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Endomorphism Rings and Category Equivalences

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INTRODUCTION

The use of category equivalences for the study of endomorphism rings stems from the Morita theorem. In a sense, this theorem can be viewed as stating that if P is a finitely generated projective generator of R-mod and $S = \mathrm{End}({}_RP)$, then properties of P correspond to properties of S through the equivalence between the categories R-mod and S-mod given by the functor $\mathrm{Hom}_R(P, -)$. Generalizations of this theorem were given in [4, 5]. In [4], P is only assumed to be finitely generated and projective, and $\mathrm{Hom}_R(P, -)$ gives in this case an equivalence between S-mod and a quotient category of R-mod, while in [5] it is shown that if P is a finitely generated quasiprojective self-generator, then the equivalence induced by the same functor is now defined between the category $\sigma[P]$ of all the R-modules subgenerated by P and S-mod.

Later on, other category equivalences were constructed, in an analogous way to those already mentioned, by replacing S-mod by a certain quotient category of S-mod. Thus, in [14] Morita contexts are used to obtain a category equivalence between quotient categories of both R-mod and S-mod for an arbitrary R-module M. On the other hand, if M is a Σ-quasiprojective module, then it is shown in [8] that the functor $\mathrm{Hom}_R(M, -)$ induces an equivalence between quotient categories of $\sigma[M]$ and S-mod, and the latter quotient category coincides with S-mod when M is finitely generated.

All the above constructions can be considered as particular cases of the following: if $\mathscr{C}$ is a locally finitely generated Grothendieck category and M is an object of $\mathscr{C}$ with $S = \mathrm{End}_{\mathscr{C}}(M)$, then the class of the M-distinguished objects of $\mathscr{C}$ (in the sense of [10]) is the torsionfree class of a torsion

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theory (**T**, **F**) of $\mathscr{C}$ and the functor $\mathrm{Hom}_{\mathscr{C}}(M, -): \mathscr{C} \to S$-mod induces an equivalence between the quotient category of $\mathscr{C}$ modulo **T** and a certain quotient category $(S, \mathscr{F})$-mod of S-mod (Theorem 1.7). Moreover, this latter quotient category consists of all the S-modules if and only if M is a finitely generated quasiprojective object of $\mathscr{C}$ which is CQF-3 in the sense of [16]. In the first section of this paper, the properties of the foregoing construction are studied.

On the other hand, Ohtake [16] considers a situation which is slightly different from ours: $\mathscr{C}$ is assumed to be a cocomplete abelian category with exact direct limits and the object M of $\mathscr{C}$ is supposed to be CQF-3. The above-mentioned class **T** is also, in this case, a torsionfree class corresponding to a cohereditary torsion theory (**D**, **T**). If M is codivisible with respect to this torsion theory, then another equivalence of categories is obtained between a co-Giraud subcategory of $\mathscr{C}$ and the quotient category $(S, \mathscr{F})$-mod to which we referred in the preceding paragraph. In Section 2, we show that if M is CQF-3, then the full subcategory of $\mathscr{C}$ whose objects belong simultaneously to **D** and **F** is also equivalent to $(S, \mathscr{F})$-mod (Proposition 2.1). Thus, if M is codivisible there are three different full subcategories of $\mathscr{C}$ which are equivalent to $(S, \mathscr{F})$-mod. As a consequence, we give a short proof of [16, Theorem 2.5] for the case of $\mathscr{C}$ being a Grothendieck category (Proposition 2.7).

Finally, the preceding results are applied in Section 3 to characterize the modules M such that the endomorphism ring S of M is a left semihereditary ring (Theorem 3.2), a left CS-ring (Theorem 3.5), or a left continuous ring (Proposition 3.7). This is done provided the module M satisfies certain conditions such as being weakly **T**-closed (this class of modules includes, for instance, all the M-distinguished and quasi-injective modules M, the Σ-quasiprojective modules, the codivisible CQF-3 modules, or the quasiprojective and trace-accessible modules).

Throughout this paper, all rings will be associative with 1 and all modules are left modules unless otherwise stated. A composition $s \circ t$ of morphisms will be written, alternatively (in particular, when dealing with endomorphism rings), as ts. However, if **F** and **G** are functors, then **FG** will always mean the composition $\mathbf{F} \circ \mathbf{G}$. The injective hull of a module N will be denoted by $E(N)$. A submodule L of N is said to be essentially closed when L has no proper essential extension within N. On the other hand, if I is a left ideal of a ring R and $a \in R$, $(I: a)$ stands for $\{x \in R \mid xa \in I\}$.

We assume that all functors between abelian categories are additive. A Grothendieck category $\mathscr{C}$ is said to be locally finitely generated when it has a family of finitely generated generators. An object X of $\mathscr{C}$ is called (Σ-)quasiprojective when for every finite (arbitrary) set I and every epimorphism $p: X^{(I)} \to Y$ of $\mathscr{C}$, the induced morphism $p_*: \mathrm{Hom}_{\mathscr{C}}(X, X^{(I)})$

$\to \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is surjective. An object X of $\mathscr{C}$ is called *CQF*-3 [16] when for every epimorphism $p: Y \to Z$ of $\mathscr{C}$, the induced morphism $p_*: \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ is zero if and only if $\mathrm{Hom}_{\mathscr{C}}(X, Z) = 0$.

Let M be an object of the Grothendieck category $\mathscr{C}$. An object X of $\mathscr{C}$ is (finitely) M-generated if it is an epimorphic image of a (finite) direct sum $M^{(I)}$ of copies of M. For each X in $\mathscr{C}$ there is a greatest M-generated subobject of X, which is the sum of all the M-generated subobjects of X, and will be denoted by X_M. For $\mathscr{C} = R$-mod, a module N is subgenerated by M if it is isomorphic to a submodule of an M-generated module and the full subcategory of R-mod whose objects are all the modules subgenerated by M is denoted by $\sigma[M]$. This category is a locally finitely generated Grothendieck category [20].

Recall from the definition of a torsion theory in a Grothendieck category $\mathscr{C}$ [18] that a class $\mathbf{T}$ (resp., $\mathbf{F}$) of objects of $\mathscr{C}$ is said to be a torsion (resp., a torsionfree) class if it is closed under epimorphic images, extensions, and direct sums (resp., subobjects, extensions, and products). The torsion theory $(\mathbf{T}, \mathbf{F})$ is called hereditary (resp., cohereditary) when $\mathbf{T}$ is closed under subobjects (resp., $\mathbf{F}$ is closed under epimorphic images). The torsion radical associated to $(\mathbf{T}, \mathbf{F})$ will be denoted by $\mathbf{t}_{\mathbf{T}}$ (or $\mathbf{t}$ if the torsion class $\mathbf{T}$ is clear from the context). Unless otherwise stated, the torsion theories we consider in this paper are hereditary. A subobject L of an object X of $\mathscr{C}$ will be called $\mathbf{T}$-saturated when $X/L \in \mathbf{F}$, and the $\mathbf{T}$-saturated subobjects of X form a complete lattice which we denote by $\mathrm{Sat}_{\mathbf{T}}(X)$. If $(\mathbf{T}, \mathbf{F})$ is a (not necessarily hereditary) torsion theory and X is an object of $\mathscr{C}$, then X is called $\mathbf{T}$-injective (resp., $\mathbf{T}$-codivisible) if for each short exact sequence $0 \to L \xrightarrow{u} Y \xrightarrow{p} N \to 0$ in $\mathscr{C}$ such that $N \in \mathbf{T}$ (resp., $L \in \mathbf{F}$), the induced homomorphism $u^*: \mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Hom}_{\mathscr{C}}(L, X)$ (resp., $p_*: \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, N)$) is surjective.

