Pseudomonotone and Copositive Star Matrices

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ABSTRACT

We study general and complementarity properties of matrices which are either pseudomonotone or copositive star on a closed convex cone. In particular we show that if a matrix is pseudomonotone on the nonnegative orthant of $\mathbb{R}^n$, then it belongs to $P_0 \cap Q_0$. We also show that if a matrix $T$ is copositive star on a closed convex cone $K$, then the linear complementarity problem LCP$(T, K, q)$ is solvable for all $q$ in $\mathbb{R}^n$ iff zero is the only solution of the (homogeneous) problem LCP$(T, K, 0)$. We characterize pseudomonotonicity of a matrix $T$ by the nonnegativity of the quadratic form $\langle Tx, x \rangle$ on a certain set.

1. INTRODUCTION

In this article, we establish certain complementarity properties of matrices which are either pseudomonotone or copositive star on a cone. Given a closed convex cone $K$ (i.e., $K$ is a closed set satisfying $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda \geq 0$) and a matrix $T \in \mathbb{R}^{n \times n}$, we say that $T$ is pseudomonotone on $K$ if

$$x, y \in K, \quad \langle Tx, y - x \rangle \geq 0 \quad \Rightarrow \quad \langle Ty, y - x \rangle \geq 0. \quad (1.1)$$

Pseudomonotonicity is a natural generalization of monotonicity, which is defined by the condition

$$x, y \in K \quad \Rightarrow \quad \langle Ty - Tx, y - x \rangle \geq 0. \quad (1.2)$$


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We show that a matrix pseudomonotone on $K$ has the copositive star property:

$$\langle Tx, x \rangle \geq 0 \quad \forall \ x \in K,$$  \hspace{1cm} (1.3a)

and

$$x \in K, \ Tx \in K^*, \ \langle Tx, x \rangle = 0 \implies x \in - [T(K)]^*, \hspace{1cm} (1.3b)$$

where, for any set $E$ in $\mathbb{R}^n$, the polar of $E$ is defined by

$$E^* := \{ y \in \mathbb{R}^n : \langle y, z \rangle > 0 \ \forall \ z \in E \}.$$

As we shall see, matrices satisfying the copositive star property have nice complementarity properties. Corresponding to a matrix $T \in \mathbb{R}^{n \times n}$, a closed convex cone $K \subseteq \mathbb{R}^n$, and a vector $q \in \mathbb{R}^n$, the linear complementarity problem, LCP($T, K, q$), is to find an $x$ such that

$$x \in K, \ Tx + q \in K^* \hspace{1cm} (1.4a)$$

and

$$\langle Tx + q, x \rangle = 0. \hspace{1cm} (1.4b)$$

We say that LCP($T, K, q$) is feasible if there is an $x$ satisfying (1.4a) [and solvable if (1.4) is satisfied for some $x$].

Here are our main results:

(A) If $T$ is pseudomonotone on $\mathbb{R}^n_+$, then $T \in P_0 \cap Q_0$, i.e., every principal minor of $T$ is nonnegative, and for any $q$, the feasibility of LCP($T, \mathbb{R}^n_+, q$) implies its solvability.

(B) If $T$ is copositive star on a polyhedral cone $K$, then for any $q$, the feasibility of LCP($T, K, q$) implies its solvability.

(C) If $T$ is copositive star on $K$ (which is an arbitrary closed convex cone), then LCP($T, K, q$) is solvable for all $q$ iff \{ $x : x \in K, \ Tx \in K^*, \ \langle Tx, x \rangle = 0$ \} = \{ 0 \}.

Result (B) can be regarded as a generalization of the classical result of Lemke [11] that for a matrix copositive plus on $\mathbb{R}^n_+$, feasibility of the LCP implies its solvability. In the classical situation (i.e., when $K = \mathbb{R}^n_+$), result (C) can be deduced from a result of Pang [14].
The conditions (1.1)–(1.4) make sense (even) for nonlinear mappings. In this regard, a generalization of result (C), stated for positive homogeneous mappings, appears in [7].

Apart from the main results, we prove several minor ones. For example, we show that a matrix copositive star on $\mathbb{R}^n_+$ belongs to the class $L$ of Eaves [4]. We characterize matrices pseudomonotone on $\mathbb{R}^n_+$ by the nonnegativity of their quadratic forms on certain sets. Another result we prove is that a symmetric matrix pseudomonotone on a convex cone is monotone. This shows that on a convex cone, a quadratic form $\langle Tx, x \rangle$ is convex iff it is pseudoconvex.

