Note

The diameter of directed graphs

Peter Dankelmann

School of Mathematical Sciences, University of KwaZulu-Natal, Durban, South Africa

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Abstract

Let $D$ be a strongly connected oriented graph, i.e., a digraph with no cycles of length 2, of order $n$ and minimum out-degree $\delta$. Let $D$ be eulerian, i.e., the in-degree and out-degree of each vertex are equal. Knyazev (Mat. Z. 41(6) 1987 829) proved that the diameter of $D$ is at most \( \frac{5}{2\delta+2} n \) and, for given $n$ and $\delta$, constructed strongly connected oriented graphs of order $n$ which are $\delta$-regular and have diameter greater than \( \frac{4}{2\delta+1} n - 4 \). We show that Knyazev’s upper bound can be improved to \( \text{diam}(D) \leq \frac{4}{2\delta+1} n + 2 \), and this bound is sharp apart from an additive constant.

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The problem of determining a sharp upper bound on the diameter of an undirected graph $G$ in terms of its order and minimum degree was solved by several authors (e.g. [2,4]), who proved that

\[
\text{diam}(G) \leq \frac{3}{\delta + 1} n + c,
\]

where $n$ is the order of $G$, $\delta$ is the minimum degree of $G$ and $c$ is a constant.

We consider the corresponding problem for directed graphs. If the minimum degree is replaced by the minimum out-degree, then (1) does not generalize to directed graphs: Soares [5] constructed strongly connected digraphs of order $n$ and minimum out-degree $\delta$ of diameter $n - 2\delta + 1$ and showed that this is the maximum diameter of strong digraphs.
of order \(n\) and minimum degree \(\delta\). However, Knyazev [3] and, independently, Soares [5] demonstrated that for eulerian digraphs, i.e., each vertex has equal in-degree and out-degree, (1) holds and that the factor \(3/(\delta + 1)\) is best possible. Essentially the same bound was given by Knyazev [3]. For oriented graphs, i.e., digraphs with no 2-cycle, Knyazev [3] improved the factor \(3/(\delta + 1)\) to \(5/(2\delta + 2)\).

**Theorem 1** (Knyazev [3]). Let \(D\) be a strong, eulerian digraph of order \(n\) and minimum out-degree \(\delta, 2^{} \leq \delta \leq n/2\). If \(D\) contains no 2-cycle then

\[
\text{diam}(D) \leq \frac{5}{2\delta + 2} n.
\]

For given \(n, \delta\) with \(2^{} \leq \delta \leq n/2\) there exists a \(\delta\)-regular eulerian oriented graph \(D\) of order \(n\) and

\[
\text{diam}(D) \geq \frac{4}{2\delta + 1} n - 4 + \frac{1}{2\delta + 1}.
\]

In this note we prove that the factor \(\frac{5}{2\delta + 2}\) can be improved to \(\frac{4}{2\delta + 1}\). In conjunction with the second part of Theorem 1, it follows that for fixed \(\delta \geq 2\) and large \(n\), the maximum diameter of an eulerian oriented graph of order \(n\) and minimum degree \(\delta\) is \(\frac{4}{2\delta + 1} n + O(1)\).

The digraphs considered in this note have no multiple arcs or cycles of length 2 and are strongly connected. The out-degree of a vertex \(v\) is denoted by \(\text{deg}^+(v)\). We denote the minimum out-degree by \(\delta\). For subsets \(A, B\) of the vertex set of a digraph, we denote the number of arcs with tail in \(A\) and head in \(B\) by \(q(A, B)\). For \(q(A, A)\) we write \(q(A)\). The diameter \(\text{diam}(D)\) is the maximum distance between any two vertices of \(D\).

