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A local global principle for regular operators in Hilbert C^*-modules ☆

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Abstract

Hilbert C^*-modules are the analogues of Hilbert spaces where a C^*-algebra plays the role of the scalar field. With the advent of Kasparov's celebrated KK-theory they became a standard tool in the theory of operator algebras. While the elementary properties of Hilbert C^*-modules can be derived basically in parallel to Hilbert space theory the lack of an analogue of the Projection Theorem soon leads to serious obstructions and difficulties. In particular the theory of unbounded operators is notoriously more complicated due to the additional axiom of regularity which is not easy to check. In this paper we present a new criterion for regularity in terms of the Hilbert space localizations of an unbounded operator. We discuss several examples which show that the criterion can easily be checked and that it leads to nontrivial regularity results.

Keywords: Hilbert C^*-module; Unbounded operator; Semiregular and regular operator

Contents

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1. Introduction

A Hilbert C^*-module E over a C^*-algebra $\mathscr{A}$ is an $\mathscr{A}$-right module equipped with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle$ and such that E is complete with respect to the norm $\|x\| := \|\langle x, x\rangle^{1/2}\| = \|\langle x, x\rangle\|^{1/2}$. The notion was introduced by Kaplansky in the commutative case [9] and in general independently by Paschke [16], Rieffel [19] and Takahashi (for the latter cf. [19, p. 179] and [7, p. 364]). Kasparov's celebrated KK-theory makes extensive use of Hilbert C^*-modules [10] and by now Hilbert C^*-modules are a standard tool in the theory of operator algebras. They are covered in several textbooks, Blackadar [3, Sec. 13], [4, Sec. II.7], Manuilov and Troitsky [14], Raeburn and Williams [17], Wegge-Olsen [20, Chap. 15]; our standard reference will be Lance [13].

The elementary properties of C^*-modules can be derived basically in parallel to Hilbert space theory. However, there is no analogue of the Projection Theorem which soon leads to serious obstructions and difficulties.

A Hilbert C^*-module E comes with a natural C^*-algebra $\mathscr{L}(E)$ of bounded adjointable module endomorphisms. As for Hilbert spaces one soon needs to consider unbounded adjointable operators, Baaj and Julg [2], Guljaš [6], Kucerovsky [11], Pal [15], Woronowicz [21]; see also [13, Chap. 9/10].

The lack of a Projection Theorem in Hilbert C^*-modules causes the theory of unbounded operators to be notoriously more complicated. To explain this let us introduce some terminology: following Pal [15] by a *semiregular* operator in a Hilbert C^*-module E over $\mathscr{A}$ we will understand an operator $T : \mathscr{D}(T) \longrightarrow E$ defined on a dense $\mathscr{A}$-submodule $\mathscr{D}(T) \subset E$ and such that the adjoint T^* is densely defined, too. One now easily deduces that T is $\mathscr{A}$-linear and closable and that T^* is closed. Besides this semiregular operators can be rather pathologic (see the discussion in Section 2.3 and Section 6).

To have a reasonable theory (e.g. with a functional calculus for selfadjoint operators) one has to introduce the additional *axiom of regularity*: a closed semiregular operator T in E is called *regular* if $I + T^*T$ is invertible. Regular operators behave more or less as nicely as closed densely defined operators in Hilbert space. In particular for selfadjoint regular operators there is a continuous functional calculus [1,12,21,22].

While in a *Hilbert space* every densely defined closed operator is regular in general Hilbert C^*-modules there exist closed semiregular operators which are not regular, see Proposition 6.3.

There is, however, a considerable drawback of the regularity axiom. We quote here from [15, p. 332]:

> ... But when one deals with specific unbounded operators on concrete Hilbert C^*-modules, it is usually extremely difficult to verify the regularity condition, though the semiregularity conditions are relatively easy to check. So it would be interesting to find other more easily

manageable conditions that are equivalent to the last condition above. In [21], Woronowicz gave a criterion based on the graph of an operator for it to be regular, and to this date, this remains the only attempt in this direction.

The aim of this paper is to remedy this distressing situation which has not much improved in the more than 10 years after Pal had written this. Before going into that let us briefly comment on Woronowicz's work and explain the criterion mentioned in the previous paragraph.

Woronowicz [21] works in the a priori special situation $E = \mathscr{A}$, i.e. E is the C^*-algebra $\mathscr{A}$ viewed as a Hilbert module over itself. This is not as special as it seems: for a general Hilbert C^*-module E it is shown in [15, Sec. 3] that there is a one–one correspondence between (semi)regular operators in E and (semi)regular operators in the C^*-algebra $\mathscr{K}(E)$ of $\mathscr{A}$-compact operators. Nevertheless, we do not quite agree with [15] that this fact allows "without any loss in generality" to restrict one-self to (semi)regular operators on C^*-algebras. After all changing the scalars from $\mathscr{A}$ to $\mathscr{K}(E)$ is rather substantial.

Woronowicz' criterion reads as follows: a closed semiregular operator T in E is regular if and only if its graph

$$\Gamma(T) := \{(x, y) \in E \oplus E \mid x \in \mathscr{D}(T),\ y = Tx\}$$

is complementable. This was proved in [21] for $E = \mathscr{A}$, the (straightforward) extension to the general case can be found in [13, Thm. 9.3 and Prop. 9.5].

In practical terms this criterion does not help much. One rather quickly sees that checking it boils down to solving the equation $(I + T^*T)x = y$.

Let us describe in non-technical terms the problem from which this paper arose. In our study of an approach to the KK-product for unbounded modules [8] we needed to study two selfadjoint regular operators S, T in a Hilbert C^*-module with "small" commutator (Section 7). More precisely, we were looking at unbounded odd Kasparov modules (D_1, X) and (D_2, Y) together with a densely defined connection ∇. The operator S then corresponds to $D_1 \otimes 1$ whereas T corresponds to $1 \otimes_\nabla D_2$. The Hilbert C^*-module is given by the interior tensor product of X and Y over some C^*-algebra. As an essential part of forming the unbounded Kasparov product of (D_1, X) and (D_2, Y) one needs to study the selfadjointness and regularity of the unbounded product operator

$$D := \begin{pmatrix} 0 & S - iT \\ S + iT & 0 \end{pmatrix}, \qquad \mathscr{D}(D) = \big(\mathscr{D}(S) \cap \mathscr{D}(T)\big)^2 \subset E \oplus E. \tag{1.1}$$

With some effort we could prove that this operator is selfadjoint but all efforts to prove regularity using Woronowicz' criterion failed. For a while we even started to look for counterexamples. On the other hand, in a Hilbert space regularity comes for free and the construction of D out of S and T was more or less "functorial".

So stated somewhat vaguely, the following principle should hold true: given a "functorial" construction of an operator $D = D(S, T)$ out of two selfadjoint and regular operators S, T. If then for Hilbert spaces this construction always produces a selfadjoint operator then $D(S, T)$ is selfadjoint and regular.

More rigorously, let us consider a closed, densely defined and, for simplicity, symmetric operator T in the Hilbert C^*-module E. Then for each state ω on $\mathscr{A}$ there is a canonical Hilbert space E^ω, a natural map $\iota_\omega : E \to E^\omega$ with dense range, and a symmetric operator T^ω which is defined by closing the operator defined by $T_0^\omega \iota_\omega(x) := \iota_\omega(Tx)$. We call T^ω the *localization* of T with respect to the state ω. One of the main results of this paper is the following Local–Global Principle. For the sake of brevity, it is stated here for symmetric operators. See Theorem 4.2 for the general case.

Theorem 1.1 *(Local–Global Principle). For a closed, densely defined and symmetric operator T the following statements are equivalent*:

(1) *T is selfadjoint and regular.*
(2) *For every state $\omega \in S(\mathscr{A})$ the localization T^ω is selfadjoint.*

The main tool for proving this theorem is the following separation theorem.

Theorem 1.2. *Let $L \subset E$ be a closed convex subset of the Hilbert C^*-module E over $\mathscr{A}$. For each vector $x_0 \in E \setminus L$ there exists a state ω on $\mathscr{A}$ such that $\iota_\omega(x_0)$ is not in the closure of $\iota_\omega(L)$. In particular there exists a state ω such that $\iota_\omega(L)$ is not dense in E^ω and hence $\iota_\omega(L)^\perp \neq \{0\}$.*

We will show by a couple of examples that the Local–Global Principle can easily be checked in concrete situations. We would find it aesthetically more appealing if in Theorems 1.1 and 1.2 one could replace “state” by “pure state”. We conjecture that this is true, but we can only prove it under additional assumptions on the Hilbert C^*-module. That pure states suffice in these cases turns out to be practically useful in Section 5 and in the discussion of examples of nonregular operators in Section 6. We therefore single out the following conjecture:

Conjecture 1.3. *If L is a proper submodule of the C^*-algebra $\mathscr{A}$ then there exists a pure state ω on $\mathscr{A}$ such that $\iota_\omega(L)^\perp \neq \{0\}$.*

Consequently, a closed densely defined symmetric operator in the Hilbert C^*-module E over $\mathscr{A}$ is regular if and only if for each pure state ω on $\mathscr{A}$ the localization T^ω is selfadjoint.

We close this introduction with a few remarks about the organization of the paper:

In Section 2 we collect the necessary background and notation. In particular (semi)regular operators and their localizations with respect to representations of the underlying C^*-algebra are introduced.

In Section 3 we prove the separation Theorem 1.2 and discuss various corner cases which illustrate that the separation theorem is not as obvious as it might seem. As a first application we show that a submodule $\mathscr{E} \subset \mathscr{D}(T)$ is a core for T if and only if for each state ω the subspace $\iota_\omega(\mathscr{E}) \subset \mathscr{D}(T^\omega)$ is a core for the localized operator T^ω (Theorem 3.5).

Section 4 contains the statement and proof of the main result of this paper, the Local–Global Principle characterizing the regularity of a semiregular operator T in terms of the Hilbert space localizations T^ω. The proof uses crucially the separation Theorem 3.1 in Section 3. To illustrate the power of the Local–Global Principle we generalize Wüst’s extension of the Kato–Rellich Theorem to Hilbert C^*-modules (Theorem 4.6).

Section 5 discusses various aspects of the conjectural refinement of the Local–Global Principle, Conjecture 1.3. We prove the conjecture for Hilbert C^*-modules over commutative C^*-algebras (Theorem 5.8) as well as for the Hilbert C^*-module $E = \mathscr{A}$ for any C^*-algebra (Theorem 5.10). Furthermore, it is shown that for a finitely generated Hilbert module over a commutative C^*-algebra every semiregular operator is regular (Theorem 5.9); this was earlier proved by Pal [15, Sec. 4] for the special module $E = \mathscr{A}$ for $\mathscr{A}$ commutative.

In Section 6 we will recast in a slightly more general context the known constructions for nonregular operators. Propositions 6.3 and 6.4 give a precise measure theoretic characterization for the regularity of a large class of semiregular operators acting on the Hilbert module $C(X, H)$ (X some compact space and H some Hilbert space). These results contain the known examples of nonregular operators as special cases.

Finally, Section 7 contains the regularity result which was the main motivation to write this paper, as explained above. We will study the regularity of sums $S \pm iT$ where S, T are selfadjoint regular operators in some Hilbert C^*-module with a technical condition on the size of the commutator $[S, T]$. We will make crucial use of this result in a subsequent publication on the unbounded Kasparov product, see [8].

2. Regular operators and their localizations

2.1. Notations and conventions

Script letters $\mathscr{A}, \mathscr{B}, \ldots$ denote (not necessarily unital) C^*-algebras. Hilbert C^*-modules over a C^*-algebra will be denoted by letters $E, F, \ldots$; H usually denotes a Hilbert space, i.e. a Hilbert C^*-module over $\mathbb{C}$. Recall that a Hilbert C^*-module over $\mathscr{A}$ is an $\mathscr{A}$-right module equipped with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle$. Furthermore, it is assumed that E is complete with respect to the induced norm $\|x\| := \|\langle x, x\rangle^{1/2}\| = \|\langle x, x\rangle\|^{1/2}$.

We will adopt the convention that inner products are conjugate *$\mathscr{A}$-linear* in the *first* variable and *linear* in the *second*. This convention is also adopted for Hilbert spaces. We let $\mathscr{L}(E)$ denote the C^*-algebra of bounded adjointable operators on E. Our standard reference for Hilbert C^*-modules is Lance [13].

