On Some Fixed Point Theorems on Uniformly Convex Banach Spaces

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We introduce a notion of T-regularity to generalize a well known fixed point theorem of Browder. We also give some related results in this direction. © 1992 Academic Press, Inc.

We introduce the following definition.

**Definition 1.1.** Let $X$ be a vector space and $A$ be a subset of $X$. $A$ is said to be a $T$-regular set if and only if

(i) $T: A \to A$

(ii) $(x + Tx)/2 \in A$, for each $x$ in $A$.

**Remark 1.1.** Obviously every convex set invariant under a map $T$ is a $T$-regular set. But a $T$-regular set need not be a convex set.

**Example 1.1.** Let $X$ be a nonzero vector space and $x, y \in X$, $x \neq y$. Let

$$z = \frac{x + y}{2}, \quad A = \{x, y, z\}.$$ 

Define $T: A \to A$ as $Tx = y$, $Ty = x$, and $Tz = z$. Then $A$ is a $T$-regular set.

**Some Properties of $T$-Regular Sets:**

(i) Let $X$ be a vector space and $\{A_x\}_{x \in I}$ be a collection of $T$-regular subsets of $X$. Then

$$\bigcap_{x \in I} A_x \quad \text{and} \quad \bigcup_{x \in I} A_x$$

are $T$-regular sets.
(ii) Let $X$ be a vector space and $T: X \to X$ be a linear transformation. Suppose $A, B$ are $T$-regular subsets of $X$. Then $T(A)$, $A + B$ are $T$-regular sets.

(iii) Let $X$ be a topological vector space and $T: X \to X$ be a continuous map. Suppose $A$ is a $T$-regular subset of $X$. Then $\overline{A}$, the closure of $A$, is also a $T$-regular set.

(iv) Let $X$ be a uniformly convex Banach space and $F$ be a bounded $T$-regular subset of $X$. Then either $Tx = x$, for all $x$ in $F$ or there exists $x_0$ in $F$ such that

$$\text{Sup}\{\|x - x_0\| : x \in F\} = \delta(x_0, F) < \delta(F), \quad \text{the diameter of } F.$$

**Proof.** Parts (i), (ii), and (iii) are easy to prove. We give below the proof of (iv).

Suppose for some $x \in F$, $x \neq Tx$. For any $y \in F$, $\|y - Tx\| \leq \delta(F)$, $\|y - x\| \leq \delta(F)$.

Let $x_0 = (x + Tx)/2$.

As $F$ is a $T$-regular set $Tx \in F$ and $x_0 \in F$.

The uniform convexity of the space implies the existence of a number $\alpha$, $0 < \alpha < 1$ such that

$$\|x_0 - y\| \leq \alpha \delta(F)$$

implies $\delta(x_0, F) \leq \alpha \delta(F)$. □

**Remark 1.2.** Using some properties of $T$-regular sets we prove the following theorem.

**Theorem 1.1.** Let $K$ be a nonempty weakly compact $T$-regular subset of a uniformly convex Banach space $X$. Further for each weakly closed $T$-regular subset $F$ of $K$, with $\delta(F) > 0$, there exists some $\beta(F)$, $0 < \beta(F) < 1$, such that

$$\|Tx - Ty\| \leq \max\{\|x - y\|, \beta \delta(F)\},$$

for all $x, y$ in $F$.

Then $T$ has a fixed point in $K$.

**Proof.** Let $H$ be the collection of all nonempty weakly closed, $T$-regular subsets of $K$. Because of property (i) one can use Zorn's Lemma to get a minimal element say $F$ of $H$.

Suppose for some $x$ in $F$, $x \neq Tx$. $F$ is a bounded $T$-regular set implies there exists $x_0$ in $F$ and $\alpha$, $0 < \alpha < 1$, such that

$$\delta(x_0, F) \leq \alpha \delta(F) \quad (\text{by property (iv)}).$$
By hypothesis there exists $\beta, 0 < \beta < 1,$ such that
$$\|Tx - Ty\| \leq \beta \delta(F).$$

Let
$$x_0 = \max\{x, \beta\}$$
$$E_0 = \{x \in X: \delta(x, F) \leq x_0 \delta(F)\}$$
$$F_0 = E_0 \cap F.$$

$x_0 \in E_0 \cap F$ implies $F_0$ is nonempty, and $E_0, F$ are weakly closed sets implies $F_0$ is weakly closed.

Let $x \in F_0, \|Tx - Ty\| \leq x_0 \delta(F)$, for all $y \in F$.