**Theorem 2.** Let \(D\) be a strong, eulerian digraph of order \(n\) and minimum out-degree \(\delta, 2^{} \leq \delta \leq n/2\). If \(D\) contains no 2-cycle then

\[
\text{diam}(D) \leq 4 \frac{2\delta + 1}{n + 2}.
\]

**Proof.** Let \(v\) be an arbitrary vertex and for an integer \(i\) let \(V_{\leq i}, V_i, V_{\geq i}\) be the set of vertices at distance at most \(i\), exactly \(i\) and at least \(i\), respectively. Let \(n_i = |V_i|\) and let \(\text{ex}(v)\) be the largest \(i\) with \(n_i > 0\). We prove that for \(i = 1, 2, \ldots, \text{ex}(v) - 2,\)

\[
n_{i-1} + n_i + n_{i+1} + n_{i+2} \geq 2\delta + 1.
\]

Since \(D\) is eulerian, we have \(q(V_{\geq i}, V_{\leq i-1}) = q(V_{\leq i-1}, V_{\geq i})\) and thus

\[
q(V_i, V_{\leq i-1}) + q(V_{i+1}, V_{\leq i-1}) \leq q(V_{\geq i}, V_{\leq i-1}) = q(V_{\leq i-1}, V_{\geq i}) = q(V_{i-1}, V_i) \leq n_{i-1} n_i.
\]

Since \(D\) is an oriented graph we have

\[
q(V_i, V_{i+1}) + q(V_{i+1}, V_i) \leq n_i n_{i+1}.
\]
Adding all out-degrees and applying (4) and (3), we obtain

\[ 0 \leq \sum_{x \in V_i} \deg^+(x) + \sum_{x \in V_{i+1}} \deg^+(x) - \delta(n_i + n_{i+1}) \]

\[ = \left( q(V_i, V_{i-1}) + q(V_i) + q(V_i, V_{i+1}) \right) \]

\[ + \left( q(V_{i+1}, V_{i-1}) + q(V_{i+1}, V_i) + q(V_{i+1}) \right) \]

\[ + q(V_{i+1}, V_{i+2}) - \delta(n_i + n_{i+1}) \]

\[ \leq n_{i-1}n_i + \frac{1}{2} n_i(n_i - 1) + n_in_{i+1} + \frac{1}{2} n_{i+1}(n_{i+1} - 1) \]

\[ + n_{i+1}n_{i+2} - \delta(n_i + n_{i+1}) \]

\[ = \frac{1}{2} \left( n_i + n_{i+1} \right) \left( n_i + n_{i+1} - 2\delta - 1 \right) + n_{i-1}n_i + n_{i+1}n_{i+2}. \] (5)

Define the function \( g(n_{i-1}, n_i, n_{i+1}, n_{i+2}) \) to be the right-hand side of (5). In order to prove (2) we minimize the function \( f(n_{i-1}, n_i, n_{i+1}, n_{i+2}) = n_{i-1} + n_i + n_{i+1} + n_{i+2} \) subject to the constraints \( n_{i-1}, n_i, n_{i+1}, n_{i+2} \geq 1 \) and, by (5), \( g(n_{i-1}, n_i, n_{i+1}, n_{i+2}) \geq 0 \). If \( \text{max}\{n_{i-1}, n_i, n_{i+1}, n_{i+2}\} > 2\delta \) then (2) holds, hence we can assume that \( n_{i-1}, n_i, n_{i+1}, n_{i+2} \) are in the closed interval \([1, 2\delta]\). Since the set of all solutions of \( g \geq 0 \) with \( n_{i-1}, n_i, n_{i+1}, n_{i+2} \in [1, 2\delta] \) is compact, \( f \) attains its minimum on it. Let \( n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^* \) be chosen such that \( f(n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) \) is minimum subject to the above conditions. We first show that

\[ g(n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) = 0. \] (6)

Suppose to the contrary that \( g > 0 \). Then at least one of \( n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^* \) is strictly greater than 1. Reducing it by a sufficiently small value will leave \( g > 0 \) but reduce \( f \), a contradiction. Hence (6) holds.

We have to minimize \( f \) subject to \( g = 0 \) and \( n_{i-1}, n_i, n_{i+1}, n_{i+2} \geq 1 \).