2.2. Localizations of Hilbert C^*-modules, cf. [13, Chap. 5]

Let π be a representation of $\mathscr{A}$ on the Hilbert space H_π. We then get an induced representation π_E of $\mathscr{L}(E)$ on the *interior* tensor product $E \widehat{\otimes}_{\mathscr{A}} H_\pi$ [13, Chap. 4]. The latter is the Hilbert space obtained as the completion of the algebraic tensor product $E \otimes_{\mathscr{A}} H_\pi$ with respect to the inner product

$$\langle x \otimes h, x' \otimes h'\rangle = \langle h, \pi\big(\langle x, x'\rangle_{\mathscr{A}}\big) h'\rangle, \tag{2.1}$$

and for $T \in \mathscr{L}(E)$ one has $\pi_E(T)(x \otimes h) = (Tx) \otimes h$. We emphasize that by [13, Prop. 4.5] the inner product (2.1) on $E \otimes_{\mathscr{A}} H_\pi$ is indeed positive definite and hence $E \otimes_{\mathscr{A}} H_\pi$ may be viewed as a dense subspace of $E \widehat{\otimes}_{\mathscr{A}} H_\pi$. We call the Hilbert space $E \widehat{\otimes}_{\mathscr{A}} H_\pi$ the *localization* of E with respect to the representation π. If π is faithful then so is the induced representation π_E of $\mathscr{L}(E)$.

For cyclic representations one has a slightly different but equivalent description of $E \widehat{\otimes}_{\mathscr{A}} H_\pi$. Namely, let $\omega \in S(\mathscr{A})$ be a state. Then one can mimic the GNS construction for E as follows: ω gives rise to a (possibly degenerate) scalar product

$$\langle x, y\rangle_\omega := \omega\big(\langle x, y\rangle\big) \tag{2.2}$$

on E. $\mathscr{N}_\omega := \{x \in E \mid \langle x, x\rangle_\omega = 0\}$ is a subspace of E. $\langle\cdot,\cdot\rangle_\omega$ induces a scalar product on the quotient $E/\mathscr{N}_\omega$ and we denote by E^ω the Hilbert space completion of $E/\mathscr{N}_\omega$. We let $\iota_\omega : E \to E^\omega$

denote the natural map. Clearly ι_ω is continuous with dense range; it is injective if and only if ω is faithful.

Now let $(\pi_\omega, H_\omega, \xi_\omega)$ be the cyclic representation of $\mathscr{A}$ with cyclic vector ξ_ω associated with the state ω. One then has $\langle \xi_\omega, \pi_\omega(a)\xi_\omega\rangle = \omega(a)$ for $a \in \mathscr{A}$. Furthermore, the map

$$E^\omega \to E \mathbin{\widehat{\otimes}_A} H_\omega, \qquad \iota_\omega(e) \mapsto e \otimes \xi_\omega \tag{2.3}$$

is a unitary isomorphism. We will from now on tacitly identify E^ω with $E \mathbin{\widehat{\otimes}_{\mathscr{A}}} H_\omega$ and hence identify $\iota_\omega(e)$ with $e \otimes \xi_\omega$ where convenient.

2.3. *Semiregular and regular operators*

Following Pal [15] by a *semiregular* operator in E we will understand an operator $T : \mathscr{D}(T) \longrightarrow E$ defined on a dense $\mathscr{A}$-submodule $\mathscr{D}(T) \subset E$ and such that the adjoint T^* is densely defined, too.

This definition is the adaption of the notion of a densely defined closable operator in the Hilbert space setting. Pal also requires that T is closable but, as for Hilbert spaces, this indeed follows from the other assumptions:

Lemma 2.1. *Let T be a semiregular operator in E. Then T is $\mathscr{A}$-linear and closable. The adjoint T^* is closed and $T^* = (\overline{T})^*$. Here $\overline{T}$ denotes the closure of T.*

Proof. $\mathscr{A}$-linearity and closability are simple consequences of the fact that T^* is densely defined. E.g. let $(x_n) \subset \mathscr{D}(T)$ be a sequence such that $x_n \to 0$ and $Tx_n \to y$. Then for all $z \in \mathscr{D}(T^*)$

$$\langle y, z\rangle = \lim_{n\to\infty} \langle Tx_n, z\rangle = \lim_{n\to\infty} \langle x_n, T^*z\rangle = 0$$

and hence $y = 0$. This proves that T is closable. The remaining claims follow easily. □

Besides this one should not take for granted any of the properties one is used to from unbounded operators in Hilbert space. Semiregular operators in Hilbert C^*-modules can be rather pathologic, see e.g. [13, Chap. 9] and Section 6 below. We mention as a warning that in general $\overline{T} \subsetneqq T^{**}$, see Proposition 6.3 and the discussion thereafter.

A *closed semiregular* operator T is called *regular* if in addition $I + T^*T$ has dense range. It then follows that $I + T^*T$ is densely defined [13, Lemma 9.1] and invertible. Regular operators behave more or less as nicely as closed densely defined operators in Hilbert space. In particular for selfadjoint regular operators there is a continuous functional calculus [1,12,21,22].

Since the functional calculus will be needed, let us briefly describe it. Let $C_\infty(\mathbb{R})$ denote the algebra of continuous functions f on the real line such that f has limits as $x \to \pm\infty$. This algebra is isomorphic to the continuous functions on the compact interval $[-1, 1]$ via $C_\infty(\mathbb{R}) \ni f \mapsto \tilde{f} \in C[-1, 1]$, $\tilde{f}(x) := f(x/\sqrt{1-x^2})$. For a selfadjoint regular operator T the bounded transform $T(I + T^2)^{-1/2}$ is in $\mathscr{L}(E)$ (cf. [13, Chap. 10]). Putting $f(T) := \tilde{f}(T(I + T^2)^{-1/2})$ then yields a $*$-homomorphism $C_\infty(\mathbb{R}) \to \mathscr{L}(E)$ which sends the function $x \mapsto (1 + x^2)^{-1}$ to $(I + T^2)^{-1}$ and $x \mapsto x(1 + x^2)^{-1}$ to $T(I + T^2)^{-1}$.

While in a *Hilbert space* every densely defined closed operator is regular in general Hilbert C^*-modules there exist closed semiregular operators which are not regular, see Proposition 6.3.

Lemma 2.2. *(Cf. [13, Cor. 9.6].) Let T be a regular operator. Then T^* is regular, too. Furthermore $T = T^{**}$.*

Proof. Lance states this as an if and only if condition. As pointed out by Pal [15, Rem. 2.4 (ii)] Corollary 9.6 in [13] is not correct as stated. Indeed [15, Props. 2.2 and 2.3] shows that there exists a semiregular nonregular symmetric operator S such that S^* is selfadjoint and regular.

An inspection of the arguments preceding [13, Cor. 9.6] shows that the regularity of T indeed implies the regularity of T^*. Furthermore, then $T = T^{**}$ by [13, Cor. 9.4].

In case of the operator S one can still conclude the regularity of S^{**}. There is no contradiction here, it just follows that $S^{**} \neq \overline{S}$. □

Symmetry and selfadjointness are defined as usual as $T \subset T^*$ resp. $T = T^*$. The following reduction of the regularity problem to selfadjoint operators will be convenient.

Lemma 2.3. *Let T be a closed and semiregular operator and define*

$$\hat{T} := \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}. \tag{2.4}$$

Then $\hat{T}$ is a closed symmetric operator. Moreover, T is regular if and only if $\hat{T}$ is selfadjoint and *regular.*

Proof. That $\hat{T}$ is closed and symmetric is immediate.

If T is regular then by Lemma 2.2 T^* is also regular and $T^{**} = T$. Thus $\hat{T}$ is selfadjoint and

$$I + \hat{T}^2 = I + \hat{T}^*\hat{T} = \begin{pmatrix} I + T^*T & 0 \\ 0 & I + TT^* \end{pmatrix} = \begin{pmatrix} I + T^*T & 0 \\ 0 & I + T^{**}T^* \end{pmatrix} \tag{2.5}$$

is invertible.

Conversely, if $\hat{T}$ is selfadjoint and regular then the first two equalities in (2.5) hold and they show that T is regular. □

For closed operators in Hilbert space the domain equipped with the graph scalar product is in itself a Hilbert space. We briefly discuss the analogous construction for a semiregular operator T. For $x, y \in \mathscr{D}(T)$ put

$$\langle x, y\rangle_T := \langle x, y\rangle + \langle Tx, Ty\rangle. \tag{2.6}$$

It is straightforward to check that this turns $\mathscr{D}(T)$ into a pre-Hilbert C^*-module which is complete if and only if T is a closed operator. Furthermore, the natural inclusion $\iota_T : \mathscr{D}(T) \hookrightarrow E$ is a continuous $\mathscr{A}$-module homomorphism. Furthermore, we have

Proposition 2.4. *For a closed semiregular operator T the map ι_T is adjointable if and only if T is regular. In that case one has $\iota_T^* = (I + T^*T)^{-1}$, where the latter is viewed as a map $E \longrightarrow \mathscr{D}(T)$.*

We leave the simple proof to the reader, cf. also [13, Chap. 9].

2.4. Localizations of semiregular operators

Let E be a Hilbert C^*-module over some C^*-algebra $\mathscr{A}$. Furthermore, let π be a representation of $\mathscr{A}$ on the Hilbert space H_π. The construction of π_E in Section 2.2 can be extended to semiregular operators. Let T be a semiregular operator in E. We define T_0^π as unbounded operator in $E\widehat{\otimes}_{\mathscr{A}} H_\pi$ by

$$\mathscr{D}\big(T_0^\pi\big) := \mathscr{D}(T) \otimes_{\mathscr{A}} H_\pi, \qquad T_0^\pi(x \otimes h) := (Tx) \otimes h \in E \mathbin{\widehat{\otimes}_{\mathscr{A}}} H_\pi. \tag{2.7}$$

T_0^π is certainly well defined on the dense $\mathscr{A}$-submodule $\mathscr{D}(T) \otimes_{\mathscr{A}} H_\pi$. Furthermore, for $x \in \mathscr{D}(T)$, $y \in \mathscr{D}(T^*)$, $h_1, h_2 \in H_\pi$

$$\begin{aligned}\big\langle T_0^\pi(x \otimes h_1), y \otimes h_2\big\rangle &= \big\langle (Tx) \otimes h_1, y \otimes h_2\big\rangle = \big\langle h_1, \pi\big(\langle Tx, y\rangle\big)h_2\big\rangle \\ &= \big\langle h_1, \pi\big(\langle x, T^*y\rangle\big)h_2\big\rangle = \big\langle x \otimes h_1, \big(T^*\big)_0^\pi(y \otimes h_2)\big\rangle. \end{aligned} \tag{2.8}$$

This shows that the densely defined operator $(T^*)_0^\pi$ is contained in $(T_0^\pi)^*$. Let us summarize

Lemma 2.5. *For any representation (π, H_π) of $\mathscr{A}$ the operator T_0^π is densely defined and closable. Furthermore, $(T^*)_0^\pi \subset (T_0^\pi)^*$. We let T^π be the closure of T_0^π and call it the localization of T with respect to the representation (π, H_π). We have $(T^*)^\pi \subset (T^\pi)^*$.*

In particular if T is symmetric then the localization T^π is symmetric, too.

Finally we note that if $(\pi_\omega, H_\omega, \xi_\omega)$ is the cyclic representation associated to the state ω we write T_0^ω resp. T^ω for the localization viewed as an operator in E^ω. It follows from (2.3) that $\mathscr{D}(T_0^\omega) = \iota_\omega(\mathscr{D}(T))$ and $T_0^\omega(\iota_\omega x) = \iota_\omega(Tx)$.[1]

3. A separation theorem for Hilbert C^*-modules

In this section we are going to prove the following separation theorem which will be the main tool for proving the Local–Global Principle, Theorem 4.2, for regular operators.

Theorem 3.1. *Let $L \subset E$ be a closed convex subset of the Hilbert C^*-module E over $\mathscr{A}$. For each vector $x_0 \in E \setminus L$ there exists a state ω on $\mathscr{A}$ such that $\iota_\omega(x_0)$ is not in the closure of $\iota_\omega(L)$. In particular there exists a state ω such that $\iota_\omega(L)$ is not dense in E^ω and thus, when L is a submodule, $\iota_\omega(L)^\perp \neq \{0\}$.*

Remark 3.2. We emphasize that even if L is a submodule it is not necessarily complementable. If it is complementable then the statement of the Theorem is obvious. Namely, write $x_0 = x_0' + x_0''$ with $x_0' \in L$ and $x_0'' \in L^\perp$. $x_0'' \neq 0$ since $x_0 \notin L$. Furthermore, $\iota_\omega(x_0'') \in \iota_\omega(L)^\perp$ for any state and choosing ω such that $\omega(\langle x_o'', x_0''\rangle) \neq 0$ we have $\iota_\omega(x_0'') \neq 0$.