This article is organized as follows. After dealing with preliminaries in Section 2, we describe some general properties of pseudomonotone and copositive star matrices in Section 3. We characterize matrices pseudomonotone on $\mathbb{R}^n_+$ in Section 4. Section 5 deals with complementarity results, and in Section 6 we give examples.

2. PRELIMINARIES

In the Euclidean space $\mathbb{R}^n$, we denote the usual inner product between $n$-vectors $x$ and $y$ by $\langle x, y \rangle$. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$. For $x \in \mathbb{R}^n_+$ we write $x \geq 0$. For a $\lambda \in \mathbb{R}$, $\lambda^+ := \max\{\lambda, 0\}$ and $\lambda^- := \max\{0, -\lambda\}$. For any $z = (z_1, \ldots, z_n)$ we write $z^+ = (z_1^+, \ldots, z_n^+)$ and $z^- = (-z)^+$. We note that $z = z^+ - z^-$. For any set $E \subseteq \mathbb{R}^n$, the polar of $E$ is defined by $E^* := \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \forall \ x \in E\}$. $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices, and for $T \in \mathbb{R}^{n \times n}$, $T^*$ denotes the transpose. A nonempty set $K \subseteq \mathbb{R}^n$ is a convex cone if $\lambda x + \mu y \in K$ for all $\lambda, \mu \in \mathbb{R}_+$ and $x, y \in K$. In this paper, $K$ always denotes a closed convex cone. A convex cone $K$ is polyhedral if there is a set $\{x_1, \ldots, x_m\} \subseteq K$ such that $K = \{x : x = \sum \lambda_i x_i, \lambda_i \geq 0\}$.

We say that $T$ is

(a) copositive on $K$ if $\langle Tx, x \rangle \geq 0 \ \forall \ x \in K$,

(b) monotone on $K$ (or positive semidefinite on $K - K$) if

$$\langle Tx - Ty, x - y \rangle \geq 0 \ \forall \ x, y \in K,$$

(c) pseudomonotone on $K$ if

$$x, y \in K, \ \langle Tx, y - x \rangle \geq 0 \ \Rightarrow \ \langle Ty, y - x \rangle \geq 0,$$
(d) **copositive plus** on $K$ if $T$ is copositive on $K$ and

$$x \in K, \quad \langle Tx, x \rangle = 0 \Rightarrow \langle Tx, u \rangle + \langle Tu, x \rangle = 0 \quad (\forall u \in K),$$

(e) **copositive star** on $K$ if $T$ is copositive on $K$ and

$$x \in K, \quad Tx \in K^*, \quad \langle Tx, x \rangle = 0 \Rightarrow -x \in [T(K)]^*.$$

We note that the above definitions are valid even for nonlinear mappings. However, since our $T$ is linear, some of the definitions can be simplified; for example, in (d) we can write $\langle (T + T^*)x, u \rangle = 0$ for $\langle Tx, u \rangle + \langle Tu, x \rangle = 0$, and in (e), $-x \in [T(K)]^*$ can be replaced by $-T^*x \in K^*$.

We list some more definitions. We say that $T$

(f) a $P_0$-matrix (or $T \in P_0$) if every principal minor of $T$ is nonnegative,

(g) an $L$-matrix (or $T \in L$) if

(i) for each nonzero $x \geq 0$, there is an index $i$ such that $x_i > 0$ and $(Tx)_i \geq 0$,

(ii) for each nonzero $x$ with $x > 0$, $Tx \geq 0$, and $\langle Tx, x \rangle = 0$, there are nonnegative diagonal matrices $\Lambda$ and $\Gamma$ such that $\Gamma x \neq 0$ and $(\Lambda T + T^*\Gamma)x = 0$.

$T$ is said to be a $Q_0$-matrix (or $T \in Q_0$) if for all $q \in \mathbb{R}^n$, feasibility of $\text{LCP}(T, \mathbb{R}^n_+, q)$ implies its solvability.

3. **GENERAL PROPERTIES OF PSEUDOMONOTONE AND COPOSITIVE STAR MATRICES**

**Proposition 1.** Suppose that $T$ is either copositive plus on $K$ or pseudomonotone on $K$. Then $T$ is copositive star on $K$.