**Case 1:** \( \min\{n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*\} > 1 \).

Then \( (n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) \) is a local minimum of \( f \) subject to \( g = 0 \). By the Lagrange multiplier theorem (see for example [1, p.461]) there exists a real \( \lambda \) such that at \( (n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) \) we have

\[ \frac{\partial f}{\partial n_j} + \lambda \frac{\partial g}{\partial n_j} = 0 \]

for \( j = i - 1, i, i + 1, i + 2 \). Hence

\[
\begin{pmatrix}
1 & n_i^* \\
1 & n_{i-1}^* + n_i^* + n_{i+1}^* - \delta - \frac{1}{2} \\
1 & n_i^* + n_{i+1}^* + n_{i+2}^* - \delta - \frac{1}{2} \\
1 & n_{i+1}^*
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\lambda \\
\lambda \\
\lambda
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
which implies $\lambda = -1/n_i^*, n_i^* = n_{i+1}^*$ and $n_{i-1}^* = n_{i+2}^*$. Hence $1 - (n_{i-1}^* + 2n_i^* - \delta - 1/2)/n_i^* = 0$, which, after simplification, yields

$$n_{i-1}^* + n_i^* = n_{i+1}^* + n_{i+2}^* = \delta + \frac{1}{2}.$$ 

Hence $f \geq 2\delta + 1$ and (2) holds.

Case 2: $(n_{i-1}^* = 1$ and $n_{i+1}^*, n_{i+1}^*, n_{i+2}^* > 1)$ or $(n_{i+1}^* = 1$ and $n_{i-1}^*, n_i^*, n_{i+1}^* > 1)$. We assume that $n_{i-1}^* = 1$; the case $n_{i-1}^* = 1$ is analogous. Then $(n_i^*, n_{i+1}^*, n_{i+2}^*)$ is a local minimum of $f$ subject to $g = 0$ and $n_{i-1} = 1$. Again by the Lagrange multiplier method, there exist reals $\lambda_1, \lambda_2$ with

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} n_i^* \\ n_i^* + n_{i+1}^* - \frac{\delta}{2} \\ n_i^* - \frac{1}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

Equality of the second and third row above implies $n_{i-1}^* = n_{i+2}^* = 1$, a contradiction.

Case 3: $n_{i-1}^* = n_{i+2}^* = 1$ and $n_i^*, n_{i+1}^* > 1$.

In this case we have $g = g(1, n_i^*, n_{i+1}^*, 1) = \frac{1}{2}(n_i^* + n_{i+1}^*)(n_i^* + n_{i+1}^* - 2\delta) \geq 0$, which implies $n_i^* + n_{i+1}^* - 2\delta \geq 1$ and thus $f \geq 2\delta + 3$, so (2) holds.

Case 4: $(n_i^* = 1$ and $n_{i-1}^*, n_{i+1}^*, n_{i+2}^* > 1)$ or $(n_{i+1}^* = 1$ and $n_{i-1}^*, n_i^*, n_{i+2}^* > 1)$.

This case is analogous to Case 2; we obtain the contradiction $n_i^* = n_{i+1}^* = 1$.

Case 5: $n_i^* = n_{i+1}^*$.

In this case we have $g = g(n_{i-1}^*, 1, 1, n_{i+2}^*) = n_{i-1}^* + n_{i+2}^* + 1 - 2\delta \geq 0$, which implies $n_{i-1}^* + n_{i+2}^* > 2\delta - 1$ and thus $f \geq 2\delta + 1$.

In each case we have $f(n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) \geq 2\delta + 1$, which implies (2).

Now let $v$ be a vertex such that the eccentricity of $v$ is maximum and thus equals $\text{diam}(D) =: d$. Let $a = \lfloor \frac{d+1}{4} \rfloor$. Then

$$n = \sum_{i=0}^{d} n_i \geq \sum_{i=0}^{a-1} (n_{4i} + n_{4i+1} + n_{4i+2} + n_{4i+3}) \geq a(2\delta + 1).$$ 

Hence, by $\frac{d-2}{4} \leq a$, we obtain $d \leq \frac{4}{2\delta+1} n + 2$, as desired.

References