We mention two more pathologies of Hilbert C^*-modules which underline that the theorem should not be viewed as obvious.

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[1] Originally we considered only the localizations T^ω constructed on the Hilbert space E^ω (cf. (2.2)). We are indebted to Ryszard Nest for pointing out to us the more general construction via the interior tensor product.

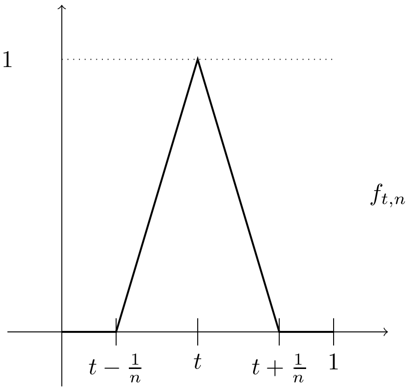

Fig. 1. The graph of the function $f_{t,n}$.

3.1. $\iota_\omega(L)$ can be dense for faithful ω

Let $\mathscr{A} = C[0,1]$, $E = C[0,1]$, $L = \{f \in C[0,1] \mid f(0) = 0\}$. L is a closed nontrivial submodule of E. The Lebesgue state $\omega(f) = \int_0^1 f(t)\,dt$ is faithful, $E^\omega \simeq L^2[0,1]$ and $\iota_\omega(L)$ is dense in E^ω. So even for faithful states, and hence for faithful representations, it may happen that $\iota_\omega(L)^\perp = \{0\}$.

3.2. Convex hulls of closed subsets of $\mathscr{A}_+ \setminus \{0\}$ may contain 0

The proof of Theorem 3.1 will proceed by applying the Hahn–Banach Theorem to the convex hull of the set

$$A := \bigl\{\langle y - x_0, y - x_0\rangle \bigm| y \in L\bigr\} \subset \mathscr{A}_+. \tag{3.1}$$

The closedness of L implies that $\inf\{\|a\| \mid a \in A\} > 0$. It will be crucial to show that the closure of the convex hull does not contain 0.

We illustrate by example that in general we cannot hope that if $A \subset \mathscr{A}_+$ with $\inf\{\|a\| \mid a \in A\} > 0$ that then $0 \notin \overline{\operatorname{co}(A)}$. Namely, we will construct a subset $A \subset \mathscr{A} = C[0,1]$ such that

- $A \subset \mathscr{A}_+$,
- $\|a\| \geqslant 1$ for all $a \in A$,
- $0 \in \overline{\operatorname{co}(A)}$.

For $0 < t < 1$ and $n \in \mathbb{Z}_+$ such that $0 < t - 1/n$, $t + 1/n < 1$ let (cf. Fig. 1)

$$f_{t,n}(x) := \begin{cases} 0, & |x - t| \geqslant 1/n, \\ 1 - n|x - t|, & |x - t| \leqslant 1/n. \end{cases} \tag{3.2}$$

Let $A = \{f_{t,n} \mid (t,n) \in \mathbb{Q} \times \mathbb{Z}_+\ 0 < t - 1/n < t < t + 1/n < 1\}$. Then A is a countable subset of $\mathscr{A}_+$ and

$$\inf\bigl\{\|f\| \bigm| f \in A\bigr\} = 1. \tag{3.3}$$

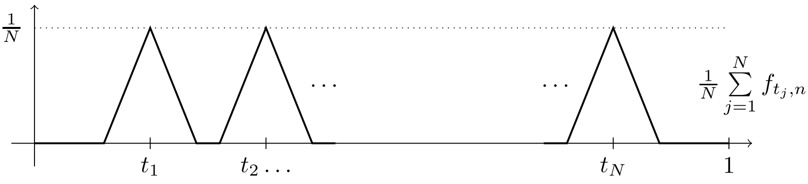

Fig. 2. Convex combination of $f_{t_j,n}$ with arbitrarily small norm.

Now let $\varepsilon > 0$ be given. Choose a natural number $N > 1/\varepsilon$ and put $t_j := j/(N+1)$, $j = 1, \dots, N$, $n := 2N + 2$. Then for $x \in [0,1]$ there is at most one index j with $f_{t_j,n}(x) \neq 0$. Hence for the convex combination $\frac{1}{N}\sum_{j=1}^{N} f_{t_j,n}$ we have

$$0 \leqslant \frac{1}{N}\sum_{j=1}^{N} f_{t_j,n}(x) \leqslant \frac{1}{N} < \varepsilon \tag{3.4}$$

showing that $0 \in \overline{\operatorname{co}(A)}$, cf. Fig. 2.

3.3. Counterexample for pure states

The previous construction can also be used to show that in Theorem 3.1 "state" cannot be replaced by "pure state". Namely, let $\mathscr{A} = E = C[0,1]$ and let L be the closed convex hull of the two functions $f_{1/4,5}$, $f_{3/4,5}$. Then certainly for each $f \in L$ we have $\|f\| \geqslant 1/2$, hence $x_0 = 0 \notin L$.

Now let ω be a pure state of $\mathscr{A}$. Then there is $p \in [0,1]$ such that $\omega(f) = f(p)$. Let $1 \geqslant \varepsilon > 0$ be given. If $p \leqslant 1/2$ then for $f = \varepsilon f_{1/4,5} + (1-\varepsilon) f_{3/4,5}$ we have $\omega(\langle f, f\rangle) \leqslant \varepsilon^2$. If $p \geqslant 1/2$ then put $f = (1-\varepsilon) f_{1/4,5} + \varepsilon f_{3/4,5}$. This argument shows that $0 = \iota_\omega(x_0)$ is in the closure if $\iota_\omega(L)$.

3.4. Proof of Theorem 3.1

Let now A be the set defined in Eq. (3.1). Since L is closed we have

$$\delta := \inf\bigl\{\|y - x_0\|^2 \bigm| y \in L\bigr\} = \inf\bigl\{\|a\| \bigm| a \in A\bigr\} > 0. \tag{3.5}$$

To apply the Hahn–Banach Theorem we need to show that $0 \notin \overline{\operatorname{co}(A)}$.

To this end we consider arbitrary $y_1, \dots, y_n \in E$ and real numbers $\lambda_j \geqslant 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. Then (cf. [13, Lemmas 4.2 and 4.3])

$$\begin{aligned}
\sum_{k,l=1}^{n} \lambda_k \lambda_l \langle y_k, y_l\rangle &= \sum_{k=1}^{n} \lambda_k^2 \langle y_k, y_k\rangle + \sum_{k<l} \lambda_k \lambda_l \bigl(\langle y_k, y_l\rangle + \langle y_l, y_k\rangle\bigr) \\
&\leqslant \sum_{k=1}^{n} \lambda_k^2 \langle y_k, y_k\rangle + \sum_{k<l} \lambda_k \lambda_l \bigl(\langle y_k, y_k\rangle + \langle y_l, y_l\rangle\bigr) \\
&= \sum_{k=1}^{n} \lambda_k \langle y_k, y_k\rangle.
\end{aligned} \tag{3.6}$$

Here we have used

$$\langle x, y\rangle + \langle y, x\rangle \leqslant \langle x, x\rangle + \langle y, y\rangle \tag{3.7}$$

which can be seen by expanding $\langle x-y, x-y\rangle \geqslant 0$.

Consider the convex combination $\sum_{j=1}^n \lambda_j\langle y_j - x_0, y_j - x_0\rangle$, $y_1,\dots,y_n \in L$, of elements of A. Using (3.6) we find

$$\begin{aligned}\sum_{j=1}^n \lambda_j\langle y_j - x_0, y_j - x_0\rangle &= \langle x_0, x_0\rangle - \sum_{j=1}^n \lambda_j\big(\langle y_j, x_0\rangle + \langle x_0, y_j\rangle\big) + \sum_{j=1}^n \lambda_j\langle y_j, y_j\rangle \\ &\geqslant \langle x_0, x_0\rangle - \sum_{j=1}^n \lambda_j\big(\langle y_j, x_0\rangle + \langle x_0, y_j\rangle\big) + \sum_{k,l=1}^n \lambda_k\lambda_l\langle y_k, y_l\rangle \\ &= \left\langle x_0 - \sum_{j=1}^n \lambda_j y_j, x_0 - \sum_{j=1}^n \lambda_j y_j\right\rangle. \end{aligned} \tag{3.8}$$

Since L is assumed to be convex, $\sum \lambda_j y_j \in L$, hence (3.5) and (3.8) give

$$\left\| \sum_{j=1}^n \lambda_j\langle y_j - x_0, y_j - x_0\rangle \right\| \geqslant \delta. \tag{3.9}$$

This shows that each element b in the closure of the convex hull $\overline{\mathrm{co}(A)}$ of A satisfies $\|b\| \geqslant \delta$. This proves that $0 \notin \overline{\mathrm{co}(A)}$.

The Hahn–Banach separation theorem now implies the existence of a continuous linear functional $\varphi : \mathscr{A}_{\mathrm{sa}} \to \mathbb{R}$ and an $\varepsilon > 0$ such that $\varphi(b) > \varepsilon$ for all $b \in \overline{\mathrm{co}(A)}$. Here $\mathscr{A}_{\mathrm{sa}}$ denotes the real Banach space of selfadjoint elements in the C^*-algebra $\mathscr{A}$. We extend the linear functional φ to a selfadjoint linear functional on the C^*-algebra $\mathscr{A}$ by defining

$$\tau : \mathscr{A} \to \mathbb{C}, \qquad \tau(x) := \varphi\left(\frac{x + x^*}{2}\right) + i\varphi\left(\frac{x - x^*}{2i}\right).$$

By Jordan decomposition for C^*-algebras we can then find two positive linear functionals $\omega_\pm \in \mathscr{A}_+^*$ such that $\tau = \omega_+ - \omega_-$. Hence $\omega_+(b) \geqslant \varphi(b) > \varepsilon$ for all $b \in \overline{\mathrm{co}(A)} \subseteq \mathscr{A}_+$. Putting $\omega = \omega_+/\|\omega_+\|$ we see that in E^ω the vector $\iota_\omega(x_0)$ and the subspace $\iota_\omega(L)$ have distance at least $\sqrt{\varepsilon/\|\omega_+\|} > 0$ which proves the claim. □

3.5. Application: A core-criterion for semiregular operators

Theorem 3.3. *Suppose that T is a closed and semiregular operator in the Hilbert $\mathscr{A}$-module E. Let $\mathscr{E} \subseteq \mathscr{D}(T)$ be a submodule of the domain of T. The following statements are then equivalent*:

(1) *The submodule $\mathscr{E}$ is a core for $\mathscr{D}(T)$.*
(2) *For every representation (π, H_π) of $\mathscr{A}$ the subspace $\mathscr{E} \otimes_{\mathscr{A}} H_\pi$ is a core for T^π.*
(3) *For every state $\omega \in S(\mathscr{A})$ the subspace $\iota_\omega(\mathscr{E}) \subseteq \mathscr{D}(T^\omega)$ is a core for the localization T^ω.*

Proof. Firstly, the implication (2) $\Rightarrow$ (3) is clear. Secondly, we note that for any representation (π, H_π) the scalar product on $\mathscr{D}(T) \otimes_{\mathscr{A}} H_\pi$ (induced by the graph scalar product on $\mathscr{D}(T)$) equals the graph scalar product of T^π. Namely, for $x, y \in \mathscr{D}(T)$, $h, h' \in H_\pi$ we have

$$\begin{aligned}
\langle x \otimes h, y \otimes h' \rangle_{\mathscr{D}(T) \otimes_{\mathscr{A}} H_\pi} &= \langle h, \pi(\langle x, y \rangle_T) h' \rangle \\
&= \langle h, \pi(\langle x, y \rangle) h' \rangle + \langle h, \pi(\langle Tx, Ty \rangle) h' \rangle \\
&= \langle x \otimes h, y \otimes h \rangle_{E \otimes_{\mathscr{A}} H_\pi} + \langle T^\pi(x \otimes h), T^\pi(y \otimes h) \rangle_{E \otimes_{\mathscr{A}} H_\pi} \\
&= \langle x \otimes h, y \otimes h' \rangle_{T^\pi}.
\end{aligned} \tag{3.10}$$

This shows that $\mathscr{D}(T) \widehat{\otimes}_{\mathscr{A}} H_\pi = \mathscr{D}(T^\pi)$ as Hilbert spaces.

In light of this if $\mathscr{E}$ is dense in $\mathscr{D}(T)$ then so is $\mathscr{E} \otimes_{\mathscr{A}} H_\pi$ in $\mathscr{D}(T^\pi)$ proving (1) $\Rightarrow$ (2).