Hence $T(F)$ is contained in a closed ball $U$ of centre $Tx$ and radius $x_0 \delta(F).$ This implies $T(F \cap U) \subset F \cap U.$ Now $F$ is a $T$-regular set, $U$ is a convex set implies $F \cap U$ is a $T$-regular set. Hence by the minimality of $F$, $T(F_0) \subset F_0$. Also $F_0$ is a $T$-regular set. Hence $F_0 \in H.$ But $\delta(F_0) < \delta(F)$, a contradiction. \[\]

**Corollary 1.2.** Let $K$ be a nonempty weakly compact $T$-regular subset of a uniformly convex Banach space and $T: K \to K$ a nonexpansive map. Then $T$ has a fixed point.

**Corollary 1.3** [1]. Let $K$ be a nonempty weakly compact convex subset of a uniformly convex Banach space and $T: K \to K$ a nonexpansive map.

Then $T$ has a fixed point.

**Definition 1.2** [3]. If $D$ is a subset of a Banach space $X$, $T$ is a mapping from $D$ into $X$, and $x_1 \in D$, then $M(x_1, t_n, T)$ is the sequence $\{x_n\}_{n=1}^{\infty}$ defined by
$$x_{n+1} = (1 - t_n) x_n + t_n Tx_n,$$

where $\{t_n\}_{n=1}^{\infty}$ is a real sequence.

If a point $x_1$ and a sequence $\{t_n\}_{n=1}^{\infty}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} t_n = \infty$.

(ii) $0 \leq t_n \leq b < 1$, for all positive integers $n$, and $x_n \in D$ for all positive integers $n$, then $x_1$ and $\{t_n\}_{n=1}^{\infty}$ will be said to satisfy Condition A.

**Definition 1.3** [3]. Let $D$ be a subset of a Banach space $X$. A mapping $T: D \to X$ with a nonempty fixed points set $F$ in $D$ will be said to satisfy Condition B if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with
ON SOME FIXED POINT THEOREMS

\[ f(0) = 0, \ f(r) > 0 \ \text{for} \ r \in (0, \infty) \ \text{such that} \ \|x - Tx\| \geq f(d(x, F)), \ \text{for all} \ x \in D, \ \text{where} \ d(x, F) = \inf\{\|x - z\| : z \in F\}. \]

**Lemma 1.4 [3].** Let \( D \) be a subset of a Banach space \( X \) and \( T \) be a nonexpansive mapping from \( D \) into \( X \). If there exist \( x_1 \) and \( \{t_n\}_{n=1}^{\infty} \) that satisfy Condition A and \( M(x_1, t_n, T) \) is bounded then \( x_n - Tx_n \) converges to zero as \( n \to \infty \).

Using Lemma 1.4, Ishikawa [3] has proved the following theorem.

**Theorem 1.5 [3].** Let \( D \) be a closed subset of a Banach space \( X \) and \( T: D \to X \) be a nonexpansive mapping with a nonempty fixed points set \( F \) in \( D \). If \( T \) satisfies Condition B and there exist \( x_1 \) and \( \{t_n\}_{n=1}^{\infty} \) that satisfy Condition A, then \( M(x_1, t_n, T) \) converges to a member of \( F \).

The following Theorem 1.6 is a generalization of a result of Outlaw [5].

**Theorem 1.6.** Let \( K \) be a nonempty weakly compact \( T \)-regular subset of a uniformly convex Banach space \( X \). Let \( T: K \to K \) be a nonexpansive map satisfying condition B. Then \( M(x_1, \frac{1}{2}, T) \) converges to a fixed point of \( T \) for any \( x_1 \) in \( K \).

**Proof.** By Corollary 1.2., \( F = \{x \in K: Tx = x\} \) is nonempty. Hence the result follows from Theorem 1.5. \( \square \)

Gillespie and Williams [2] have given the following.

**Theorem 1.7 [2].** Let \( K \) be a closed, bounded, convex subset of a Banach space \( X \) and \( T: K \to K \) be a map satisfying:

1. \( \|Tx - Ty\| \leq \|x - y\| \), for all \( x, y \in K; \)
2. For some \( \alpha > 0 \)
   \[ \|Tx - Ty\| \leq \alpha(\|x - Tx\| + \|y - Ty\|), \quad \text{for} \ x, y \in K. \]

Then \( T \) has a unique fixed point.

The following generalization of Theorem 1.7 is immediate by an application of Lemma 1.4.

**Theorem 1.8.** Let \( K \) be a closed, bounded subset of a Banach space \( X \) and \( T: K \to K \) be a map satisfying

1. \( \|Tx - Ty\| \leq \|x - y\| \), for \( x, y \in K; \)
2. For some \( \alpha > 0 \),
   \[ \|Tx - Ty\| \leq \alpha(\|x - Tx\| + \|y - Ty\|), \quad \text{for} \ x, y \in K. \]
Further suppose there exist $x_1$ and $t_n$ that satisfy Condition A, then $M(x_1, t_n, T)$ converges to the unique fixed point of $T$.