**Proof.**

(1) Let $T$ be copositive plus on $K$. For any $x \in K$ with $Tx \in K^*$ and $\langle Tx, x \rangle = 0$ we get $\langle Tx, k \rangle + \langle Tk, x \rangle = 0$ for all $k \in K$, i.e., $-\langle Tk, x \rangle = \langle Tx, x \rangle = 0$. Thus, for any $u \in K$, we have $\langle Tu, x \rangle = 0$. Therefore, $\langle Tx, x \rangle = 0$ for all $x \in K$. Since $T$ is linear, we have $\langle Tx, x \rangle = 0$ for all $x \in K$. Therefore, $T$ is copositive star on $K$. 

(d) **copositive plus** on $K$ if $T$ is copositive on $K$ and

$$x \in K, \quad \langle Tx, x \rangle = 0 \Rightarrow \langle Tx, u \rangle + \langle Tu, x \rangle = 0 \quad (\forall u \in K),$$

(e) **copositive star** on $K$ if $T$ is copositive on $K$ and

$$x \in K, \quad Tx \in K^*, \quad \langle Tx, x \rangle = 0 \Rightarrow -x \in [T(K)]^*.$$
\( \langle Tx, k \rangle \geq 0 \) \((k \in K)\). Hence \(-x \in [T(K)]^*\). Since \(T\) is also copositive, \(T\) is copositive star on \(K\).

(2) Suppose that \(T\) is pseudomonotone on \(K\). Fix any \(x \in K\) such that \(Tx \in K^*\) and \(\langle Tx, x \rangle = 0\) (zero is such an element). For any \(u \in K\) and \(\lambda > 0\) we have \(\langle Tx, \lambda u - x \rangle = -\lambda \langle Tx, u \rangle \geq 0\) and hence \(\langle T(\lambda u), \lambda u - x \rangle \geq 0\), i.e., \(\lambda \langle Tu, u \rangle \geq \langle Tu, x \rangle\). If we divide this inequality by \(\lambda\) and let \(\lambda \to \infty\), we get \(\langle Tu, u \rangle \geq 0\), giving us the copositivity of \(T\) on \(K\). If we let \(\lambda \to 0\), we get \(\langle Tu, x \rangle \leq 0\), giving us the copositive star property of \(T\) on \(K\).

When \(K = \mathbb{R}^n_+\), more can be said about pseudomonotone matrices.

**Proposition 2.** Suppose that \(T\) is pseudomonotone on \(\mathbb{R}^n_+\). Then \(T\) is a \(P_0\)-matrix.

**Proof.** In view of Fiedler and Ptak’s result in [6], it is enough to show that for any nonzero \(z\), \(\max_{i: z_i > 0} z_i (T^* z)_i \geq 0\). Suppose, if possible, that for some nonzero \(z\), \(\max_{i: z_i > 0} z_i (T^* z)_i < 0\). Let \(I = \{i: z_i > 0\}\) and \(J = \{j: z_j < 0\}\). Since \(T\) is copositive on \(\mathbb{R}^n_+\) (from Proposition 1), \(I\) and \(J\) are both nonempty. We have \(\langle Tz^-, z^+ - z^- \rangle = \langle z^-, T^* z \rangle = -\Sigma_j z_j (T^* z)_j < 0\) and \(\langle Tz^+, z^+ - z^- \rangle = \langle z^+, T^* z \rangle = \Sigma_j z_j (T^* z)_j < 0\), contradicting the pseudomonotonicity of \(T\) on \(\mathbb{R}^n_+\). Hence \(T\) is a \(P_0\)-matrix.

Since a symmetric \(P_0\)-matrix is positive semidefinite, we have

**Corollary 1.** If \(T\) is symmetric and pseudomonotone on \(\mathbb{R}^n_+\), then \(T\) is positive semidefinite.

The next proposition shows that the above result holds for any convex cone.

**Proposition 3.** Suppose that \(T\) is symmetric and pseudomonotone on a convex cone \(S\) (which need not be closed). Then \(T\) is monotone on \(S\).