$\neg$(1) $\Rightarrow$ $\neg$(3). If $\mathscr{E}$ is not a core for T then there exists a vector $x_0 \in \mathscr{D}(T) \setminus \overline{\mathscr{E}}$. Hence by Theorem 3.1 there exists a state ω such that $\iota_\omega^T(x_0) = x_0 \otimes \xi_\omega$ is not in the closure of $\iota_\omega^T(\mathscr{E})$. Here ι_ω^T denotes the natural map $\mathscr{D}(T) \longrightarrow \mathscr{D}(T)^\omega$. Thus $\iota_\omega(\mathscr{E})$ is not a core for T^ω. $\square$

4. The Local–Global Principle

Before we prove the main theorem of this section we recall the characterization of selfadjoint regular operators in terms of the range of the operators $T \pm i$, [13, Lemmas 9.7 and 9.8]:

Proposition 4.1. *Let T be a closed, densely defined and symmetric operator in the Hilbert C^*-module E over $\mathscr{A}$. Then for $\mu \in \mathbb{R} \setminus \{0\}$ the operator $T \pm i\mu$ is injective and has closed range. Furthermore, the following statements are then equivalent*:

(1) *The unbounded operator T is selfadjoint and regular.*
(2) *There exists $\mu > 0$ such that each of the operators $T + i\mu$ and $T - i\mu$ has dense range.*

It then follows that $T \pm i\mu$ is invertible for all $\mu \in \mathbb{R} \setminus \{0\}$. In [13] (2) is stated for $\mu = 1$. The slight extension to arbitrary nonzero μ is proved as in the Hilbert space setting and left to the reader.

We remark that regularity is a consequence of selfadjointness when the Hilbert C^*-module is a Hilbert space. This property and the separation theorem for Hilbert C^*-modules proved in Section 3 are applied in the proof of the next theorem.

Theorem 4.2 *(Local–Global Principle).*

1. *For a closed semiregular operator T in a Hilbert C^*-module the following statements are equivalent*:
   (1) *T is regular.*
   (2) *For every representation (π, H_π) of $\mathscr{A}$ the localizations T^π and $(T^*)^\pi$ are adjoints of each other, i.e. $(T^*)^\pi = (T^\pi)^*$.*
   (3) *For every state $\omega \in S(\mathscr{A})$ the localizations T^ω and $(T^*)^\omega$ are adjoints of each other.*
2. *For a closed, densely defined and symmetric operator T the following statements are equivalent*:

(1) *T is selfadjoint and regular.*
(2) *For every representation (π, H_π) of $\mathscr{A}$ the localization T^π is selfadjoint.*
(3) *For every state $\omega \in S(\mathscr{A})$ the localization T^ω is selfadjoint.*

Remark 4.3. We note that under 1.(2) the identity $(T^*)^\pi = (T^\pi)^*$ implies $T^\pi = ((T^*)^\pi)^*$ since T^π is a closed operator in a Hilbert space and therefore $T^\pi = (T^\pi)^{**}$.

Proof. In light of Lemma 2.3 it suffices to prove 2. The implication (2) $\Rightarrow$ (3) is obvious.

(1) $\Rightarrow$ (2). Assume that T is selfadjoint and regular and let (π, H_π) be a representation of $\mathscr{A}$. By Proposition 4.1 we only need to prove that $T^\pi + i$ and $T^\pi - i$ have dense range. W.l.o.g. consider $T^\pi + i$. Since $E \otimes_{\mathscr{A}} H_\pi$ is dense in E^π and by linearity it suffices to show that $x \otimes h \in \operatorname{ran}(T^\pi + i)$ for $x \in E$ and $h \in H_\pi$. Since T is selfadjoint and regular $T + i$ is surjective and hence $y := (T + i)^{-1} x \in \mathscr{D}(T)$ exists. Then $(T^\pi + i)(y \otimes h) = x \otimes h$.

(3) $\Rightarrow$ (1). Next we prove that the selfadjointness of all the localized operators imply the selfadjointness and regularity of the global operator.

Thus assume that the localized operator T^ω is selfadjoint for each state $\omega \in S(A)$. Assume by contradiction that the range of $T + i$ is not dense in E. By Proposition 4.1 the range $\operatorname{ran}(T + i)$ is a proper closed submodule of E. By Theorem 3.1 there exists a state $\omega \in S(\mathscr{A})$ such that

$$\overline{i_\omega\big(\operatorname{ran}(T+i)\big)} \neq E^\omega. \tag{4.1}$$

However, we also have the identities of subspaces

$$\overline{i_\omega\big(\operatorname{ran}(T+i)\big)} = \overline{\operatorname{ran}\big(T_0^\omega + i\big)} = \operatorname{ran}\big(T^\omega + i\big).$$

Thus $T^\omega + i$ does not have dense range which is in contradiction with the selfadjointness of T^ω. The same argument shows that the operator $T - i$ has dense range as well and the theorem is proved. $\square$

Remark 4.4. The PhD-thesis of Baaj [1] seems to be the earliest detailed treatment of regular operators, though only for the special case where the Hilbert C^*-module is the C^*-algebra itself. This work contains the functional calculus as well as both of the implications (1) $\Rightarrow$ (2) in Theorem 4.2.

4.1. Application: Wüst's extension of the Kato–Rellich Theorem

The Kato–Rellich Theorem [18, Thm. X.12] extends to Hilbert C^*-modules without any difficulty.

Theorem 4.5 *(Kato–Rellich). Let $T : \mathscr{D}(T) \to E$ be a selfadjoint regular operator and let $V : \mathscr{D}(V) \to E$ be a symmetric operator such that $\mathscr{D}(T) \subseteq \mathscr{D}(V)$. Suppose that V is relatively T-bounded with relative bound < 1. That is there exist $a \in (0,1)$, $b \in \mathbb{R}_+$ such that for $x \in \mathscr{D}(T)$*

$$\|Vx\| \leqslant a\|Tx\| + b\|x\|.$$

Then $T + V$ with domain $\mathscr{D}(T)$ is selfadjoint and regular.

Proof. The standard Hilbert space proof extends to this situation: namely, for $\mu \in \mathbb{R}_+$ large enough the operators $T + V \pm i\mu = (I + V(T \pm i\mu)^{-1})(T \pm i\mu)$ is invertible and hence $T + V$ is selfadjoint and regular. $\square$

The proof of Wüst's extension to the case of relative bound 1 [18, Theorem X.14] makes heavy use of the fact that Hilbert spaces are self-dual and of weak compactness of the unit ball. These tools are not available for Hilbert C^*-modules. Our Local–Global Principle allows us to generalize Wüst's Theorem as follows:

Theorem 4.6 *(Wüst). Let $T : \mathscr{D}(T) \to E$ be a selfadjoint regular operator and let $V : \mathscr{D}(V) \to E$ be a symmetric operator such that $\mathscr{D}(T) \subseteq \mathscr{D}(V)$. Suppose that there exists a $b \in \mathbb{R}_+$ such that for $x \in \mathscr{D}(T)$*

$$\langle Vx, Vx\rangle \leqslant \langle Tx, Tx\rangle + b\langle x, x\rangle.$$

Then $T + V$ with domain $\mathscr{D}(T)$ is essentially selfadjoint and regular.

Proof. Let $\omega \in S(\mathscr{A})$ be a state of $\mathscr{A}$. Then we have for $x \in \mathscr{D}(T)$

$$\begin{aligned} \big\| V_0^\omega\big(\iota_\omega(x)\big)\big\|^2 &= \omega\big(\langle Vx, Vx\rangle\big) \leqslant \omega\big(\langle Tx, Tx\rangle\big) + b \cdot \omega\big(\langle x, x\rangle\big) \\ &= \big\| T_0^\omega \iota_\omega(x)\big\|^2 + b \cdot \big\|\iota_\omega(x)\big\|^2, \end{aligned} \tag{4.2}$$

thus V_0^ω is relative T_0^ω-bounded with relative bound 1. Taking closures shows that V^ω is relative T^ω-bounded with relative bound 1, too. By Wüst's Theorem [18, Thm. X.14] it follows that $T^\omega + V^\omega$ is essentially selfadjoint on any core for T^ω. In particular it is essentially selfadjoint on $\iota_\omega(\mathscr{D}(T))$. For $\iota_\omega(x)$, $x \in \mathscr{D}(T)$, however, we have $(T^\omega + V^\omega)\iota_\omega(x) = \iota_\omega(Tx + Vx) = (T + V)_0^\omega \iota_\omega(x)$. Thus the localization $(T + V)^\omega$ of $T + V$ is selfadjoint.

The claim now follows from Theorem 4.2. $\square$

5. Pure states, commutative algebras and involutive Hilbert C^*-modules

Section 3.3 shows that in Theorem 3.1 one cannot conclude that ω can be chosen to be pure.

Definition 5.1. 1. $\{\varrho_j\}_{j=1}^n$ is called a partition of unity if

(1) $\varrho_j \in \mathscr{A}$, $\quad j = 1, \ldots, n-1$ and $\varrho_n \in \mathscr{A}^+$,
(2) $\sum_{j=1}^n \varrho_j^* \varrho_j = I$.

Here $\mathscr{A}^+$ is $\mathscr{A}$ if $\mathscr{A}$ is unital and otherwise it denotes the unitalization of $\mathscr{A}$; I is the unit in $\mathscr{A}^+$.

2. A subset $A \subset \mathscr{A}$ is called $\mathscr{A}$-convex if for any $x_1, \ldots, x_n \in A$ and a partition of unity $\varrho_j \in \mathscr{A}$, $j = 1 \ldots, n$ one has

$$\sum_{j=1}^n \varrho_j^* x_j \varrho_j \in A.$$

Conjecture 5.2. *If in the situation of Theorem* 3.1 *L is an $\mathscr{A}$-submodule then there exists a pure state ω such that $\iota_\omega(x_0)$ is not in the closure of $\iota_\omega(L)$. In particular there exists a pure state ω such that $\iota_\omega(L)$ is not dense in E^ω and hence $\iota_\omega(L)^\perp \neq \{0\}$.*

Conjecture 5.3. *Let $\mathscr{A}$ be a C^*-algebra and let $A \subset \mathscr{A}_+$ be a closed $\mathscr{A}$-convex subset of the positive cone of $\mathscr{A}$. If $0 \notin A$ then there exist an $\varepsilon > 0$ and a pure state ω such that $\omega(a) \geqslant \varepsilon$ for all $a \in A$.*

Conjecture 5.4. *Conjecture* 5.3 *implies Conjecture* 5.2.

Remark 5.5. Theorem 3.1 and its proof show that if one replaces "$\mathscr{A}$-convex" by the weaker condition "convex" then all three conjectures hold true if one replaces "pure state" by the weaker conclusion "state". Section 3.3 shows that under the weaker condition "convex" the statement of Conjecture 5.3 becomes false for pure states.

Theorem 5.6. *If Conjecture* 5.2 *holds for a C^*-algebra $\mathscr{A}$ then in statement* (3) *under* 1. *and* 2. *of Theorem* 4.2 *"state" can be replaced by "pure state".*

Remark 5.7. Actually, for this conclusion to hold the second sentence in Conjecture 5.2 suffices.

Proof. One argues as in the proof of Theorem 4.2 replacing the conclusion of Theorem 3.1 by that of Conjecture 5.2. With regard to the previous remark we emphasize that indeed in (4.1) only the last sentence of Theorem 3.1 was used. □

5.1. Commutative algebras

We are now going to prove that all three conjectures are true for commutative C^*-algebras.

Theorem 5.8. *If $\mathscr{A}$ is commutative then Conjectures* 5.2*,* 5.3*, and* 5.4 *hold.*

Proof. 5.3 $\Rightarrow$ 5.2 The inequalities (3.6) and (3.8) are proved verbatim for $\lambda_j = \varrho_j^* \varrho_j$ and $y_j \in L$. Since $\mathscr{A}$ is commutative they can be checked pointwise on the Gelfand spectrum of $\mathscr{A}$. Thus as in the proof of Theorem 3.1 one concludes that the $\mathscr{A}$-convex hull $\widetilde{A}$ of the set A defined in (3.1) does not have 0 in its closure. Conjecture 5.3 now gives us a pure state ω which implies the validity of Conjecture 5.2.

We will now prove Conjecture 5.3 for $\mathscr{A}$ commutative. Let X be the Gelfand spectrum of $\mathscr{A}^+$. This is a compact Hausdorff space. If $\mathscr{A}$ is unital then $\mathscr{A} \simeq C(X)$ and if $\mathscr{A}$ is non-unital then there is a distinguished point $\infty \in X$ such that $\mathscr{A} \simeq \{f \in C(X) \mid f(\infty) = 0\}$.