If $(\mathbf{T}, \mathbf{F})$ is a torsion theory in $\mathscr{C}$, the full subcategory of $\mathscr{C}$ determined by $\mathbf{T}$ is a localizing subcategory (in the sense of [6]) and thus there exists an associated quotient category $\mathscr{C}/\mathbf{T}$, which is a Grothendieck category, with canonical functor $\mathbf{a}: \mathscr{C} \to \mathscr{C}/\mathbf{T}$, which is exact. $\mathbf{a}$ has a right adjoint $\mathbf{i}: \mathscr{C}/\mathbf{T} \to \mathscr{C}$ which is full and faithful, and hence $\mathscr{C}/\mathbf{T}$ can be identified with a full subcategory of $\mathscr{C}$ consisting of all the objects X of $\mathscr{C}$ that are $\mathbf{T}$-torsionfree and $\mathbf{T}$-injective (these are called $\mathbf{T}$-closed objects). The composition $\mathbf{i} \circ \mathbf{a}: \mathscr{C} \to \mathscr{C}$ is usually known as the localization functor and $\psi: 1_{\mathscr{C}} \to \mathbf{i} \circ \mathbf{a}$ will denote the associated natural transformation. For further details about localization in Grothendieck categories, we refer the reader to [6, 18].

In the particular case $\mathscr{C} = R$-mod, each torsion theory $(\mathbf{T}, \mathbf{F})$ is given by a Gabriel filter $\mathscr{F}$ of left ideals of R [18, VI.5.1]. In this case, $\mathbf{T}$ will be replaced by $\mathscr{F}$ in our notation (e.g., we write $\mathscr{F}$-injective, $\mathscr{F}$-torsionfree, instead of $\mathbf{T}$-injective or $\mathbf{T}$-torsionfree), and the corresponding quotient category will be denoted by $(R, \mathscr{F})$-mod. Also, for a given module N, $N_{\mathscr{F}}$ will stand for $\mathbf{a}(N)$ (or $\mathbf{i} \circ \mathbf{a}(N)$).

1. Torsion Theories over Endomorphism Rings

Let R be a ring, M a left R-module, and $S = \mathrm{End}(_R M)$, its endomorphism ring. The question of how properties of M are related to properties of S has been studied in many papers, through the construction of equivalences between certain subcategories of R-mod and of S-mod, as stated in the Introduction. Amongst them, we single out the following: (1) the study of the derived context of an arbitrary module [14]; (2) if M is CQF-3 and codivisible, there is an equivalence of categories between full subcategories of R-mod and of S-mod [16]; (3) when M is a Σ-quasiprojective R-module, up to three different subcategories of R-mod are equivalent to a single full subcategory of S-mod [8, 9]. Our aim in this section is to obtain a generalization of the foregoing constructions.

Definition 1.1. Let $\mathscr{C}$ be a Grothendieck category and M an object of $\mathscr{C}$. An object X of $\mathscr{C}$ is called M-distinguished if for any nonzero morphism $f\colon Y \to X$ there is a morphism $g\colon M \to Y$ such that $f \circ g \neq 0$.

The preceding definition was given by Kato [10] for the particular case of a category of modules.

Proposition 1.2. *Let $\mathscr{C}$ be a Grothendieck category and M an object of $\mathscr{C}$. The class* $\mathbf{F}$ *of M-distinguished objects is a torsionfree class of $\mathscr{C}$. The corresponding torsion class* $\mathbf{T}$ *is the smallest (hereditary) torsion class of $\mathscr{C}$ containing all objects of the form X/X_M for X an object of $\mathscr{C}$. If U is a generator of $\mathscr{C}$, then* $\mathbf{T}$ *is the smallest torsion class containing U/U_M.*

Proof. The fact that $\mathbf{F}$ is a torsionfree class is proved in a straightforward way. Analogously to [9, Proposition 1.1] one can then show that $\mathbf{T}$ is generated by all the objects of the form X/X_M. Finally, it is easy to see that for each X in $\mathscr{C}$, X/X_M is a quotient of a direct sum of copies of U/U_M, if U is a generator, from which the last statement of the proposition follows.

From now on, we will assume in this section that a Grothendieck category $\mathscr{C}$ is given and M is a fixed object of $\mathscr{C}$, with $S = \mathrm{End}_{\mathscr{C}}(M)$, the endomorphism ring of M. The torsion theory of Proposition 1.2 will be denoted by $(\mathbf{T}, \mathbf{F})$ and its associated torsion radical by $\mathbf{t}$, while $\bar{X}$ will stand for $X/\mathbf{t}(X)$. The quotient category $\mathscr{C}/\mathbf{T}$ will be written $\mathscr{C}_M$. $\mathscr{C}_M$ can be identified with the full subcategory of $\mathscr{C}$ whose objects are all the $\mathbf{T}$-closed objects of $\mathscr{C}$. The canonical functor $\mathbf{a}: \mathscr{C} \to \mathscr{C}_M$ is exact and has a right adjoint $\mathbf{i}$, which can be identified with the inclusion functor. These identifications will be assumed in the sequel. The canonical morphism $\psi_M: M \to \mathbf{i} \circ \mathbf{a}(M)$ will be denoted by ψ.

LEMMA 1.3. *$\mathbf{a}(M)$ is a generator of $\mathscr{C}_M$.*

Proof. Let $h: M^{(\mathrm{Hom}(M,X))} \to X$ be the canonical morphism for each $\mathbf{T}$-closed object X in $\mathscr{C}$. Since $\mathrm{Coker}\, h \in \mathbf{T}$, it follows from the exactness of the functor $\mathbf{a}$ and the fact that $\mathbf{a}$ commutes with direct sums that $\mathbf{a}(h): \mathbf{a}(M)^{(\mathrm{Hom}(M,X))} \to X$ is an epimorphism.

Given an object X of a Grothendieck category $\mathscr{A}$, let us put $R = \mathrm{End}_{\mathscr{A}}(X)$. By [13, Theorem VI.3.1], the functor $\mathrm{Hom}_{\mathscr{A}}(X, -)$ from $\mathscr{A}$ to R-mod has a left adjoint which we denote by $X \otimes_R -: R\text{-mod} \to \mathscr{A}$. If I is a left ideal of R, then XI will denote the image of the canonical morphism $X \otimes_R I \to X \otimes_R R \cong X$. Then it is clear that $XI = \sum\{\mathrm{Im}\, \alpha \mid \alpha \in I\}$. Henceforth, we will use S' to denote the endomorphism ring of $\mathbf{a}(M)$, $S' = \mathrm{End}_{\mathscr{C}_M}(\mathbf{a}(M)) \cong \mathrm{End}_{\mathscr{C}}(\mathbf{ia}(M))$. There is a canonical ring homomorphism $\mu: S \to S'$ given by $\mu(f) = \mathbf{a}(f)$ $(= \mathbf{ia}(f))$.