**Proof.** We can approximate \(S\) by a sequence \(\{K_m\}\) of polyhedral cones each contained in \(S\) (in the sense that for any \(x \in S\) there exists \(x_m \in K_m\) such that \(\lim x_m = x\)). By continuity of \(T\), it is enough to show that \(T\) is monotone on each \(K_m\). Fix \(m\), and write \(K = K_m\) (just for convenience). Let \(n\) be the order of \(T\). Since \(K\) is polyhedral, there exists some \(r\) and a \(n \times r\) matrix \(B\) such that \(B(\mathbb{R}^r_+) = K\). It is easily seen from \(\langle B^* TBu, v - u \rangle = \langle T(Bu), Bu - Bu \rangle \) \((u, u \in \mathbb{R}^r_+)\) that \(B^* TB\) is symmetric and pseudomonotone on \(\mathbb{R}^r_+\), which gives (by Corollary 1) the monotonicity of \(T\) on \(K\). This completes the proof.
From Proposition 3, we can get a result about pseudoconvex quadratic forms. A quadratic form $\langle Tx, x \rangle$ (corresponding to a symmetric matrix $T$) is said to be *pseudoconvex* on a convex set $C$ if

$$x, y \in C, \quad \langle Tx, y - x \rangle \geq 0 \quad \Rightarrow \quad \langle Ty, y \rangle \geq \langle Tx, x \rangle.$$ 

It is known that (cf. Karamardian [9]) pseudoconvexity of $\langle Tx, x \rangle$ on $C$ is equivalent to pseudomonotonicity of $T$ on $C$. Proposition 3 yields a result (which is perhaps known) that on a convex cone, a quadratic form is pseudoconvex iff it is convex. However, on the interior of a convex cone, pseudoconvexity need not be the same as convexity. For example, on the positive orthant of $\mathbb{R}^n$ (= interior of $\mathbb{R}^+_n$) there are pseudoconvex quadratic forms (coming from so-called positive subdefinite matrices; see Martos [12]) which are nonconvex. (For a discussion on pseudoconvex functions we refer the reader to the articles by Ferland [5] and Schiable [15].)

The next result is an analogue of Proposition 3 for copositive star matrices.

**Proposition 4.** If $T$ is symmetric and copositive star on $K$, then it is copositive plus on $K$.

**Proof.** Let $x \in K$ such that $\langle Tx, x \rangle = 0$. From the copositivity of $T$ on $K$ we have for any $u \in K$ and $\lambda \geq 0$,

$$\langle (T + T^*)x, x + \lambda u \rangle \geq 0.$$ 

This leads to $(T + T^*)x \in K^*$ and by symmetry to $Tx \in K^*$. Since $T$ is copositive star on $K$, we have $-T^*x \in K^*$, i.e., $-Tx \in K^*$. Thus $\langle (T + T^*)x, u \rangle = 2\langle Tx, u \rangle = 0 \forall u \in K$, i.e., $T$ is copositive plus on $K$.

We include the next (isolated) result about real eigenvalues of a pseudomonotone matrix to indicate that pseudomonotone matrices and monotone ( = positive semidefinite) matrices share many properties. (The $P_0$-property is one such common property.)

**Proposition 5.** Suppose that $T$ is pseudomonotone on a self-polar cone $K$ (i.e., $K^* = K$). Then every real eigenvalue of $T$ is nonnegative.

**Proof.** Let $\lambda$ be a real eigenvalue of $T$. Then $\lambda$ is also a real eigenvalue of $T^*$. Suppose that $\lambda < 0$, and let $z$ be an eigenvector of $T^*$ corresponding
to $\lambda$. By Moreau [13], $z = y - x$, where $y \in K$, $x \in K^*$ ($= K$), and $\langle y, x \rangle = 0$. We have

$$\langle Tx, y - x \rangle = \langle x, T^* z \rangle = \lambda \langle x, z \rangle = (-\lambda) \langle x, x \rangle \geq 0,$$

implying, by pseudomonotonicity,

$$\langle Ty, y - x \rangle \geq 0,$$

i.e.,

$$\langle y, T^*, z \rangle = \lambda \langle y, z \rangle = \lambda \langle y, y \rangle \geq 0.$$

Hence $y = 0$. By considering $-z$, we get $x = 0$. Thus $z = 0$, a contradiction. Hence $\lambda \geq 0$.

4. A CHARACTERIZATION OF PSEUDOMONOTONICITY ON $\mathbb{R}^n_+$

**Proposition 6.** $T$ is pseudomonotone on $\mathbb{R}^n_+$ iff the following hold for any $z$:

(i) $(T^* z)_i < 0$ for some $i$ and $\langle T z^-, z \rangle \geq 0 \Rightarrow \langle T z, z \rangle \geq 0$.