Furthermore, each $p \in X$ ($X \setminus \{\infty\}$ in the non-unital case) gives rise to a pure state $\omega_p(f) = f(p)$ and every pure state arises in this way.

We proceed by contradiction and assume that the conclusion of Conjecture 5.3 does not hold. Then for given $\varepsilon > 0$ and each $p \in X$ there exist an open neighborhood U_p of p and an $f_p \in A$ with $f_p(q) \leqslant \varepsilon$ for $q \in U_p$. By compactness there exist finitely many $p_1, \dots, p_n$ such that $X = U_{p_1} \cup \dots \cup U_{p_n}$. In the non-unital case we may choose and enumerate them such that $\infty \in U_{p_n}$ and $\infty \notin U_{p_j}$ for $j = 1, \dots, n-1$. Since compact spaces are paracompact there exists a subordinated partition of unity $\chi_1, \dots, \chi_n$, $\chi_j \in C_c(U_{p_j})$. The set $\{\sqrt{\chi_j}\}_{j=1}^n$ is then a partition of unity in the sense of Definition 5.1. Hence $f := \sum_{j=1}^n \chi_j f_{p_j} = \sum_{j=1}^n \sqrt{\chi_j}\, f_{p_j} \sqrt{\chi_j}$ is in A by $\mathscr{A}$-convexity and it satisfies $0 \leqslant f \leqslant \varepsilon$. This shows that 0 is in A contradicting the assumption $0 \notin A$. □

We can now easily deduce the following generalization of a result of Pal [15, Prop. 4.1] about regular operators over commutative C^*-algebras.

Theorem 5.9. *Let E be a* finitely *generated Hilbert C^*-module over the commutative C^*-algebra $\mathscr{A}$. Then every semiregular operator T in E is regular.*

Proof. As before let X be the Gelfand spectrum of $\mathscr{A}^+$. By Theorem 5.8 it suffices to show that for each pure state ω_p, $p \in X$, the localized operator T^ω satisfies $(T^*)^\omega = (T^\omega)^*$.

Let $f_1, \dots, f_N \in E$ be a generating set over $\mathscr{A}$. Recall that E^ω is the Hilbert space completion of $E/\mathscr{N}_\omega$ with respect to the scalar product $\langle f, g\rangle(p)$. Given a fixed $f \in E$. Then for $\varphi \in \mathscr{A}$ the vector $\iota_\omega(f\varphi)$ depends only on the value $\varphi(p)$ and hence E^ω is the vector space spanned by $\iota_\omega(f_1), \dots, \iota_\omega(f_N)$ and thus is finite-dimensional. T^ω and $(T^*)^\omega$ are, as densely defined closed operators in a finite–dimensional vector space, everywhere defined and bounded. The inclusion $(T^*)^\omega \subset (T^\omega)^*$ (Lemma 2.5) then implies $(T^*)^\omega = (T^\omega)^*$. □

5.2. Regular operators over C^*-algebras and involutive Hilbert C^*-modules

Theorem 5.10. *Let $\mathscr{A}$ be a C^*-algebra. Then for the Hilbert C^*-module $E = \mathscr{A}$ the conclusion of the last sentence in Conjecture* 5.2 *holds and hence in statement* (3) *under* 1. *and* 2. *of Theorem* 4.2 *"state" can be replaced by "pure state".*

Remark 5.11. 1. So for unbounded semiregular operators over C^*-algebras regularity can be checked by looking at the localizations with respect to pure states. We emphasize that [15, Sec. 3] does not help in extending this result to semiregular operators over general Hilbert C^*-modules because [15] only shows that a (semi)regular operator in E is equivalent to an operator in $\mathscr{K}(E)$. Our theorem then says that one could check regularity now by looking at the localizations of the equivalent operator with respect to the pure states of $\mathscr{K}(E)$.

2. Theorem 5.10 can in principle be extracted from [21, Prop. 2.5], although there it is not stated explicitly. Our proof, however, is basically the same as the one in [21].

Proof. Let L be a proper $\mathscr{A}$ right submodule of E. Then L is a proper *right ideal* in $\mathscr{A}$ and hence by [5, Thm. 2.9.5] there exists a pure state $\omega \in S(\mathscr{A})$ such that $\omega \restriction L = 0$. Hence $\iota_\omega(L) = \{0\}$ but since ω is a state certainly $E^\omega \neq \{0\}$ proving $\iota_\omega(L)^\perp \neq \{0\}$. □

The argument of the previous proof exploits that the Hilbert C^*-module $E = \mathscr{A}$ has a little more structure: Namely, it is a left- and a right module. The inner product compatible with the right module structure is $\langle a, b\rangle = a^*b$ and the inner product compatible with the left module structure is $\langle a, b\rangle_l = ab^*$. Obviously, the involution $* : \mathscr{A} \to \mathscr{A}$ has the property $\langle a, b\rangle = \langle a^*, b^*\rangle_l$. This motivates

Definition 5.12. An involutive Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module E together with a bounded involution $* : E \to E$.

Putting $ax := (xa^*)^*$ and $\langle x, y\rangle_l := \langle x^*, y^*\rangle$ for $a \in \mathscr{A}$, $x, y \in E$ gives E the structure of an $\mathscr{A}$ Bimodule such that $(E, \langle\cdot,\cdot\rangle_l)$ is a left Hilbert $\mathscr{A}$-module and $(E, \langle\cdot,\cdot\rangle)$ is a right Hilbert $\mathscr{A}$-module.

For any C^*-algebra $\mathscr{A}$ the space $E = \mathscr{A}^n$ is an involutive Hilbert module via $(a_j)_j^* := (a_j^*)_j$. However, for noncommutative $\mathscr{A}$ the countably generated Hilbert module $H_{\mathscr{A}}$ is not necessarily involutive since $(a_j)_j \mapsto (a_j^*)_j$ is not necessarily bounded. (Except for the trivial commutative case we do not know of other interesting involutive Hilbert C^*-modules.)

We believe that Theorem 5.10 extends to full involutive Hilbert C^*-modules. But in the lack of good examples we do not follow this path any further and leave the details to the reader.

6. Examples of nonregular operators

In this section we will recast in a slightly more general context the known constructions of nonregular operators [13, Chap. 9], [15].

Fix a separable Hilbert space and a symmetric closed operator D with deficiency indices $(1, 1)$. E.g. $H = L^2[0, 1]$,

$$\mathscr{D}(D) := H_0^1[0,1] = \left\{ f \in L^2[0,1] \;\middle|\; f' \in L^2[0,1],\ f(0) = f(1) = 0 \right\}$$

and $Df := -if'$ will do.

As usual we put $D_{\min} := D$, $D_{\max} := D^*$. The domain $\mathscr{D}(D_{\max})$ is a Hilbert space in its own right with respect to the graph scalar product (cf. (2.6)) and there are two normalized vectors $\phi_\pm$ such that

$$\begin{aligned} D_{\max}\phi_\pm &= \pm i\phi_\pm, \\ \mathscr{D}(D_{\max}) &= \mathscr{D}(D_{\min}) \oplus \mathbb{C}\phi_+ \oplus \mathbb{C}\phi_-, \end{aligned} \tag{6.1}$$

where the orthogonal sum and the normalization of $\phi_\pm$ are understood with respect to the graph scalar product of $D_{\max}$. Therefore,

$$1 = \|\phi_\pm\|_D^2 = \|\phi_\pm\|^2 + \|\pm i\phi_\pm\|^2 = 2\|\phi_\pm\|^2, \qquad \|\phi_\pm\| = \frac{1}{\sqrt{2}}. \tag{6.2}$$

We introduce two continuous linear functionals

$$\alpha_\pm : \mathscr{D}(D_{\max}) \longrightarrow \mathbb{C}, \qquad \alpha_\pm(\xi) := \langle \phi_\pm, \xi\rangle_D. \tag{6.3}$$

With these notations the selfadjoint extensions of D are parametrized by $\lambda \in S^1$: for $\lambda \in S^1$ the operator D_λ is $D_{\max}$ restricted to

$$\mathscr{D}(D_\lambda) = \left\{ \xi \in \mathscr{D}(D_{\max}) \;\middle|\; \alpha_+(\xi) = \lambda\alpha_-(\xi) \right\}. \tag{6.4}$$

With

$$\eta_\lambda := \frac{1}{\sqrt{2}}(\lambda\phi_+ + \phi_-), \qquad \eta_\lambda^\perp := \frac{1}{\sqrt{2}}(\phi_+ - \bar{\lambda}\phi_-), \tag{6.5}$$

we have

$$\mathscr{D}(D_{\max}) = \mathscr{D}(D_{\min}) \oplus \mathbb{C}\eta_\lambda \oplus \mathbb{C}\eta_\lambda^\perp = \mathscr{D}(D_\lambda) \oplus \mathbb{C}\eta_\lambda^\perp, \tag{6.6}$$

hence $\xi \in \mathscr{D}(D_{\max})$ lies in $\mathscr{D}(D_\lambda)$ iff $\xi \perp \eta_\lambda^\perp$ with respect to the graph scalar product, equivalently if

$$\bigl\langle \xi, D_{\max}(\lambda\phi_+ + \phi_-)\bigr\rangle = \langle D_{\max}\xi, \lambda\phi_+ + \phi_-\rangle. \tag{6.7}$$

Next let X be a locally compact Hausdorff space, $C_0(X)$ the C^*-algebra of continuous functions which vanish at infinity. $E := C_0(X, H) = H \widehat{\otimes}_{C_0(X)} C_0(X)$ is the standard Hilbert C^*-module over $C_0(X)$ modeled on H with inner product

$$\langle f, g\rangle(x) := \bigl\langle f(x), g(x)\bigr\rangle_H. \tag{6.8}$$

We are now going to introduce semiregular operators $T_{\min}, T_{\max}$ as follows: let $\mathscr{D}(T_{\max/\min}) := C_0(X, \mathscr{D}(D_{\max/\min})) = \mathscr{D}(D_{\max/\min}) \widehat{\otimes}_{C_0(X)} C_0(X)$. These are Hilbert C^*-modules and we have natural continuous inclusions

$$\mathscr{D}(T_{\min}) \hookrightarrow \mathscr{D}(T_{\max}) \hookrightarrow E. \tag{6.9}$$

For $f \in \mathscr{D}(T_{\max/\min})$ put $(T_{\max/\min} f)(x) := D_{\max/\min}(f(x))$.

Lemma 6.1. *$T_{\max/\min}$ are closed regular operators in E with $T_{\max}^* = T_{\min}$, $T_{\min}^* = T_{\max}$. Furthermore, the natural inner products on $C_0(X, \mathscr{D}(D_{\max/\min}))$ coincide with the graph inner products of $T_{\max/\min}$. Hence by Proposition* 2.4 *the inclusion maps* (6.9) *are adjointable.*

Proof. Certainly for $f, g \in C_0(X, \mathscr{D}(D_{\max}))$ we have

$$\begin{aligned}
\langle f, g\rangle_{C_0(X,\mathscr{D}(D_{\max}))}(x) &= \bigl\langle f(x), g(x)\bigr\rangle_{\mathscr{D}(D_{\max})} \\
&= \bigl\langle f(x), g(x)\bigr\rangle_H + \bigl\langle D_{\max}\bigl(f(x)\bigr), D_{\max}\bigl(g(x)\bigr)\bigr\rangle_H \\
&= \langle f, g\rangle_E(x) + \langle T_{\max} f, T_{\max} g\rangle_E(x) \\
&= \langle f, g\rangle_{T_{\max}}(x),
\end{aligned} \tag{6.10}$$

proving the claim about graph inner products. Since $C_0(X, \mathscr{D}(D_{\max/\min}))$ are Hilbert C^*-modules this also shows that $T_{\max/\min}$ are closed operators.

If $f \in \mathscr{D}(T_{\min}^*)$ then for each $g \in \mathscr{D}(T_{\min}) = C_0(X, \mathscr{D}(D_{\min}))$ and each $x \in X$

$$\bigl\langle \bigl(T_{\min}^* f\bigr)(x), g(x)\bigr\rangle = \bigl\langle f(x), D_{\min}\bigl(g(x)\bigr)\bigr\rangle, \tag{6.11}$$

hence $f(x) \in \mathscr{D}(D_{\max})$ and $D_{\max}(f(x)) = (T_{\min}^* f)(x)$. This proves that $f \in \mathscr{D}(T_{\max})$ and $T_{\max} f = T_{\min}^* f$. This argument proves $T_{\min}^* \subset T_{\max}$. The inclusion $T_{\max} \subset T_{\min}^*$ is obvious and hence we have equality. The equality $T_{\max}^* = T_{\min}$ now follows similarly.