PROPOSITION 1.4. *The class $\mathscr{T}$ of all the left S'-modules X such that $\mathbf{ia}(M) \otimes_{S'} X$ is a torsion object of $\mathscr{C}$ is a torsion class of S'-mod. If $\mathscr{G}$ is the Gabriel filter on S' corresponding to this torsion theory, then the functor $H = \mathrm{Hom}_{\mathscr{C}}(\mathbf{ia}(M), -): \mathscr{C} \to S'$-mod induces an equivalence of categories between $\mathscr{C}_M$ and the quotient category $(S', \mathscr{G})$-mod.*

Proof. By [18, Theorem X.4.1], the functor $H \circ \mathbf{i}: \mathscr{C}_M \to S'$-mod induces an equivalence of categories between $\mathscr{C}_M$ and the quotient category $(S', \mathscr{G})$-mod of S'-mod corresponding to a certain torsion theory $(\mathbf{T}', \mathbf{F}')$ of S'-mod. Then, the composition of the localization functor from S'-mod to $(S', \mathscr{G})$-mod followed by the equivalence is a left adjoint of $H \circ \mathbf{i}$, thus it can be identified with $\mathbf{a} \circ G$, where $G = \mathbf{ia}(M) \otimes_{S'} -: S'\text{-mod} \to \mathscr{C}$. Therefore a left S'-module X is $\mathbf{T}'$-torsion if and only if $\mathbf{a}(\mathbf{ia}(M) \otimes_{S'} X)) = 0$, that is, if and only if $\mathbf{ia}(M) \otimes_{S'} X$ is a torsion object of $\mathscr{C}$, i.e., $\mathbf{T}' = \mathscr{T}$.

Note that the Gabriel filter $\mathscr{G}$ consists of all the left ideals I of S' such that $\mathbf{ia}(M)/\mathbf{ia}(M)I$ is a torsion object of $\mathscr{C}$.

It is natural to ask under what conditions the category equivalence of Proposition 1.4 results in an equivalence between $\mathscr{C}_M$ and a quotient category of S-mod. To answer this, we will need the following lemma.

LEMMA 1.5. *Let I be a left ideal of S. Then, M/MI is a $\mathbf{T}$-torsion object of $\mathscr{C}$ if and only if $S'\mu(I) \in \mathscr{G}$.*

Proof. Let $h: M^{(I)} \to M$ be the unique morphism such that, for each

$f \in I$, we have $h \circ q_f = f$, $q_f: M \to M^{(I)}$ being the canonical injections. Then $\operatorname{Im} h = MI$ and we have an induced commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccc} M^{(I)} & \xrightarrow{h} & M \\ {\scriptstyle \psi^{(I)}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbf{ia}(M)^{(I)} & \xrightarrow{h'} & \mathbf{ia}(M) \end{array}$$

where the existence of a unique h' follows from the facts that $\mathbf{ia}(M)$ is $\mathbf{T}$-closed and $\operatorname{Ker} \psi^{(I)}$ and $\operatorname{Coker} \psi^{(I)}$ are $\mathbf{T}$-torsion objects. Then, h' verifies $h' \circ q'_f = \mu(f)$, where the q'_f are the canonical injections from $\mathbf{ia}(M)$ to $\mathbf{ia}(M)^{(I)}$. From this it follows that $\operatorname{Im} h' = \Sigma\{\operatorname{Im}(\mu(f)) \mid f \in I\} = \mathbf{ia}(M)\, S'\mu(I)$. Thus we obtain a unique morphism $g: MI \to \mathbf{ia}(M)\, S'\mu(I)$ such that $\operatorname{Coker} g$ is $\mathbf{T}$-torsion (because it is a quotient of $\operatorname{Coker} \psi^{(I)}$) and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & MI & \longrightarrow & M & \longrightarrow & \operatorname{Coker} h \cong M/MI & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & \mathbf{ia}(M)\, S'\mu(I) & \longrightarrow & \mathbf{ia}(M) & \longrightarrow & \operatorname{Coker} h' \cong \mathbf{ia}(M)/\mathbf{ia}(M)\, S'\mu(I) & \longrightarrow & 0 \end{array}$$

is commutative with exact rows. Inasmuch as $\operatorname{Ker} \psi$ and $\operatorname{Coker} \psi$ are $\mathbf{T}$-torsion, $\operatorname{Ker} k$ and $\operatorname{Coker} k$ are $\mathbf{T}$-torsion as well, by the Ker–Coker lemma. Therefore, M/MI is $\mathbf{T}$-torsion if and only if so is $\mathbf{ia}(M)/\mathbf{ia}(M)\, S'\mu(I)$, that is, M/MI is $\mathbf{T}$-torsion if and only if $S'\mu(I)$ belongs to $\mathscr{G}$.

We are now ready to prove the main result of this section.

THEOREM 1.6. *Let* $j: (S', \mathscr{G})$-mod $\to S'$-mod *be the inclusion functor and* $\mu_*: S'$-mod $\to S$-mod *the restriction of scalars functor. The following conditions are equivalent.*

(i) $\mu_* \circ j: (S', \mathscr{G})$-mod $\to S$-mod *has an exact left adjoint.*

(ii) *For each S-monomorphism* $L \to N$, *the kernel of the induced morphism* $M \otimes_S L \to M \otimes_S N$ *is a* $\mathbf{T}$*-torsion object of* $\mathscr{C}$.

(iii) *The class of all the left S-modules X such that* $M \otimes_S X$ *is* $\mathbf{T}$*-torsion is a (hereditary) torsion class of* S-mod.

(iv) $\mathscr{F} = \{I \leqslant_S S \mid M/MI$ *is a* $\mathbf{T}$*-torsion object of* $\mathscr{C}\}$ *is a left Gabriel topology of S.*

Moreover, when these equivalent conditions hold, the functor $\operatorname{Hom}_{\mathscr{C}}(M, -)$: $\mathscr{C} \to S$-mod *induces an equivalence of categories between* $\mathscr{C}_M$ *and* $(S, \mathscr{F})$-mod, *and* S' *is the ring of quotients of S with respect to* $\mathscr{F}$, $S' = S_{\mathscr{F}}$.

Proof. (i) ⇒ (ii) Since $(S', \mathscr{G})$-mod and $\mathscr{C}_M$ are equivalent categories, the hypothesis implies that the functor $\mu_* \circ H \circ \mathbf{i}$ from $\mathscr{C}_M$ to S-mod has an exact left adjoint. By using the fact that, for all objects X in $\mathscr{C}_M$, we have $\mathrm{Hom}_{\mathscr{C}}(\mathbf{ia}(M), \mathbf{i}(X)) \cong \mathrm{Hom}_{\mathscr{C}}(M, \mathbf{i}(X))$, we see that the above functor is naturally equivalent to the composition $\mathscr{C}_M \xrightarrow[\mathbf{i}]{} \mathscr{C} \xrightarrow[\mathrm{Hom}_{\mathscr{C}}(M,\,-)]{} S$-mod. A left adjoint to this composition is precisely $S\text{-mod} \xrightarrow[M \otimes_S -]{} \mathscr{C} \xrightarrow[\mathbf{a}]{} \mathscr{C}_M$. But, by (i), this functor is exact and hence, bearing in mind that $\mathbf{a}$ is also exact, we easily obtain that (ii) holds.