**Corollary 1.9.** Let $K$ be a closed, bounded, $T$-regular subset of a Banach space $X$ and $T: K \rightarrow K$ be a map satisfying

(i) $\|Tx - Ty\| \leq \|x - y\|$, for $x, y \in K$,

(ii) for some $\alpha > 0$,

$$\|Tx - Ty\| \leq \alpha(\|x - Tx\| + y - Ty\|), \text{ for } x, y \in K.$$ 

Then for each $x_1 \in K$, $M(x_1, \frac{1}{T}, T)$ converges to the unique fixed point of $T$.

**Corollary 1.10.** Let $K$ be a weakly compact convex subset of a Banach space $X$ and $T: K \rightarrow K$ be a nonexpansive affine map. Then $T$ has a fixed point in $K$.

**Proof.** Let $\alpha_0 = \inf\{\|x - Tx\| + \|y - Ty\| : x, y \in K\}$. If $\alpha_0 > 0$ then there exists an $\alpha > 0$, take $\alpha \geq \bar{\delta}(K)/\alpha_0$, such that

$$\|Tx - Ty\| \leq \alpha(\|x - Tx\| + \|y - Ty\|), \text{ for } x, y \in K.$$ 

Suppose $\alpha_0 = 0$.

$$(x, y) \rightarrow \|x - Tx\| + \|y - Ty\|$$ 

is a continuous convex function, $K$ is a weakly compact set implies

$$\|x - Tx\| + \|y - Ty\| = 0,$$

for some $(x, y) \in K \times K$. 

We use Corollary 1.2 to prove the following results in optimization and approximation theory.

**Theorem 2.1.** Let $K$ be a nonempty weakly compact $T$-regular subset of a uniformly convex Banach space $X$. Let

$$T: K \rightarrow K$$

$$h: X \rightarrow \mathbb{R}, \text{ the set of reals, satisfy}$$

(i) $T$ is nonexpansive

(ii) $h$ is a lower semicontinuous (l.s.c.) convex function

(iii) $h \cdot T \leq h$ on $K$.

Then there exists $x_0$ in $K$ such that $Tx_0 = x_0$ and $h(x_0) = \inf\{h(x) : x \in K\}$.
Proof. Let \( A = \{ x \in K : h(x) = \inf_{y \in K} h(y) \} \). \( h \) is a l.s.c. convex function implies \( h \) is weakly l.s.c. Now \( K \) is a weakly compact set implies \( A \neq \emptyset \). Also \( \{ x \in X : h(x) \leq \inf_{y \in K} h(y) \} \) is a weakly closed set implies

\[
A = \{ x \in X : h(x) = \inf_{y \in K} h(y) \} \cap K
\]

is a weakly closed set.

Now \( h \cdot T \leq h \) implies \( T(A) \subset A \).

For \( x \in A \),

\[
\frac{h\left(\frac{x + T x}{2}\right)}{2} \leq \frac{h(x)}{2} + \frac{h(T x)}{2}
\]

implies

\[
\frac{h(x + T x)}{2} = \inf_{y \in K} h(y)
\]

implies

\[
\frac{x + T x}{2} \in A.
\]

Hence \( A \) is a nonempty weakly compact \( T \)-regular subset of the uniformly convex Banach space \( X \).

Therefore by Corollary 1.2 there exists \( x_0 \) in \( A \) such that

\[
Tx_0 = x_0.
\]

**COROLLARY 2.2.** Let \( K \) be a nonempty weakly compact \( T \)-regular subset of a uniformly convex Banach space \( X \) and \( y_0 \in X \setminus K \). Suppose

1. \( \| Tx - y_0 \| \leq \| x - y_0 \| \), for \( x \) in \( K \);
2. \( \| Tx - Ty \| \leq \| x - y \| \), for \( x, y \) in \( K \).

Then \( T \) has a fixed point \( x_0 \) in \( K \) which is a best approximation to \( y_0 \) from \( K \).

**Proof.** Define \( h : X \to \mathbb{R} \) as \( h(x) = \| x - y_0 \| \).

We give the following simple example to justify the above result.

**EXAMPLE 2.1.** Let \( K = [-2, -1] \cup [1, 2] \). Define \( T : K \to K \) as

\[
Tx = \begin{cases} 
-1, & \text{for } x \in [-2, -1] \\
1, & \text{for } x \in [1, 2].
\end{cases}
\]

Then all the conditions of Corollary 2.2 are satisfied.
Here $-1, 1 \in K$ are such that

(i) $T(-1) = -1, T(1) = 1$ and

(ii) $-1, 1$ are both best approximations to 0 from $K.$

REFERENCES