To prove regularity we only have to note that for given $f \in E$ the elements defined by $g_1(x) := (I + D_{\max} D_{\min})^{-1} f(x)$ resp. $g_2(x) := (I + D_{\min} D_{\max})^{-1} f(x)$ are in E and even more lie in the domain of $T_{\max} T_{\min}$ resp. $T_{\min} T_{\max}$ and $(I + T_{\max} T_{\min}) g_1 = f$ resp. $(I + T_{\min} T_{\max}) g_1 = f$. $\square$

We are now going to study semiregular extensions $T_{\min} \subset T_\Lambda \subset T_{\max}$ which depend on a Borel function $\Lambda : X \longrightarrow S^1$. For a Borel function Λ we let T_Λ be the operator $T_{\max}$ restricted to

$$\mathscr{D}(T_\Lambda) := \bigl\{ f \in \mathscr{D}(T_{\max}) \bigm| \alpha_+ \circ f = \Lambda \cdot (\alpha_- \circ f) \bigr\}. \tag{6.12}$$

T_Λ is a closed operator, $T_{\min} \subset T_\Lambda \subset T_{\max}$, hence $T_\Lambda^* \supset T_{\max}^* = T_{\min}$. Thus T_Λ is a semiregular operator. In view of Theorem 5.8 for characterizing the regularity of T_Λ it suffices to study its localizations T_Λ^p with respect to the points $p \in X$ (i.e. the pure states on $C_0(X)$). As a preparation we define the following subsets of X depending on the function Λ:

$$\operatorname{reg}(\Lambda) := \bigl\{ p \in X \bigm| \Lambda \text{ continuous in a neighborhood of } p \bigr\}, \tag{6.13}$$

$$\text{sing-supp}(\Lambda) := X \setminus \operatorname{reg}(\Lambda). \tag{6.14}$$

$\operatorname{reg}(\Lambda)$ is the largest open subset of X on which Λ is continuous, $\text{sing-supp}(\Lambda)$ is the closed singular support of Λ. We furthermore distinguish two kinds of points in the common boundary $\partial \operatorname{reg}(\Lambda) = \partial\, \text{sing-supp}(\Lambda)$. For a point in $\partial \operatorname{reg}(\Lambda)$ we say that $p \in \operatorname{reg}_\infty(\Lambda)$ if there exists an open neighborhood U of p and a continuous function $\tilde{\Lambda} : U \to S^1$ such that

$$\tilde{\Lambda} \restriction U \cap \operatorname{reg}(\Lambda) = \Lambda \restriction U \cap \operatorname{reg}(\Lambda). \tag{6.15}$$

For these p the limit

$$\tilde{\Lambda}(p) := \lim_{q \to p,\, q \in \operatorname{reg}(\Lambda)} \Lambda(q) \in S^1 \tag{6.16}$$

exists and hence the value $\tilde{\Lambda}(p) \in S^1$ is uniquely determined. However, the existence of the limit (6.16) does in general not imply that $p \in \operatorname{reg}_\infty(\Lambda)$. Finally,

$$\text{sing-supp}_r(\Lambda) := \bigl(\partial\, \text{sing-supp}(\Lambda)\bigr) \setminus \operatorname{reg}_\infty(\Lambda) \tag{6.17}$$

denotes the complement of $\operatorname{reg}_\infty(\Lambda)$ in $\partial\, \text{sing-supp}(\Lambda)$.

Lemma 6.2. *The localizations of T_Λ and T_Λ^* with respect to pure states are given as follows*:

$$T_\Lambda^p := \begin{cases} D_{\min}, & p \in \text{sing-supp}\,\Lambda, \\ D_{\Lambda(p)}, & p \in \operatorname{reg} \Lambda. \end{cases} \tag{6.18}$$

$$\bigl(T_\Lambda^*\bigr)^p := \begin{cases} D_{\max}, & p \in (\text{sing-supp}(\Lambda))^\circ, \\ D_{\Lambda(p)}, & p \in \operatorname{reg}(\Lambda), \\ D_{\tilde{\Lambda}(p)}, & p \in \operatorname{reg}_\infty \Lambda, \\ D_{\min}, & p \in \text{sing-supp}_r \Lambda. \end{cases} \tag{6.19}$$

Proof. We first note that $\mathscr{D}(D_{\min}) \subset \mathscr{D}(T^p_{\Lambda,0}) \subset \mathscr{D}(D_{\Lambda(p)})$. The second inclusion follows from the definition (2.7). To see the first inclusion let $\xi \in \mathscr{D}(D_{\min})$. Then the constant function $f(x) := \xi$ lies in $\mathscr{D}(T_{\min}) \subset \mathscr{D}(T_\Lambda)$ and $f(p) = \xi$. Since $\mathscr{D}(D_{\Lambda(p)})/\mathscr{D}(D_{\min})$ is one-dimensional it follows that either $\mathscr{D}(T^p_{\Lambda,0}) = \mathscr{D}(T^p_\Lambda) = \mathscr{D}(D_{\min})$ or $\mathscr{D}(T^p_{\Lambda,0}) = \mathscr{D}(T^p_\Lambda) = \mathscr{D}(D_{\Lambda(p)})$. Suppose that $f \in \mathscr{D}(T_\Lambda)$ with $\alpha_-(f(p)) \neq 0$. Then there exists an open neighborhood U of p such that $\alpha_-(f(q)) \neq 0$ for $q \in U$ and hence

$$\Lambda \restriction U = \frac{\alpha_+ \circ f}{\alpha_- \circ f} \restriction U \tag{6.20}$$

is continuous on U, proving $p \in \operatorname{reg}(\Lambda)$. Thus if $p \notin \operatorname{reg}(\Lambda)$ we have $\mathscr{D}(T^p_\Lambda) = \mathscr{D}(D_{\min})$. Continuing with $p \in \operatorname{reg}(\Lambda)$ choose a function $\varphi \in C_c(\operatorname{reg}(\Lambda))$ with $\varphi(p) = 1$ and put

$$f(q) := \varphi(q)\big(\Lambda(q)\phi_+ + \phi_-\big). \tag{6.21}$$

Then $f \in C_0(X, \mathscr{D}(D_{\max}))$ with $\alpha_+ \circ f = \Lambda \cdot (\alpha_- \circ f)$, hence $f \in \mathscr{D}(T_\Lambda)$ and thus $f(p) \in \mathscr{D}(T^p_{\Lambda,0})$. $\alpha_-(f(p)) = 1$ proving $\mathscr{D}(T^p_{\Lambda,0}) = \mathscr{D}(D_{\Lambda(p)})$.

Next consider $T^*_\Lambda \subset T^*_{\min} = T_{\max}$. The inclusion $(T^*_\Lambda)^p \subset (T^p_\Lambda)^*$ (Lemma 2.5) implies

$$\mathscr{D}\big((T^*_\Lambda)^p\big) \subset \begin{cases} \mathscr{D}(D_{\max}), & p \in \text{sing-supp}(\Lambda), \\ \mathscr{D}(D_{\Lambda(p)}), & p \in \operatorname{reg}(\Lambda). \end{cases} \tag{6.22}$$

The construction of Eq. (6.21) can now be adapted to prove the claim for the case $p \in \operatorname{reg}(\Lambda)$.

If $p \in (\text{sing-supp}(\Lambda))^\circ$ then choose a continuous compactly supported function $\psi \in C_c((\text{sing-supp}(\Lambda))^\circ)$ with $\psi(p) = 1$. Then the functions $q \mapsto \psi(q) \cdot \phi_\pm$ lie in the domain of $\mathscr{D}(T^*_\Lambda)$ since $(\alpha_+ \circ f)(q) = 0 = (\alpha_- \circ f)(q) = 0$ for all $q \in (\text{sing-supp}(\Lambda))^\circ$ and all $f \in \mathscr{D}(T_\Lambda)$. This proves that the vectors ϕ_+ and ϕ_- are contained in $\mathscr{D}((T^*_\Lambda)^p)$.

If $p \in \partial(\operatorname{reg}(\Lambda))$ and $f \in \mathscr{D}(T^*_\Lambda)$ with $\alpha_-(f(p)) \neq 0$ then $f \in C_0(X, \mathscr{D}(D_{\max}))$, hence there is an open neighborhood U of p such that $\alpha_-(f(q)) \neq 0$ for $q \in U$. Furthermore $\alpha_+(f(q)) = \Lambda(q) \cdot \alpha_-(f(q))$ for $q \in U \cap \operatorname{reg}(\Lambda)$. Thus

$$\tilde{\Lambda}(p) := \lim_{q \to p,\, q \in \operatorname{reg}(\Lambda)} \Lambda(q) = \lim_{q \to p,\, q \in \operatorname{reg}(\Lambda)} \frac{\alpha_+(f(q))}{\alpha_-(f(q))} \in S^1 \tag{6.23}$$

exists and thus $\alpha_+(f(p)) = \tilde{\Lambda}(p) \cdot \alpha_-(f(p)) \neq 0$, too. Thus after possibly making U smaller we may assume that also $\alpha_+(f(q)) \neq 0$ for q in U and hence the continuous function

$$\tilde{\Lambda}(q) := \frac{\alpha_+(f(q))|\alpha_-(f(q))|}{\alpha_-(f(q))|\alpha_+(f(q))|}, \quad q \in U \tag{6.24}$$

is a continuous extension of $\Lambda \restriction U \cap \operatorname{reg}(\Lambda)$, proving that necessarily $p \in \operatorname{reg}_\infty(\Lambda)$. This argument proves that $(T^*_\Lambda)^p = D_{\min}$ for $p \in \text{sing-supp}_r(\Lambda)$. Continuing with $p \in \operatorname{reg}_\infty(\Lambda)$ we first note that taking limits $q \to p$, $q \in U \cap \operatorname{reg}(\Lambda)$ we see that necessarily $\alpha_+(f(p)) = \tilde{\Lambda}(p) \cdot \alpha_-(f(p))$ proving $\mathscr{D}((T^*_\Lambda)^p) \subset \mathscr{D}(D_{\tilde{\Lambda}(p)})$. To prove the converse inclusion put (cf. (6.21))

$$f(q) := \varphi(q)\big(\tilde{\Lambda}(q)\phi_+ + \phi_-\big). \tag{6.25}$$

Then it easily follows that $f \in \mathscr{D}(T_\Lambda^*)$. Since $\alpha_-(f(p)) = 1$ we conclude that indeed $(T_\Lambda^*)^p = D_{\tilde{\Lambda}(p)}$ and the lemma is proved. $\square$

Proposition 6.3.

(1) T_Λ *is regular if and only if* $(\text{sing-supp}(\Lambda))^\circ = \text{sing-supp}(\Lambda)$.
(2) T_Λ *is selfadjoint if and only if* $\text{sing-supp}\,\Lambda = \text{sing-supp}_r\,\Lambda$.
(3) T_Λ *is selfadjoint and regular if and only if* Λ *is continuous* (*i.e.* $\text{sing-supp}\,\Lambda = \emptyset$).
(4) T_Λ^* *is selfadjoint and regular if and only if* $\tilde{\Lambda}$ *is continuous* (*i.e.* $\text{sing-supp}\,\Lambda = \text{reg}_\infty\,\Lambda$).

Hence if $\text{sing-supp}\,\Lambda = \text{sing-supp}_r\,\Lambda \neq \emptyset$ then T_Λ is selfadjoint and not regular (cf. [13, Chap. 9]).

If $\text{sing-supp}\,\Lambda = \text{reg}_\infty\,\Lambda \neq \emptyset$ then T_Λ^* is regular and selfadjoint, but $T_\Lambda \subsetneq T_\Lambda^* = T_\Lambda^{**}$ is not regular (cf. [15]).

Proof. 1. By Theorem 5.8 T_Λ is regular if and only if $(T_\Lambda^p)^* = (T_\Lambda^*)^p$ for all $p \in X$. In view of (6.18), (6.19) this is the case if and only if $(\text{sing-supp}(\Lambda))^\circ = \text{sing-supp}(\Lambda)$.

2. For T_Λ being selfadjoint it is necessary that $T_\Lambda^p = (T_\Lambda^*)^p$ for all $p \in X$. By Lemma 6.2 the latter is only true if $\text{sing-supp}(\Lambda) = \text{sing-supp}_r(\Lambda)$. If that is the case let us consider $f \in \mathscr{D}(T_\Lambda^*)$. Then by (6.19) $f \in C_0(X, \mathscr{D}(D_{\max}))$, $f(p) \in \mathscr{D}(D_{\min})$ if $p \in \text{sing-supp}_r(\Lambda)$, and $f(p) \in \mathscr{D}(D_{\Lambda(p)})$ if $p \in \text{reg}(\Lambda)$. But then, since $(\text{sing-supp}(\Lambda))^\circ = \emptyset$ we have $\alpha_+ \circ f = \Lambda \cdot \alpha_- \circ f$, thus $f \in \mathscr{D}(T_\Lambda)$.

