is characterized by $k = 0$ and $Bk + TC = 0$ [see (4.12a), (4.12b) for $\bar{q}_1 = 0$ and case 1 of Section V]. Similarly, we see that cases 2 and 3 are characterized by the conditions that $\bar{q}_1$ vanishes and k approaches infinity and zero, respectively.

Now, we shall extend the above results for limiting speeds to the case of *any permissible speed in the two ranges* which are bounded by the limiting speeds for which k approaches zero or infinity. We shall show: *if* k, T, G, F

[see (2.7), (2.14), (2.15)] *have the same nonzero and finite values in the non-magnetic and magnetic cases and $\bar{q}_1$ vanishes in both cases then corresponding to every nonmagnetic speed E, there exists two magnetic speeds, the solutions of (2) when H, $\bar{K}$ are specified.* Assuming F, G, k do not vanish and dividing (4.7) by $\rho^2 FGQ\Phi$, we obtain by using (3.11b), (4.8), (4.13), (4.14) when $\bar{q}_1$ vanishes

$$\frac{k}{G}(c^2 - {}''c^2) + \frac{T}{F}(c^2 - {}'''c^2) = 0 \tag{3}$$

Similarly, if we factor L out of the first term of the left hand side of (5.9) then divide this relation by $\rho^2 FGQ\Phi^2$, we obtain by use of (3.11b), (4.8), (5.6)-(5.8), (4.13), (4.14) when $\bar{q}_1$ vanishes

$$\begin{aligned} &\frac{k}{G}\left\{c^4 - c^2\left({}''c^2 + \frac{\eta}{\rho}H^2\right) + \frac{\eta}{\rho}({}''c\bar{K})^2\right\} \\ &+ \frac{T}{F}\left\{c^4 - c^2\left({}'''c^2 + \frac{\eta}{\rho}H^2\right) + \frac{\eta}{\rho}({}'''c\bar{K})^2\right\} = 0 \end{aligned} \tag{4}$$

By our assumption, the values of k, T, F, G in (3) and (4) are identical. If we eliminate k/G in (4) by use of (3), with c of (3) replaced by E, we obtain (2) after a direct but lengthy algebraic calculation.

References

1. Stupochenko, E. V., and Stakhanov, I. P. The equations of relaxation hydrodynamics. *Soviet Phys.-Doklady.* **4**, 782-785 (1960).
2. Coburn, N. Discontinuity relations for changed, compressible, relativistic fluids. *J. Math. and Mech.* **3**, 361-392 (1961).
3. Friedrichs, K. O., and Kranzer, H. Non-linear wave motion. A.E.C. Research and Development Report, NYO-6486 (July, 1958).
4. Coburn, N. "Vector and Tensor Analysis." Macmillan, New York, 1955.
5. Hadamard, J. "Leçons sur la propagation des ondes." Hermann, Paris, 1903.
6. Thomas, T. Y. Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics. *J. Math. and Mech.* **3**, 311-322 (1957).
7. Schouten, J. A. "Ricci-Calculus," 2nd ed. Springer, Berlin, 1954.
8. Eisenhart, L. P. "Riemannian Geometry." Princeton Univ. Press, 1926.