(ii) ⇒ (iii) Since the class $\{X \in S\text{-mod} \mid M \otimes_S X \text{ is } \mathbf{T}\text{-torsion}\}$ is always closed under extensions, direct sums, and epimorphic images, it is only left to show that it is closed for submodules. But this follows from (ii) in a straightforward manner.

(iii) ⇒ (iv) By hypothesis, $\{I \leqslant_S S \mid M \otimes_S (S/I) \text{ is } \mathbf{T}\text{-torsion}\}$ is a left Gabriel topology [18, Theorem VI.5.1]. Since $M \otimes_S (S/I) \cong M/MI$, the result is clear.

(iv) ⇒ (i) We first prove that $\mu: S \to S'$ has $\mathscr{F}$-torsion kernel and cokernel. Note that, as a left S-module, S' may be identified with $\mathrm{Hom}_{\mathscr{C}}(M, \mathbf{ia}(M))$. Let then $s \in \operatorname{Ker} \mu$, so that $\operatorname{Im} s \subseteq \mathbf{t}(M)$ and $M/\operatorname{Ker} s$ is a $\mathbf{T}$-torsion object. Hence if $I = \mathrm{Hom}_{\mathscr{C}}(M, \operatorname{Ker} s)$ then $MI = (\operatorname{Ker} s)_M$ and thus $I \in \mathscr{F}$ and annihilates s. This shows that $\operatorname{Ker} \mu$ is $\mathscr{F}$-torsion. Now, let $s' \in S'$ and consider the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\beta} & M \\ \downarrow & & \downarrow{\scriptstyle s'} \\ M & \xrightarrow[\psi]{} & \mathbf{ia}(M) \end{array}$$

Since $\operatorname{Coker} \beta$ is a subobject of $\operatorname{Coker} \psi$, it is $\mathbf{T}$-torsion. Let I be the left ideal of S consisting of all the endomorphisms of M factoring through β. Then we have that $MI = \beta(X_M)$ and hence M/MI is also $\mathbf{T}$-torsion, because $\operatorname{Coker} \beta \cong M/\beta(X)$ and X/X_M are $\mathbf{T}$-torsion. Thus $I \in \mathscr{F}$ and, since $\mu(S) = \{f \in S' \mid f = \psi \circ s, \text{ for some } s \in S\}$, we see that $Is' \subseteq \mu(S)$, from which it follows that $S'/\mu(S) \cong \operatorname{Coker} \mu$ is an $\mathscr{F}$-torsion module.

We then proceed to show that S' is $\mathscr{F}$-torsionfree. Let $\mathscr{F}^e = \{I' \leqslant S' \mid \mu^{-1}(I') \in \mathscr{F}\}$. We deduce from Lemma 1.5 that $I' \in \mathscr{F}^e$ if and only if $S'\mu\mu^{-1}(I') \in \mathscr{G}$. Then, clearly $\mathscr{F}^e \subseteq \mathscr{G}$. Conversely, let $I' \in \mathscr{G}$. Then $I'/S'\mu\mu^{-1}(I')$ is $\mathscr{F}$-torsion as a left S-module and it follows from [12, Lemma 2.2] and the fact that $\mathscr{F}^e \subseteq \mathscr{G}$ that it is also $\mathscr{G}$-torsion. It is then immediate by using again Lemma 1.5 that $I' \in \mathscr{F}^e$. Therefore, $\mathscr{G} = \mathscr{F}^e$ and, since S' is $\mathscr{G}$-torsionfree, we conclude from [12, Theorem 2.5] that S' is also $\mathscr{F}$-torsionfree as a left S-module.

Let $S_{\mathscr{F}}$ be the ring of quotients of S, $\phi: S \to S_{\mathscr{F}}$ the canonical homomorphism. Then, due to the facts that μ has torsion kernel and cokernel and $S_{\mathscr{F}}$ is $\mathscr{F}$-closed, ϕ factors uniquely through μ, in the form $\phi = \sigma \circ \mu$. Since S' is $\mathscr{F}$-torsionfree, we have that $\operatorname{Ker} \mu = \mathbf{t}_{\mathscr{F}}(S) = \operatorname{Ker} \phi$, so that, bearing in mind that $\mu(S)$ is essential in S', we obtain that σ is a monomorphism. In fact, it is easy to see that σ is also a ring homomorphism, so that there is an exact sequence of S'-modules $0 \to S' \xrightarrow{\sigma} S_{\mathscr{F}} \to T \to 0$, where T, being an $\mathscr{F}$-torsion S-module, is also $\mathscr{G}$-torsion by [12, Theorem 2.5]. Inasmuch as S' is $\mathscr{G}$-closed, T must be zero and σ is an isomorphism. Then $\mu: S \to S'$ can be considered as the ring of quotients of S with respect to $\mathscr{F}$ and thus the functor $\mu_* \circ j$ from $(S', \mathscr{G})$-mod to S-mod has an exact left adjoint [18, p. 217], proving (i).

When these equivalent conditions hold, then, as we have just seen, $\mathscr{G} = \mathscr{F}^e$ and the quotient category $(S, \mathscr{F})$-mod consists exactly of the $\mathscr{G}$-closed S'-modules, viewed as left S-modules. Therefore, by Proposition 1.4, the funtor $\operatorname{Hom}_{\mathscr{C}}(M, -): \mathscr{C} \to S$-mod induces an equivalence of categories between $\mathscr{C}_M$ and $(S, \mathscr{F})$-mod.

Next we show that the conditions of Theorem 1.6 do hold under fairly general hypotheses.

THEOREM 1.7. *If $\mathscr{C}$ is a locally finitely generated Grothendieck category, then the functor* $\operatorname{Hom}_{\mathscr{C}}(M, -): \mathscr{C} \to S$-mod *induces an equivalence of categories between $\mathscr{C}_M$ and* $(S, \mathscr{F})$-mod, *$\mathscr{F}$ being the left Gabriel topology $\{I \leqslant_S S \mid M/MI$ is $\mathbf{T}$-torsion$\}$.*

Proof. It will be enough to prove that $\mathscr{F}$ is indeed a left Gabriel topology on S, and then use Theorem 1.6. In order to do this, we only need to show that $\mathscr{F}$ satisfies the following two properties (see [18, Lemma VI.5.2]):

T3. $(I: s) \in \mathscr{F}$ for every $s \in S$ and every $I \in \mathscr{F}$.

T4. If $(I: s) \in \mathscr{F}$ for every $s \in J$, with $J \in \mathscr{F}$, then $I \in \mathscr{F}$.