3. This is just the obvious combination of 1. and 2.

4. This follows as 1. and 2. by applying Lemma 6.2 to T_Λ^*. $\square$

Proposition 6.4. *Assume that* $\overline{\text{reg}(\Lambda)} = X$ *and let* μ *be a probability measure on* X *with*

(1) $\text{supp}\,\mu = \overline{\text{reg}(\Lambda)} = X$,
(2) $\partial(\text{reg}(\Lambda))$ *is a* μ*-null set.*

Then the representation π_μ *is faithful,* $T_\Lambda^\mu = (T_\Lambda^\mu)^*$, *and*

$$\mathscr{D}\big(T_\Lambda^\mu\big) = \big\{f \in T_{\max}^\mu \;\big|\; f(x) \in \mathscr{D}(D_{\Lambda(x)}) \textit{ for } \mu\textit{-a.e. } x \in X\big\} \tag{6.26}$$

$$= \big\{f \in T_{\max}^\mu \;\big|\; \langle f, \eta_\Lambda^\perp\rangle_{T,\mu} = 0\big\}, \tag{6.27}$$

where $\eta_\Lambda^\perp(x) := \eta_{\Lambda(x)}^\perp$ (*cf.* (6.5)).

However, if $\partial(\text{reg}(\Lambda)) = \text{sing-supp}_r\,\Lambda \neq \emptyset$ *then* T_Λ *is selfadjoint but not regular while if* $\partial(\text{reg}(\Lambda)) = \text{reg}_\infty\,\Lambda \neq \emptyset$ *then* T_Λ *is neither selfadjoint nor regular but* $T_\Lambda^\mu = (T_\Lambda^\mu)^*$ *for the* faithful *representation* π_μ.

As an example for the last situation we could concretely take $X = [0,1]$, $\Lambda : [0,1] \to S^1$ such that $\Lambda \restriction (0,1]$ is continuous but discontinuous at 0 (with not existing limit at 0 in the first case and existing limit at 0 in the second), and $\mu(f) := \int_0^1 f$.

This example shows that the answer to the following question, which the attentive reader might have hoped to be affirmative, is *negative*:

Problem 6.5. Does the essential selfadjointness of the localized unbounded operator of the form T^π on H_π imply the selfadjointness and regularity of the closed symmetric operator T when the presentation π is **faithful**?

One might however still be tempted to think that the above statement is true when the faithful representation is the atomic representation of $\mathscr{A}$, cf. Conjectures 5.2–5.4 and Theorem 5.6.

Proof of Proposition 6.4. Since $\operatorname{supp}\mu = X$ it follows that the representation π_μ of $C(X)$ by multiplication operators on $L^2(X, \mu; H)$ is faithful.

Now, let $\Omega := \operatorname{reg}(\Lambda)$ and let $\Theta : \Omega \to S^1$ denote the restriction of Θ to Ω. We can then consider the operator T_Θ acting on the Hilbert module $C_0(\Omega, H)$ together with the localization T_Θ^σ. Here σ denotes the restriction of the probability measure μ to Ω. We then get from Proposition 6.3 that T_Θ is selfadjoint and regular and hence from Theorem 5.6 that the localization T_Θ^σ is selfadjoint. The selfadjointness of the localization T_Λ^μ now follows by noting that we have the inclusion $T_\Theta^\sigma \subseteq T_\Lambda^\mu$ under the identification of Hilbert spaces $L^2(\Omega, \sigma; H) \cong L^2(X, \mu; H)$. Here we use that $\mu(\partial \operatorname{reg}(\Lambda)) = 0$.

The identities in Eq. (6.26) and Eq. (6.27) are now obvious. □

7. Sums of selfadjoint regular operators

Let us consider a Hilbert C^*-module E over some C^*-algebra $\mathscr{A}$. Furthermore, let S and T be two selfadjoint and regular operators with domains $\mathscr{D}(S) \subseteq E$ and $\mathscr{D}(T) \subseteq E$ respectively. The main purpose of this section is then to study the selfadjointness and regularity of the sum operator

$$D := \begin{pmatrix} 0 & S - iT \\ S + iT & 0 \end{pmatrix}, \qquad \mathscr{D}(D) = \big(\mathscr{D}(S) \cap \mathscr{D}(T)\big)^2 \subset E \oplus E. \tag{7.1}$$

As mentioned in the introduction this question is essential when dealing with the Kasparov product of unbounded modules. To be more precise we shall see that the following three assumptions are sufficient for the above sum to be a selfadjoint and regular operator.

Assumption 7.1. We will assume that we have a dense submodule $\mathscr{E} \subseteq E$ such that the following conditions are satisfied:

(1) The submodule $\mathscr{E} \subseteq \mathscr{D}(T)$ is a core for T.
(2) We have the inclusions

$$(S - i \cdot \mu)^{-1}(\xi) \in \mathscr{D}(S) \cap \mathscr{D}(T) \quad \text{and} \quad T(S - i \cdot \mu)^{-1}(\xi) \in \mathscr{D}(S) \tag{7.2}$$

for all $\mu \in \mathbb{R} \setminus \{0\}$ and all $\xi \in \mathscr{E}$.
(3) The module homomorphism

$$[S, T] \cdot (S - i \cdot \mu)^{-1} : \mathscr{E} \to E \tag{7.3}$$

extends to a bounded ($\mathscr{A}$-linear) operator $X_\mu : E \to E$ between Hilbert C^*-modules for all $\mu \in \mathbb{R} \setminus \{0\}$.

We will start by proving that the sum operator is selfadjoint. Afterwards we shall apply the Local–Global Principle to prove that the sum operator is regular as well. The following lemma will be used several times:

Lemma 7.2. *Let P be a selfadjoint regular operator in the Hilbert C^*-module E. Let $(f_n)_{n\in\mathbb{N}} \subset C_\infty(\mathbb{R})$ be a sequence of functions* (*cf. Section* 2.3) *such that*

(1) $\sup_n \|f_n\|_\infty < \infty$,
(2) $(f_n)_{n\in\mathbb{N}}$ *converges to* $f \in C_\infty(\mathbb{R})$ *uniformly on compact subsets of* $\mathbb{R}$.

Then $f_n(P)$ converges strongly *to $f(P)$ as $n \to \infty$.*

Proof. Consider first $x \in \mathscr{D}(P)$. Then with $\varphi(t) := (t+i)^{-1}$ and $y := (P+i)x$ we have $x = \varphi(P)y$. Since φ vanishes at ∞, $f_n\varphi \to f\varphi$ *uniformly*, in particular

$$f_n(P)x = \big((f_n\varphi)(P)\big)y \longrightarrow f(P)x.$$

The sequence $(f_n(P))_{n\in\mathbb{N}}$ therefore converges strongly to $f(P)$ on the dense submodule $\mathscr{D}(P)$. Since the sequence is uniformly norm bounded this implies the strong convergence. $\square$

7.1. Selfadjointness

Let us assume that the conditions in Assumption 7.1 are satisfied.

We begin by noting that the commutator $[S,T]$ is densely defined. Indeed, the domain of $[S,T]$ contains the dense submodule $(S-i)^{-1}(\mathscr{E})$. As a consequence we get that the bounded operators X_μ, $\mu \in \mathbb{R}\setminus\{0\}$, defined in Eq. (7.3) are adjointable. The adjoint of X_μ is given by the expression

$$(X_\mu)^*\xi = -(S+i\cdot\mu)^{-1}[S,T]\xi \tag{7.4}$$

for all ξ in the dense submodule $\mathscr{D}([S,T]) \subseteq E$.

We continue by showing that the core $\mathscr{E}$ can be replaced by the domain of the selfadjoint and regular operator T.

Proposition 7.3. *The conditions in Assumption* 7.1 *are satisfied for $\mathscr{E} = \mathscr{D}(T)$.*

Proof. Let us fix some vector $\xi \in \mathscr{D}(T)$ and some number $\mu \in \mathbb{R}\setminus\{0\}$. It then suffices to prove the inclusions

$$(S-i\cdot\mu)^{-1}(\xi) \in \mathscr{D}(T) \quad \text{and} \quad T(S-i\cdot\mu)^{-1}(\xi) \in \mathscr{D}(S). \tag{7.5}$$

To this end we let $(\xi_n) \subset \mathscr{E}$ be some sequence such that

$$\xi_n \to \xi, \quad \text{and} \quad T\xi_n \to T\xi$$

in the norm topology of E. It then follows that the sequences

$$T(S-i\cdot\mu)^{-1}\xi_n = (S-i\cdot\mu)^{-1}T\xi_n + (S-i\cdot\mu)^{-1}X_\mu\xi_n \quad \text{and}$$

$$ST(S-i\cdot\mu)^{-1}\xi_n = S(S-i\cdot\mu)^{-1}T\xi_n + S(S-i\cdot\mu)^{-1}X_\mu\xi_n$$

converge in E. But this proves the validity of the inclusions in (7.5) since our unbounded operators are closed. □

For later use we state and prove the following:

Lemma 7.4. *The sequence of adjointable operators*

$$R_n := \frac{i}{n}\cdot\left(\frac{i}{n}S+1\right)^{-1}\cdot[S,T]\cdot\left(\frac{i}{n}S+1\right)^{-1}\in\mathscr{L}(E)$$

as well as $(R_n^*)_n$ *converge strongly to the* 0*-operator.*

Proof. For each $n\in\mathbb{N}$ we can rewrite R_n as

$$R_n = \left(\frac{i}{n}S+1\right)^{-1}\cdot X_{-1}\cdot(S+i)(S-i\cdot n)^{-1}.$$

Since the factors in this decomposition are adjointable it follows that R_n is adjointable as well. By Lemma 7.2 the sequence of adjointable operators

$$\left((S+i)(S-i\cdot n)^{-1}\right)_{n\in\mathbb{N}}\subset\mathscr{L}(E)$$

converges strongly to zero as $n\to\infty$. Furthermore the sequence of adjointable operators

$$\left(\left(\frac{i}{n}S+1\right)^{-1}\cdot X_{-1}\right)_{n\in\mathbb{N}}\subset\mathscr{L}(E)$$

is uniformly bounded in the operator norm. But these two observations prove that $R_n\to 0$ strongly.

The same line of argument applies to the adjoint R_n^* and the lemma is proved. □

The C^*-algebra estimates of the next two lemmas will be important for proving that the sum operator D is closed.

Lemma 7.5. *There exists a constant* $C>0$ *such that we have the following inequality between selfadjoint elements of the* C^**-algebra* $\mathscr{A}$

$$\pm i\cdot\langle[S,T]\xi,\xi\rangle\leqslant\frac{1}{2}\langle S\xi,S\xi\rangle + C\langle\xi,\xi\rangle$$

for all $\xi\in\mathscr{D}([S,T])$.

Proof. The selfadjointness of S,T and Assumption 7.1 imply that $\langle[S,T]\xi,\xi\rangle$ is skewadjoint for $\xi\in\mathscr{D}([S,T])$. Furthermore, the inequalities

$$\begin{aligned}
\pm 2i \cdot \big\langle [S,T]\xi, \xi \big\rangle &= \mp \big\langle i[S,T]\mu\xi, \mu^{-1}\xi \big\rangle \mp \big\langle \mu^{-1}\xi, i[S,T]\mu\xi \big\rangle \\
&\leqslant \mu^2 \big\langle [S,T]\xi, [S,T]\xi \big\rangle + \mu^{-2}\langle \xi, \xi \rangle \\
&\leqslant \mu^2 \big\| [S,T](S+i)^{-1} \big\|^2 \cdot \big\langle (S+i)\xi, (S+i)\xi \big\rangle + \mu^{-2}\langle \xi, \xi \rangle \\
&\leqslant \mu^2 \|X_{-1}\|^2 \langle S\xi, S\xi \rangle + \big(\mu^2 \|X_{-1}\|^2 + \mu^{-2}\big) \cdot \langle \xi, \xi \rangle
\end{aligned}$$

are valid in the C^*-algebra $\mathscr{A}$ for any $\mu \in \mathbb{R} \setminus \{0\}$. Here we have used again the inequality (3.7). Letting $\mu = \frac{1}{\|X_{-1}\|}$ now proves the claim. □

Lemma 7.6. *There exists a constant $C > 0$ such that*

$$\big\langle (S \pm iT)\xi, (S \pm iT)\xi \big\rangle \geqslant \frac{1}{2}\langle S\xi, S\xi \rangle + \langle T\xi, T\xi \rangle - C\langle \xi, \xi \rangle$$

for all $\xi \in \mathscr{D}(S) \cap \mathscr{D}(T)$.