(ii) $(T^* z)_i > 0$ for some $i$ and $\langle T z^-, z \rangle \leq 0 \Rightarrow \langle T z, z \rangle \geq 0$.

**Proof.** The condition

$$x, y \geq 0, \quad \langle Tx, y - x \rangle \geq 0 \Rightarrow \langle Ty, y - x \rangle \geq 0$$

is equivalent to

$$u \geq 0, \quad z \in \mathbb{R}^n, \quad \langle T(z^- + u), z \rangle \geq 0 \Rightarrow \langle T(z - u), z \rangle \geq 0,$$

which is the same as

$$u \geq 0, \quad z \in \mathbb{R}^n, \quad \langle u, T^* z \rangle \geq \langle T z^-, z \rangle \Rightarrow \langle u, T^* z \rangle \geq \langle T z^+, z \rangle.$$

(4.1)

We show that this condition is equivalent to (i) and (ii). Suppose that (4.1) holds. If $(T^* z)_i < 0$ for some $i$ and $\langle T z^-, z \rangle \geq 0$, then we put $u = [ - \langle T z^-, z \rangle / (T^* z)_i ] e_i$, where $e_i$ is the $i$th coordinate vector in $\mathbb{R}^n$. We have $u \geq 0$, $\langle u, T^* z \rangle = - \langle T z^-, z \rangle$, and hence by (4.1), $\langle u, T^* z \rangle \geq 0$.
- \langle Tz^+, z \rangle. This gives - \langle Tz^-, z \rangle \geq - \langle Tz^+, z \rangle, i.e., \langle Tz, z \rangle \geq 0. Thus (i) holds.

Similarly (ii) holds. Now suppose that (i) and (ii) are satisfied, and let \( u \geq 0 \) with \( \langle u, T^*z \rangle \geq - \langle Tz^-, z \rangle \). Without loss of generality let \( Tz \neq 0 \).

**Case 1:** \( \langle Tz^-, z \rangle = \langle z^-, T^*z \rangle < 0 \). If \( (T^*z)_i > 0 \) for some \( i \), then by (ii), \( \langle Tz, z \rangle \geq 0 \), which gives - \langle Tz^-, z \rangle \geq - \langle Tz^+, z \rangle. Hence \( \langle u, T^*z \rangle \geq - \langle Tz^+, z \rangle \). If \( (T^*z)_i \leq 0 \), then \( \langle u, T^*z \rangle \geq - \langle Tz^-, z \rangle \) shows that \( \langle Tz^-, z \rangle = 0 \). Since \( T^*z \neq 0 \), \( (T^*z)_i < 0 \) for some \( j \). Now we can use (i) to get \( \langle Tz, z \rangle \geq 0 \), which gives \( \langle u, T^*z \rangle \geq - \langle Tz^+, z \rangle \).

**Case 2:** \( \langle Tz^-, z \rangle \geq 0 \). If \( T^*z \geq 0 \) then \( \langle u, T^*z \rangle \geq 0 \geq - \langle z^+, T^*z \rangle \). If \( (T^*z)_i < 0 \) for some \( i \), then (i) holds and hence \( \langle Tz, z \rangle \geq 0 \). As before, this yields \( \langle u, T^*z \rangle \geq - \langle Tz^+, z \rangle \). Thus (4.1) holds.

5. COMPLEMENTARITY RESULTS

**Proposition 7.** If \( T \) is copositive star on \( \mathbb{R}^n_+ \), then \( T \in L \).

**Proof.** Copositivity of \( T \) on \( \mathbb{R}^n_+ \) shows that for any nonzero \( x \) in \( \mathbb{R}^n_+ \), \( \max_{x_0 > 0} x_i(Tx)_i \geq 0 \). Now let \( x \geq 0 \) be such that \( Tx \in K^* \) and \( \langle Tx, x \rangle = 0 \). By copositivity, we get \( (T + T^*)x \geq 0 \), and the star property gives \( T^*x \leq 0 \). From \( Tx \geq - T^*x \geq 0 \) we conclude that for any index \( i \),

\[
(Tx)_i = 0 \quad \Rightarrow \quad (T^*x)_i = 0.
\]

Clearly, \( \Lambda Tx + T^*(Ix) = 0 \), where \( \Lambda \) is a diagonal matrix with \( \Lambda_{ii} = 0 \) if \( (Tx)_i = 0 \) and \( \Lambda_{ii} = -(T^*x)_i/(Tx)_i \) otherwise.