As a consequence of the fact that $\{X \in S\text{-mod} \mid M \otimes_S X \in \mathbf{T}\}$ is always closed under extensions, direct sums, and quotients, property T4 is deduced in an entirely similar way to [18, Theorem VI.5.1]. To show that property T3 holds, let us consider for each object X of $\mathscr{C}$ the following S-submodule X^* of $\operatorname{Hom}_{\mathscr{C}}(M, X)$, $X^* = \{f: M \to X \mid$ there is $X_0 \leqslant X$, X_0 finitely generated, and $\operatorname{Im} f \subseteq X_0\}$, and the morphism $\phi: M^{(X^*)} \to X$ such that $\phi \circ q_f = f$, for every $f \in X^*$, $q_f: M \to M^{(X^*)}$ being the canonical injections. Since X is the direct union of its finitely generated subobjects (because $\mathscr{C}$ is locally finitely generated), it is not hard to prove that $\operatorname{Coker} \phi$ is a quotient of a direct sum of objects of the type N/N_M, where N ranges over all the finitely generated subobjects of X. Consequently, $\operatorname{Coker} \phi$ is a

T-torsion object of $\mathscr{C}$. Now, let $I \in \mathscr{F}$ and $s \in S$, and let $\alpha: M^{(I)} \to M$ be such that $\alpha \circ q_f = f$, for every $f \in I$. Consider then the cartesian square

$$\begin{array}{ccc} X & \xrightarrow[\beta]{} & M \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle s} \\ M^{(I)} & \xrightarrow[\alpha]{} & M \end{array}$$

where $\operatorname{Coker} \alpha \cong M/MI$ is **T**-torsion and thus so is $\operatorname{Coker} \beta \cong M/\beta(X)$. Let X^* and ϕ be as before; then $X^*\beta = \{\beta \circ f \mid f \in X^*\} = J$ is a left ideal of S and $MJ = \beta(\operatorname{Im} \phi)$ is such that M/MJ is **T**-torsion, because, as we have just seen, $\operatorname{Coker} \phi$ is **T**-torsion. On the other hand, it is easy to see that if $g \in Js = X^*v\alpha$ $(= \{\alpha \circ v \circ f \mid f \in X^*\})$, then there exist a finite subset $F \subseteq I$ and a morphism $g': M \to M^{(F)}$ such that $g = \alpha \circ u \circ g'$, $u: M^{(F)} \to M^{(I)}$ being the canonical morphism. If we denote by $q'_f: M \to M^{(F)}$ and $p'_f: M^{(F)} \to M$ the canonical injections and projections for each f in F, we have $g = \alpha \circ u \circ (\Sigma_F q'_f \circ p'_f) \circ g' = \Sigma_F (\alpha \circ q_f) \circ (p'_f \circ g') = \Sigma_F f \circ s_f$, where $s_f = p'_f \circ g' \in S$. Thus $g = \Sigma_F s_f f \in I$, hence $Js \subseteq I$ and $J \subseteq (I: s)$ from which property T3 follows.

EXAMPLES 1.8. (a) Let R be a ring and $\mathscr{C} = R$-mod, so that M is a left R-module. Then (**T**, **F**) is in this case the torsion theory of R-mod determined by the trace T_M of M on R $(T_M = \Sigma\{\operatorname{Im} \alpha \mid \alpha \in \operatorname{Hom}_R(M, R)\})$ and the corresponding quotient category $\mathscr{C}_M$ coincides with the category $\mathscr{U}_R$ in [14]. By Theorem 1.7, the functor $\operatorname{Hom}_R(M, -)$: R-mod $\to$ S-mod induces an equivalence between $\mathscr{U}_R$ and $(S, \mathscr{F})$-mod. This equivalence is also obtained as a consequence of [14, Theorem 3].

(b) Let M be a left R-module and take $\mathscr{C} = \sigma[M]$, the category of all the left R-modules subgenerated by M [20]. Then the torsion theory (**T**, **F**) is just the torsion theory of $\sigma[M]$ given in [9, Proposition 1.1]. The quotient category in this case was denoted therein by $\mathscr{C}[M]$ and it is equivalent to $(S, \mathscr{F})$-mod, since $\sigma[M]$ is a locally finitely generated Grothendieck category. The **T**-torsionfree modules of $\sigma[M]$ are called M-faithful modules.

Some corollaries of the above results are worth mentioning.

COROLLARY 1.9. *Let M be a left R-module and $S = \operatorname{End}({}_RM)$. Then for each monomorphism $L \to N$ in S-*mod*, the kernel of the induced homomorphism $M \otimes_S L \to M \otimes_S N$ is a torsion R-module in the theory determined by the trace ideal T_M of M in R.*

COROLLARY 1.10. *Let M be a left R-module, $S = \operatorname{End}({}_RM)$, and N an*

M-faithful module. Then $\mathrm{Hom}_R(M, N)$ *is an injective left S-module if and only if N is M-injective.*

Proof. This is obtained from [9, Theorem 2.1], by removing the condition that all canonical homormorphisms $M \otimes_S I \to M$ have torsion kernel, since we know by Theorem 1.7 that this is indeed the case.

COROLLARY 1.11. *Let* $\mathscr{C}$ *be a locally finitely generated Grothendieck category and let us assume that M is M-distinguished and S is left nonsingular. Then the endomorphism ring of the injective hull of M is isomorphic to the maximal left ring of quotients of S.*

Proof. If M is **T**-torsionfree, then the injective hull of M, $E(M)$, coincides with the injective hull of $\mathbf{a}(M)$ in the category $\mathscr{C}_M$ [6, Proposition III.6]. Thus, in the equivalence of Theorem 1.7 it corresponds to the injective hull of $S_{\mathscr{F}}$, which is precisely $E(S)$, because S is $\mathscr{F}$-torsionfree. The endomorphism rings of these two corresponding objects are then isomorphic and, since S is left nonsingular, each of them is isomorphic to the maximal left ring of quotients of S.

Note that, in particular, if M satisfies the hypotheses of [21, Theorem 2.2], then M is clearly M-distinguished in R-mod and S is left nonsingular, so that [21, Theorem 2.2(ii)] can be deduced from Corollary 1.11.

Since we want to study properties of S by using the category equivalence of Theorem 1.7 it will be interesting to determine when the canonical homomorphism $\mu: S \to S'$ is an isomorphism, for, in this case, S is an object of $(S, \mathscr{F})$-mod. To accomplish this, we need the following definitions.

DEFINITION 1.12. M will be called weakly M-distinguished if the following two conditions are verified: (i) $\mathrm{Hom}_{\mathscr{C}}(M, \mathbf{t}(M)) = 0$; and (ii) for every morphism $f: M \to \bar{M}$ there exists an endomorphism s of M such that $p \circ s = f$, where $p: M \to \bar{M}$ is the canonical projection.

It is clear that M is weakly M-distinguished if and only if the canonical ring homomorphism $S \to \mathrm{End}_{\mathscr{C}}(\bar{M})$ is an isomorphism.

DEFINITION 1.13. Let X be an object of $\mathscr{C}$. X will be called **T**-M-injective if for each exact sequence $0 \to L \xrightarrow{u} M \to C \to 0$ such that C is a **T**-torsion object of $\mathscr{C}$, the canonical homomorphism $u^*: \mathrm{Hom}_{\mathscr{C}}(M, X) \to \mathrm{Hom}_{\mathscr{C}}(L, X)$ is a surjection.