Proof. By an application of Lemma 7.5 we get that

$$\begin{aligned}
\big\langle (S \pm iT)\xi, (S \pm iT)\xi \big\rangle &= \langle S\xi, S\xi \rangle + \langle T\xi, T\xi \rangle \mp i\big\langle [S,T]\xi, \xi \big\rangle \\
&\geqslant \frac{1}{2}\langle S\xi, S\xi \rangle + \langle T\xi, T\xi \rangle - C\langle \xi, \xi \rangle
\end{aligned} \tag{7.6}$$

for all $\xi \in \mathscr{D}([S,T])$. This proves the desired inequality on the submodule $\mathscr{D}([S,T]) \subseteq \mathscr{D}(S) \cap \mathscr{D}(T)$.

Now, let $\xi \in \mathscr{D}(S) \cap \mathscr{D}(T)$. We will then look at the elements

$$\xi_n := \left(\frac{i}{n}S + 1\right)^{-1} \xi \in \mathscr{D}\big([S,T]\big)$$

in the domain of $[S,T]$. It follows from Lemma 7.2 that $\xi_n \to \xi$ and $S\xi_n \to S\xi$ in E as $n \to \infty$. On the other hand, we have the identities

$$\begin{aligned}
T\xi_n &= \left(\frac{i}{n}S + 1\right)^{-1} T\xi + \frac{i}{n} \cdot \left(\frac{i}{n}S + 1\right)^{-1} [S,T] \left(\frac{i}{n}S + 1\right)^{-1} \xi \\
&= \left(\frac{i}{n}S + 1\right)^{-1} T\xi + R_n\xi.
\end{aligned}$$

It therefore follows from Lemma 7.4 that $T\xi_n \to T\xi$ as $n \to \infty$ in the norm of E. The lemma now follows by applying the inequality (7.6) to ξ_n and taking limits. □

We are now ready to prove the main result of this subsection.

Proposition 7.7. *Assume that the conditions in Assumption* 7.1 *are satisfied. The operators* $S \pm iT$ *with domain* $\mathscr{D}(S \pm iT) = \mathscr{D}(S) \cap \mathscr{D}(T)$ *are closed operators and adjoints of each other, i.e.* $(S \pm iT)^* = (S \mp iT)$. *In other words, the sum operator*

$$D = \begin{pmatrix} 0 & S - iT \\ S + iT & 0 \end{pmatrix}$$

with domain

$$\mathscr{D}(D) = \big(\mathscr{D}(S) \cap \mathscr{D}(T)\big)^2$$

is selfadjoint.

Proof. The inequality stated in Lemma 7.6 shows that the convergence of a sequence in the graph norm of $S \pm iT$ implies the convergence of the sequence in the graph norm of S *and* in the graph norm of T individually. Since the unbounded operators S and T are closed this proves that $S \pm iT$ are closed, too.

To show that $S \pm iT$ are adjoints of each other we first note that for each pair of elements $\xi, \eta \in \mathscr{D}(S) \cap \mathscr{D}(T)$ we have the identity

$$\big\langle (S + iT)\xi, \eta \big\rangle = \big\langle \xi, (S - iT)\eta \big\rangle.$$

We therefore only need to prove the inclusion of domains

$$\mathscr{D}\big((S + iT)^*\big) \subseteq \mathscr{D}(S) \cap \mathscr{D}(T). \tag{7.7}$$

To this end we let $\xi \in \mathscr{D}((S + iT)^*)$ be a vector in the domain of the adjoint. We then define the sequence (ξ_n) by

$$\xi_n := \left(-\frac{i}{n}S + 1\right)^{-1} \xi \in \mathscr{D}(S)$$

which converges to ξ in the norm of E. We shall prove that $\xi_n \in \mathscr{D}(S) \cap \mathscr{D}(T)$ for all $n \in \mathbb{N}$ and that $(S + iT)^*\xi_n = (S - iT)\xi_n$ converges as $n \to \infty$ in the norm of E. This will prove the inclusion (7.7) since $S - iT$ is already proved to be closed on $\mathscr{D}(S) \cap \mathscr{D}(T)$.

We start by proving that $\xi_n \in \mathscr{D}(S) \cap \mathscr{D}(T)$. Let $\eta \in \mathscr{D}(T)$. We can then calculate as follows, using Assumption 7.1 and Proposition 7.3

$$\begin{aligned}
\langle \xi_n, T\eta \rangle &= \left\langle \left(-\frac{i}{n}S + 1\right)^{-1} \xi, T\eta \right\rangle = \left\langle \xi, \left(\frac{i}{n}S + 1\right)^{-1} T\eta \right\rangle \\
&= \left\langle \xi, T\left(\frac{i}{n}S + 1\right)^{-1} \eta \right\rangle - \left\langle \xi, \frac{i}{n}\left(\frac{i}{n}S + 1\right)^{-1} [S, T]\left(\frac{i}{n}S + 1\right)^{-1} \eta \right\rangle \\
&= -i\left\langle \xi, (S + iT)\left(\frac{i}{n}S + 1\right)^{-1} \eta \right\rangle + i\left\langle \xi, S\left(\frac{i}{n}S + 1\right)^{-1} \eta \right\rangle - \big\langle R_n^*\xi, \eta \big\rangle \\
&= -i\left\langle \left(-\frac{i}{n}S + 1\right)^{-1} (S + iT)^*\xi, \eta \right\rangle + i\langle S\xi_n, \eta \rangle - \big\langle R_n^*\xi, \eta \big\rangle.
\end{aligned} \tag{7.8}$$

Since T is selfadjoint this proves that

$$\xi_n = \left(-\frac{i}{n}S + 1\right)^{-1}\xi \in \mathscr{D}(T)$$

and

$$T\xi_n = i\left(-\frac{i}{n}S + 1\right)^{-1}(S + iT)^*\xi - iS\xi_n - R_n^*\xi. \tag{7.9}$$

We end the proof of the inclusion (7.7) by showing that $(S + iT)^*\xi_n = (S - iT)\xi_n$ converges in the norm of E. From (7.9) we infer, since $\xi_n \in \mathscr{D}(S) \cap \mathscr{D}(T)$,

$$(S + iT)^*\xi_n = (S - iT)\xi_n = \left(-\frac{i}{n}S + 1\right)^{-1}(S + iT)^*\xi + i(R_n)^*\xi.$$

The convergence of the sequence $\{(S + iT)^*\xi_n\}$ now follows from Lemma 7.4 and $\xi \in \mathscr{D}(S) \cap \mathscr{D}(T)$ is proved.

The last claim about D is now clear. □

7.2. Regularity

We still assume that the conditions in Assumption 7.1 are satisfied. It is now our goal to improve Proposition 7.7 by showing that the operators $S \pm iT$ and hence the sum operator D are also regular. This will turn out to be an easy consequence of the Local–Global Principle and the results in Section 7.1. We start by noting that the localizations of T and S satisfy the conditions in Assumption 7.1. Remark that these localizations are selfadjoint by Theorem 4.2.

Lemma 7.8. *Let $\omega \in S(\mathscr{A})$ be a state on the C^*-algebra $\mathscr{A}$. Then the localizations with respect to ω of the selfadjoint regular operators T and S satisfy the conditions in Assumption* 7.1. *The desired core $\mathscr{E}^\omega \subseteq \mathscr{D}(T^\omega)$ is the subspace $\mathscr{E}^\omega := \iota_\omega(\mathscr{D}(T))$.*

Proof. This is a straightforward consequence of the definitions and Proposition 7.3. □

In order to prove the regularity of the sum operator D we study the behavior of localization with respect to the sum operation.

Lemma 7.9. *Let $\omega \in S(\mathscr{A})$ be a state on the C^*-algebra $\mathscr{A}$. We then have the identity*

$$S^\omega + iT^\omega = (S + iT)^\omega$$

between localized operators.

Proof. Recall that by definition $\mathscr{D}(S^\omega + iT^\omega) = \mathscr{D}(S^\omega) \cap \mathscr{D}(T^\omega)$. We start by noting that

$$(S + iT)^\omega \subseteq S^\omega + iT^\omega.$$

This inclusion is valid since $(S+iT)_0^\omega \subseteq S_0^\omega + iT_0^\omega$ and since $S^\omega + iT^\omega$ is closed by Proposition 7.7. Thus we only need to prove the inclusion

$$\mathscr{D}\big(S^\omega + iT^\omega\big) \subseteq \mathscr{D}\big((S+iT)^\omega\big) \tag{7.10}$$

of domains. In order to establish this we first prove that

$$\big(S^\omega + i\mu\big)^{-1}\big(\mathscr{D}\big(T^\omega\big)\big) \subseteq \mathscr{D}\big((S+iT)^\omega\big) \tag{7.11}$$

for all $\mu \in \mathbb{R} \setminus \{0\}$. Let $\mu \in \mathbb{R} \setminus \{0\}$ and let $\xi \in \mathscr{D}(T^\omega)$. We can then find a sequence $(\eta_n) \subset \mathscr{D}(T)$ such that

$$\iota_\omega(\eta_n) \to \xi \quad \text{and} \quad \iota_\omega(T\eta_n) \to T^\omega(\xi).$$

By Assumption 7.1 we get that

$$\big(S^\omega + i\mu\big)^{-1}\big(\iota_\omega(\eta_n)\big) = \iota_\omega\big((S+i\mu)^{-1}\eta_n\big) \in \iota_\omega\big(\mathscr{D}(S) \cap \mathscr{D}(T)\big) \subseteq \mathscr{D}\big((S+iT)^\omega\big)$$

and by continuity we have that

$$\big(S^\omega + i\mu\big)^{-1}\big(\iota_\omega(\eta_n)\big) \to \big(S^\omega + i\mu\big)^{-1}\xi.$$

We therefore only need to prove that the sequence

$$(S+iT)^\omega\big(S^\omega + i\mu\big)^{-1}\big(\iota_\omega(\eta_n)\big) = \iota_\omega\big((S+iT)(S+i\mu)^{-1}\eta_n\big)$$

is convergent in the norm of E^ω. But this follows by the argument given in the proof of Proposition 7.3.

To finish the proof of the inclusion in (7.10) we let

$$\xi \in \mathscr{D}\big(S^\omega + iT^\omega\big) = \mathscr{D}\big(S^\omega\big) \cap \mathscr{D}\big(T^\omega\big).$$

We then define the sequence (ξ_n) by

$$\xi_n := \left(\frac{i}{n}S^\omega + 1\right)^{-1}\xi.$$

By the argument given in the proof of Lemma 7.6 we have that

$$\xi_n \to \xi \quad \text{and} \quad \big(S^\omega + iT^\omega\big)\xi_n \to \big(S^\omega + iT^\omega\big)\xi$$

where the convergence takes place in the Hilbert space E^ω. But this proves that $\xi \in \mathscr{D}((S+iT)^\omega)$ since $\xi_n \in \mathscr{D}((S+iT)^\omega)$ for all $n \in \mathbb{N}$ by the inclusion in (7.11). $\square$

We are now ready to prove the main result of this section.

Theorem 7.10. *Assume that the conditions in Assumption* 7.1 *are satisfied. Then the sum operator*

$$D = \begin{pmatrix} 0 & S - iT \\ S + iT & 0 \end{pmatrix}$$

with domain $(\mathscr{D}(S) \cap \mathscr{D}(T))^2$ *is selfadjoint and regular.*

Proof. Let $\omega \in S(\mathscr{A})$ be a state on the C^*-algebra $\mathscr{A}$. By the Local–Global Principle proved in Theorem 4.2 we only need to prove that the localization D^ω agrees with the selfadjoint operator

$$\big(D^\omega\big)' := \begin{pmatrix} 0 & S^\omega - iT^\omega \\ S^\omega + iT^\omega & 0 \end{pmatrix}, \qquad \mathscr{D}\big(\big(D^\omega\big)'\big) := \big(\mathscr{D}\big(S^\omega\big) \cap \mathscr{D}\big(T^\omega\big)\big)^2.$$

Remark that the selfadjointness of $(D^\omega)'$ is a consequence of Lemma 7.8 and Proposition 7.7. However, by an application of Lemma 7.9 we get the identities

$$D^\omega = \begin{pmatrix} 0 & (S - iT)^\omega \\ (S + iT)^\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^\omega - iT^\omega \\ S^\omega + iT^\omega & 0 \end{pmatrix} = \big(D^\omega\big)'$$

and the theorem is proved. □

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