Since \( L \subseteq Q_0 \) (cf. Eaves [4]), we have

**Corollary 2.** If \( T \) is copositive star on \( \mathbb{R}^n_+ \), then \( T \in Q_0 \).

**Corollary 3.** If \( T \) is pseudomonotone on \( \mathbb{R}^n_+ \), then \( T \in P_0 \cap Q_0 \).

As we pointed out in the introduction, the copositive star property can be defined for any mapping (not just for linear ones) on any cone. (Such a generalization is not easily seen for the L-property, i.e., the property defining L-matrices.)

The following results show that copositive star matrices have interesting complementarity properties even on general cones. First we observe (the easy proof is omitted) that copositive star, copositive plus, and pseudomonotone
properties are "invariant," in the sense that if $T$ has one of these properties on $K$, then $B^*TB$ has the same property on $L$, where $B$ is any matrix (not necessarily square) and $L$ is a closed convex cone such that $B(L) = K$.

**Proposition 8.** Let $K$ be a polyhedral cone in $\mathbb{R}^n$. If $T$ is copositive star on $K$, then for any $q \in \mathbb{R}^n$, the feasibility of $\text{LCP}(T, K, q)$ implies its solvability.

**Proof.** There is an $n \times m$ matrix $B$ such that $B(\mathbb{R}^n_+) = K$. Put $M = B^*TB$ and $p = B^*q$. From

$$\langle Mx + p, u \rangle = \langle T(Bx) + q, Bu \rangle \quad (x, u \in \mathbb{R}^m_+)$$

we see that feasibility (solvability) of $\text{LCP}(M, \mathbb{R}^m_+, p)$ is equivalent to the feasibility (solvability) of $\text{LCP}(T, K, q)$. By an earlier remark, $M$ is copositive star on $\mathbb{R}^m_+$ and hence belongs to $Q_0$ (by Corollary 2). Now feasibility of $\text{LCP}(T, K, q)$ implies that of $\text{LCP}(M, \mathbb{R}^m_+, p)$. Since $\text{LCP}(M, \mathbb{R}^m_+, p)$ is solvable we see that $\text{LCP}(T, K, q)$ is also solvable.

The above result fails for a nonpolyhedral cone. For example, let $K = \{(x, y, z) \in \mathbb{R}^3 : x, z \geq 0, 2xz \geq y^2\}$ and $T(x, y, z) = (x, y, 0)$, $q = (1, 1, 0)$. It can be easily shown that $\text{LCP}(T, K, q)$ is feasible but not solvable (even though $T$ is monotone). However, for a general cone we have

**Proposition 9.** Suppose that $T$ is copositive star on $K$. Then the following are equivalent:

(i) $\text{LCP}(T, K, q)$ is solvable for all $q$,

(ii) $\text{LCP}(T, K, q)$ is feasible for all $q$,

(iii) $x \in K$, $Tx \in K^*$, $\langle Tx, x \rangle = 0 \Rightarrow x = 0$.

This is a special case of a more general result in [7] (proved for positive homogeneous mappings on a locally compact cone in a locally convex space using convex programming techniques.) For the sake of completeness we sketch an independent

**Proof.** The implication (i) $\Rightarrow$ (ii) is obvious. Suppose (ii) holds, and let $x \in K$ such that $Tx \in K^*$ and $\langle Tx, x \rangle = 0$. By the copositive star property, $-T^*x \in K^*$. Now the feasibility of $\text{LCP}(T, K, -x)$ gives $\langle Tx_0 - x, x \rangle \geq 0$ for some $x_0 \in K$, i.e., $-\langle x, x \rangle + \langle T^*x, x_0 \rangle \geq 0$. Since $-T^*x \in K^*$, we conclude that $\langle x, x \rangle = 0$, i.e., $x = 0$. This gives (iii). Now suppose that (iii)
holds, and let \( q \in \mathbb{R}^n \). If \( \text{LCP}(T, K, q) \) is not feasible, then \( q \) does not belong to the convex set \( K^* - T(K) \). By a separation argument we get a nonzero \( u \in K \) such that \(-T^*u \in K^*\). From copositivity of \( T \) we get \( \langle Tu, u \rangle = 0 \) and \((T + T^*)u \in K^*\), i.e., \( \langle Tu, u \rangle = 0 \) and \( Tu \in K^* \). This \( u \) contradicts (iii). Thus we have feasibility of \( \text{LCP}(T, K, q) \). Let \( x_0 \in K \) such that \( Tx_0 + q \in K^* \). We can approximate \( K \) by an increasing sequence of polyhedral cones \( K_m \) \((m = 1, 2, \ldots)\) (in the sense that for any \( z \in K \) there exists \( z_m \in K_m \) such that \( \lim z_m = z \)) such that \( x_0 \in K_m \subseteq K \). We can assume (by going through a subsequence, if necessary) that \( T \) is copositive star on each \( K_m \). Proposition 8 yields an \( x_m \) such that