Finally, let us call M weakly **T**-closed when M is weakly M-distinguished and $\bar{M}$ is **T**-M-injective. We have:

PROPOSITION 1.14. *$\mu: S \to S'$ is an isomorphism if and only if M is weakly* **T**-*closed.*

Proof. Since the canonical morphism $\psi: M \to \mathbf{ia}(M)$ factors in the form $M \xrightarrow{p} \bar{M} \xrightarrow{j} \mathbf{ia}(M)$, where $j = \psi_{\bar{M}}$ is a monomorphism, it is clear that if $\mu: S \to S'$ is an isomorphism, then $S \to \operatorname{End}({}_R\bar{M})$ is also an isomorphism. Thus, all we have to prove is that, under the assumption that M is weakly M-distinguished, $\bar{M}$ is **T**-M-injective if and only if $\alpha(\bar{M}) \subseteq \bar{M}$ for every $\alpha \in S'$. So, let $\bar{M}$ be **T**-M-injective and $\alpha \in S'$. Then, let $f = \alpha \circ \psi: M \to \mathbf{ia}(M)$ and $L = f^{-1}(\bar{M})$. It is clear that M/L is **T**-torsion and hence there exists $h: M \to \bar{M}$ such that the restrictions to L of h and f coincide. By composing h with j we have that $j \circ h - f$ vanishes on L. Since $M/L \in \mathbf{T}$ and $\mathbf{ia}(M) \in \mathbf{F}$, this implies that $j \circ h = f$, so that $\operatorname{Im} f = \alpha(\bar{M}) \subseteq \bar{M}$. Conversely, assume that this last condition holds for each $\alpha \in S'$ and let $0 \to L \xrightarrow{u} M \to C \to 0$ be an exact sequence with $C \in \mathbf{T}$. If $f: L \to \bar{M}$ is a morphism of $\mathscr{C}$, then $j \circ f: L \to \mathbf{ia}(M)$ induces $g: M \to \mathbf{ia}(M)$ with $g \circ u = j \circ f$, by the **T**-injectivity of $\mathbf{ia}(M)$. By hypothesis, since $\mathbf{ia}(g) \in S'$, there exists an endomorphism h of $\bar{M}$ such that $j \circ h = \mathbf{ia}(g) \circ j$. Then $j \circ h \circ p \circ u = \mathbf{ia}(g) \circ j \circ p \circ u = g \circ u = j \circ f$, which gives $h \circ p \circ u = f$, because j is a monomorphism. Thus $h \circ p: M \to \bar{M}$ is an extension of f to M and $\bar{M}$ is **T**-M-injective.

Remarks 1.15. (a) If M is a left R-module and $\mathscr{C} = \sigma[M]$, then M is weakly **T**-closed if M is quasi-injective and M-faithful in the sense of [9]. On the other hand, if M is a *CQF*-3 object of the Grothendieck category $\mathscr{C}$, then M is weakly **T**-closed if and only if condition (ii) in Definition 1.12 holds; this happens, for instance, if M is quasiprojective. In particular, all projective objects are weakly **T**-closed.

(b) Note that M can be a *CQF*-3 object of $\mathscr{C}$ without being weakly **T**-closed. For instance, let R be the ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ of upper triangular matrices over a field k, $N = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$, $L = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ left ideals of R, $U = R/L$, and $M = N \oplus U$. Take $\mathscr{C} = R$-mod. Then the trace of M on R is N, so that M is trace-accessible, hence *CQF*-3. But $\bar{M} \cong U \oplus U$ and it is easy to see that the homomorphism $f: M \to \bar{M}$ which is zero over N and takes $1 \in U \cong k$ to the pair $(\overline{e_{22}}, 0)$ (where $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) cannot be lifted through the projection $p: M \to \bar{M}$, so that M is not weakly M-distinguished.

On the other hand, if M is weakly **T**-closed it need not be *CQF*-3. An example of this is obtained by taking a left self-injective ring A and an idempotent two-sided ideal I of A such that the right annihilator of I in A is zero (for instance, we could take A to be an infinite product of copies of a field k, $A = \prod_J k_j$, with $k_j = k$ for all $j \in J$, and $I = \bigoplus_J k_j$). Then let R be the ring of upper triangular matrices $R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ and M be the two-sided ideal of R, $M = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$. Then, M is not trace-accessible, since $M^2 = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = N$ and the torsion theory $(\mathbf{T}, \mathbf{F})$ of R-mod associated to M is just the torsion theory whose Gabriel filter consists of all the left ideals of R containing N. Hence M is M-distinguished, as N does not annihilate any nonzero element of R, and it is not CQF-3. To see that M is weakly $\mathbf{T}$-closed, it is only left to show that it is $\mathbf{T}$-M-injective. If we call $N_1 = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $N_2 = \left(\begin{smallmatrix} 0 & A \\ 0 & I \end{smallmatrix}\right)$, and $N_3 = \left(\begin{smallmatrix} 0 & I \\ 0 & I \end{smallmatrix}\right)$, then $M = N_1 \oplus N_2$, $NN_1 = N_1$, $NN_2 = N_3$. Thus it will suffice to prove that each homomorphism $h\colon N_3 \to N_2$ can be extended to an endomorphism of N_2. But an easy computation shows that, since A is left self-injective, this is indeed the case.

(c) It may happen that M is weakly $\mathbf{T}$-closed as an object of a given category, but not when M is considered as an object of another category. For example, let M be a simple left R-module which is not isomorphic to a left ideal of R. Then, the trace of M on R is zero, so that when one takes $\mathscr{C} = R$-mod, then S' is the zero ring. But if $\mathscr{C} = \sigma[M]$, then M is a generator of $\mathscr{C}$ and hence $\mathbf{ia}(M) = M$ and $S' = S$. However, the converse situation cannot occur, that is, if M is weakly $\mathbf{T}$-closed as a left R-module, then M has the same property when considered as an object of $\sigma[M]$. This is shown in the next proposition.

PROPOSITION 1.16. *Let M be a left R-module. If M is weakly M-distinguished (resp., weakly $\mathbf{T}$-closed) in R-mod, then M is weakly M-distinguished (resp., weakly $\mathbf{T}$-closed) in $\sigma[M]$.*

Proof. Let $(\mathbf{T}, \mathbf{F})$ be the torsion theory of $\sigma[M]$ associated to M and $(\mathbf{T}_1, \mathbf{F}_1)$ the torsion theory of R-mod associated to M, and let $\mathbf{t}, \mathbf{t}_1$ be the corresponding radicals. Since each module belonging to $\mathbf{F}_1$ and $\sigma[M]$ is clearly in $\mathbf{F}$, we have that each module of $\mathbf{T}$ belongs to $\mathbf{T}_1$. Therefore, $\mathbf{t}(M) \subseteq \mathbf{t}_1(M)$. Now, if M is weakly M-distinguished in R-mod, then $\mathrm{Hom}_R(M, \mathbf{t}_1(M)) = 0$, so that $\mathbf{t}_1(M) \in \mathbf{T}$ and $\mathbf{t}_1(M) = \mathbf{t}(M)$. Thus it follows that M is also weakly M-distinguished in $\sigma[M]$. The latter assertion is now immediate.