\[
x_m \in K_m, \quad Tx_m + q \in K^*, \quad \text{and} \quad \langle Tx_m + q, x_m \rangle = 0. \tag{5.1}
\]

Now (5.1) and (iii) show that the sequence \( \{x_m\} \) is bounded. Any subsequential limit of \( \{x_m\} \) will solve \( \text{LCP}(T, K, q) \).

When \( K = \mathbb{R}^n_+ \), more can be said about the equivalence of (i) and (iii). Pang [14] shows that this equivalence holds for L-matrices.

6. EXAMPLES

The following examples show that on \( \mathbb{R}^n_+ \), the classes of pseudomonotone matrices, copositive plus matrices, and copositive star matrices are distinct. In this section, the standard basis in \( \mathbb{R}^n \) is denoted by \( \{e_1, \ldots, e_n\} \). In what follows we construct second and third order matrices. Higher order matrices with specific properties can then be constructed by inserting zero rows and columns.

(i) Let

\[
T = \begin{bmatrix}
1 & 4 \\
4 & 1
\end{bmatrix}.
\]

This \( T \) is copositive plus on \( \mathbb{R}^2_+ \), but not pseudomonotone, since \( \langle Te_2, e_1 - e_2 \rangle > 0 \) and \( \langle Te_1, e_1 - e_2 \rangle < 0 \).

(ii) Let

\[
T = \begin{bmatrix}
0 & -1 \\
2 & 0
\end{bmatrix}.
\]

We show that this \( T \) is pseudomonotone on \( \mathbb{R}^2_+ \) but not copositive plus on
From 

\[ \left\langle \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = 2uv - y(x + u) \geq 0 \]

we get 

\[ \left\langle \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = v(u + x) - 2uy \geq 0 \]

whenever all the vectors involved are nonnegative. [If \(x = 0\), the implication is easy to see. If \(x \neq 0\), then \(v(u + x) - 2uy > (2x)^{-1}y(x + u)^2 - 2uy = (2x)^{-1}y(x - u)^2 \geq 0\).] Thus \(T\) is pseudomonotone on \(\mathbb{R}_+^2\). However, \(T\) is not copositive plus on \(\mathbb{R}_+^2\), since \(\langle Te_1, e_1 \rangle = 0\) and \((T + T^*)e_1 \neq 0\).

**Note.** It can be shown that \(T^*\) is also pseudomonotone. Thus, even if \(T\) and \(T^*\) are both pseudomonotone, \(T\) need not be monotone.

(iii) Let 

\[ T = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

We show that \(T\) is copositive star on \(\mathbb{R}_+^3\) but not pseudomonotone and copositive plus. For any \(u \in \mathbb{R}^3\), we have \(\langle Tu, u \rangle = xy + y^2 + z^2\) where \(x, y,\) and \(z\) are components of \(u\). It follows that \(T\) is copositive on \(\mathbb{R}_+^3\). Further, \(u \geq 0, \langle Tu, u \rangle = 0 \Rightarrow u = xe_1\), with \(x \geq 0\). Since \(T^*e_1 \leq 0\) and \((T + T^*)e_1 \neq 0\), \(T\) is copositive star on \(\mathbb{R}_+^3\) but not copositive plus. Let \(u\) be the transpose of the row vector \([3, -1, 1]\). Then \(T^*u\) is the transpose of \([-2, -4, 1]\), \(\langle T^*u, u^- \rangle = -4\), and \(\langle Tu, u \rangle = -1\). Thus, condition (ii) in Proposition 6 is violated. Hence \(T\) is not pseudomonotone.

**REFERENCES**

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