This result suggests that in order to study the endomorphism ring of a left R-module M by using the equivalence of categories given in Theorem 1.7 it is preferable to take $\mathscr{C} = \sigma[M]$ than $\mathscr{C} = R$-mod.

When $\mathscr{F}$ is the trivial topology $\{S\}$, then S-mod is equivalent to a quotient category of $\mathscr{C}$. We study next when this is the case.

THEOREM 1.17. *Let $\mathscr{C}$ be locally finitely generated and $\mathscr{F}$ the filter $\{I \leqslant_S S \mid M/MI \text{ is } \mathbf{T}\text{-torsion}\}$. Then $\mathscr{F}$ is the trivial filter $\mathscr{F} = \{S\}$ if and only if M is a finitely generated quasiprojective and CQF-3 object of $\mathscr{C}$.*

Proof. Let us assume that $\mathscr{F} = \{S\}$. If $f\colon M \to N$ is an epimorphism and N is a nonzero $\mathbf{T}$-torsion object, then $X = \mathrm{Ker}\, f \subsetneq M$ satisfies that $M/X \in \mathbf{T}$. Consider $I = \mathrm{Hom}_{\mathscr{C}}(M, X)$ as a left ideal of S. Then $MI = X_M$ and so M/MI

is T-torsion, so that $I \in \mathscr{F}$, which is a contradiction because $I \neq S$. This shows that $\mathbf{T} = \{X \in \mathrm{Ob}(\mathscr{C}) \mid \mathrm{Hom}_{\mathscr{C}}(M, X) = 0\}$ and thus M is a CQF-3 object of $\mathscr{C}$.

To show that M is finitely generated, we take $\{M_i\}_{i \in I}$ a directed family of subobjects of M with $M = \sum_I M_i$. It is straightforward to see that $\mathbf{a}(M)$ is the direct union of the $\mathbf{a}(M_i)$ in $\mathscr{C}_M$; from the equivalence of categories between $\mathscr{C}_M$ and S-mod and the fact that $\mathbf{a}(M)$ corresponds to S in this equivalence, we get that $\mathbf{a}(M)$ is finitely generated (and projective) in $\mathscr{C}_M$ and hence $\mathbf{a}(M) = \mathbf{a}(M_i)$ for some $i \in I$. This implies that M/M_i is $\mathbf{T}$-torsion and, since M is CQF-3, one has $M = M_i$, from which we see that M is finitely generated.

Let now $p: M \to N$ be an epimorphism and $f: M \to N$ an arbitrary morphism. By the exactness of the functor $\mathbf{a}$ and the projectivity of $\mathbf{a}(M)$ in $\mathscr{C}_M$, we obtain a morphism $h: \mathbf{a}(M) \to \mathbf{a}(M)$ such that $\mathbf{a}(p) \circ h = \mathbf{a}(f)$. Since in this case M is, clearly, weakly $\mathbf{T}$-closed, we have $h = \mathbf{a}(g)$ for some $g: M \to M$. Then $\mathbf{a}(p \circ g) = \mathbf{a}(f)$ and thus $\mathrm{Im}(p \circ g - f)$ is $\mathbf{T}$-torsion. But $\mathrm{Im}(p \circ g - f)$ is a quotient of M, and hence zero. Therefore $p \circ g = f$ and M is quasiprojective.

Conversely, let us suppose that M is CQF-3, finitely generated, and quasiprojective. As seen in Remark 1.15(a), M is weakly $\mathbf{T}$-closed. Furthermore, it is easy to see, by using [18, Lemma V.3.3; 1, Proposition I.1.8], that M is Σ-quasiprojective. On the other hand, since M is CQF-3 we have that for a left ideal I of S, $M/MI \in \mathbf{T}$ if and only if $M = MI$, that is, $\mathscr{F} = \{I \leqslant_S S \mid MI = M\}$. Let $I \in \mathscr{F}$ and take $p: M^{(I)} \to M$ to be such that $p \circ q_f = f$ for every $f \in I$, the q_f being the canonical injections. The assumption that $I \in \mathscr{F}$ implies that p is an epimorphism, and hence it splits, because M is Σ-quasiprojective. Since M is finitely generated, it follows that there exists a finite subset $F \subseteq I$ such that the canonical morphism $p': M^{(F)} \to M$ is a split epimorphism. Then $1_M = p' \circ u = p' \circ (\Sigma_F q'_f \circ g_f) = \Sigma_F f \circ g_f = \Sigma_F g_f f \in I$, where $u: M \to M^{(F)}$ induces $g_f: M \to M$ for each $f \in F$, and $q'_f: M \to M^{(F)}$ are the canonical injections. Thus $I = S$ and $\mathscr{F} = \{S\}$.

Remark 1.18. If in Theorem 1.17 we drop the assumption of M being CQF-3, then the result is no longer true, as the example of a simple module in Remark 1.15(c) shows.

The preceding result is reminiscent of that of Fuller [5, Theorem 1.1] stating that if a full subcategory $\mathscr{C}$ of R-mod which is closed under submodules, quotients, and direct sums is equivalent to a module category S-mod, then there is a left R-module M such that $S \cong \mathrm{End}({}_R M)$ and M is a finitely generated and quasiprojective self-generator. In order to better study this connection we prove the following result.

THEOREM 1.19. *Let S be a ring and $\mathscr{C}$ a Grothendieck category with a* *projective generator. Assume that* $(\mathbf{T}_0, \mathbf{F}_0)$ *is a torsion theory in* $\mathscr{C}$ *such that* $\mathbf{T}_0$ *is closed under products and let* $\mathscr{F}$ *be a left Gabriel filter on S such that S is* $\mathscr{F}$*-closed. If* $F: \mathscr{C}/\mathbf{T}_0 \to (S, \mathscr{F})\text{-mod}$ *is an equivalence of categories, then there exists an object M of* $\mathscr{C}$ *such that*: $(\mathbf{T}_0, \mathbf{F}_0)$ *is the torsion theory* $(\mathbf{T}, \mathbf{F})$ *of* $\mathscr{C}$ *associated to M,* $S \cong \mathrm{End}_{\mathscr{C}}(\bar{M})$, *F is naturally equivalent to the restriction to the subcategory* $\mathscr{C}/\mathbf{T}_0$ *of the functor* $\mathrm{Hom}_{\mathscr{C}}(\bar{M}, -): \mathscr{C} \to (S, \mathscr{F})\text{-mod}$, *and* $\mathscr{F} = \{I \leqslant_S S \mid \bar{M}I = \bar{M}\}$.

Proof. For each object X of $\mathscr{C}$, take $d(X) = \bigcap \{Y \subseteq X \mid X/Y \in \mathbf{T}_0\}$. Since $\mathbf{T}_0$ is closed under products, we have that d is an epi-preserving preradical, $X/d(X) \in \mathbf{T}_0$, and $\mathrm{Hom}_{\mathscr{C}}(d(X), Y) = 0$ for every $Y \in \mathbf{T}_0$. Let U be an object of $\mathscr{C}/\mathbf{T}_0$ such that $F(U) \cong S \cong S_{\mathscr{F}}$. Then $\mathrm{End}_{\mathscr{C}}(U) = \mathrm{End}_{\mathscr{C}/T_0}(U) \cong \mathrm{End}_S(S) = S$. It follows easily that the functor F is naturally equivalent to $\mathrm{Hom}_{\mathscr{C}}(U, -) \cong \mathrm{Hom}_{\mathscr{C}/\mathbf{T}_0}(U, -)$ from $\mathscr{C}/\mathbf{T}_0$ to $(S, \mathscr{F})$-mod. Let $L = d(U)$ and $X \in \mathbf{F}_0$: we claim that $X/X_L \in \mathbf{T}_0$, i.e., $d(X) \subseteq X_L$. By using the fact that U is a generator of $\mathscr{C}/\mathbf{T}_0$, we obtain a morphism $q: U^{(I)} \to \mathbf{a}'(X)$ (where $\mathbf{a}': \mathscr{C} \to \mathscr{C}/\mathbf{T}_0$ is the canonical functor) which has a $\mathbf{T}_0$-torsion cokernel. Since $\mathbf{a}'(X)/X$ is $\mathbf{T}_0$-torsion, we have that $U^{(I)}/q^{-1}(X)$ is also $\mathbf{T}_0$-torsion and hence $d(U^{(I)}) = L^{(I)} \subseteq q^{-1}(X)$ [1, Lemma 3.7.1]. On the other hand, if $q': q^{-1}(X) \to X$ is the restriction of q, then we have an exact sequence $0 \to (X \cap \mathrm{Im}\, q)/\mathrm{Im}\, q' \to X/\mathrm{Im}\, q' \to \mathbf{a}'(X)/\mathrm{Im}\, q$. Now, $(X \cap \mathrm{Im}\, q)/\mathrm{Im}\, q'$ is a subobject of $\mathrm{Im}\, q/\mathrm{Im}\, q'$ which, in turn, is a quotient of $U^{(I)}/q^{-1}(X)$ and hence it is $\mathbf{T}_0$-torsion. Since the third member of the sequence is also $\mathbf{T}_0$-torsion, we get that $X/\mathrm{Im}\, q' \in \mathbf{T}_0$. But the fact that $q^{-1}(X)/L^{(I)}$ is $\mathbf{T}_0$-torsion implies that $\mathrm{Im}\, q'/q'(L^{(I)}) \in \mathbf{T}_0$ and, consequently, $X/q'(L^{(I)})$ is $\mathbf{T}_0$-torsion. Then $d(X) \subseteq q'(L^{(I)}) \subseteq X_L$, establishing the claim.

Note that, by [18, Theorem X.4.1], the category $\mathscr{C}$ is equivalent, via the exact functor $\mathrm{Hom}_{\mathscr{C}}(G, -): \mathscr{C} \to A\text{-mod}$ (where G is a projective generator of $\mathscr{C}$ and $A = \mathrm{End}_{\mathscr{C}}(G)$), to a Giraud subcategory of A-mod, and hence the objects, morphisms, and exact sequences of $\mathscr{C}$ may be considered as being in A-mod. Now, let $I = \mathrm{Hom}_{\mathscr{C}}(L, G/\mathbf{t}_0(G))$ and for each $i \in I$, consider the cartesian square

$$\begin{array}{ccc} X_i & \longrightarrow & G \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{\ i\ } & G/\mathbf{t}_0(G) \end{array}$$

where p is the canonical projection. Then f_i is an epimorphism, with $\mathbf{T}_0$-torsion kernel. Let M be the limit of the morphisms f_i from X_i to L, so that there are morphisms $g_i: M \to X_i$ such that $f_i \circ g_i$ is a fixed $h: M \to L$. Considering the above diagrams as in A-mod, we see that every f_i is an epimorphism and therefore h is also an epimorphism. Furthermore, Ker h, being a product of $\mathbf{T}_0$-torsion objects, is $\mathbf{T}_0$-torsion, so that there is an exact sequence $0 \to K \to M \to L \to 0$, with $K \in \mathbf{T}_0$. If $\phi: L^{(I)} \to G/\mathbf{t}_0(G)$ is induced by the $i \in I$, then Coker $\phi \in \mathbf{T}_0$, by our previous claim. Thus we have induced morphisms $h': M^{(I)} \to L^{(I)}$ and $g: M^{(I)} \to G$ such that h' is an epimorphism and $\phi \circ h' = p \circ g$, from which it follows that $\mathrm{Coker}(p \circ g) \cong G/\mathrm{Im}\, g + \mathbf{t}_0(G)) \in \mathbf{T}_0$ and hence $G/\mathrm{Im}\, g \in \mathbf{T}_0$ and $G/G_M \in \mathbf{T}_0$. Let $(\mathbf{T}, \mathbf{F})$ be the torsion theory of $\mathscr{C}$ associated to M. By Proposition 1.4, $\mathbf{T} \subseteq \mathbf{T}_0$. Since clearly M can be assumed to verify $d(M) = M$, we have also that $\mathbf{T}_0 \subseteq \mathbf{T}$ and thus $(\mathbf{T}, \mathbf{F}) = (\mathbf{T}_0, \mathbf{F}_0)$.

On the other hand, note that $\bar{M} = L$ and $\mathbf{a}'(M) \cong U$, so that the functor $\mathrm{Hom}_{\mathscr{C}}(U, -): \mathscr{C}/\mathbf{T}_0 \to (S, \mathscr{F})$-mod is naturally equivalent to $\mathrm{Hom}_{\mathscr{C}}(M, -)$: $\mathscr{C}/\mathbf{T}_0 \to (S, \mathscr{F})$-mod and hence this functor is equivalent to F. Besides, $\mathrm{Hom}_{\mathscr{C}}(L, U/L) = 0$ implies that $S \cong \mathrm{End}_{\mathscr{C}}(U) \cong \mathrm{End}_{\mathscr{C}}(L) = \mathrm{End}_{\mathscr{C}}(\bar{M})$. The final assertion of the theorem follows from the fact that $\bar{M}$ has no nonzero torsion quotients, along with Theorem 1.6.

Remark 1.20. It follows from the proof of the theorem that the condition of $\mathscr{C}$ having a projective generator may be replaced by either the existence of a $\mathbf{T}_0$-torsionfree generator of $\mathscr{C}$ or the condition that $\mathscr{C}$ is equivalent to a quotient category $(R, \mathscr{H})$-mod such that $\mathscr{H}$ is the left Gabriel filter of the ring R generated by an idempotent ideal. In the first case, $M = L = \bar{M}$ and $S \cong \mathrm{End}_{\mathscr{C}}(M)$. To complete the proof in the second case, note that, with the notation used above, $\mathrm{Im}\, h = \bigcap \{\mathrm{Im} f_i \mid i \in I\}$ in R-mod, so that h is also an epimorphism of $\mathscr{C}$.

On the other hand, if we delete the hypothesis of $\mathbf{T}_0$ being closed under products, then the result is no longer true, as the example of [10, Example 5] shows. Finally, the assumption of S being $\mathscr{F}$-closed is not restrictive, because in the general case a similar result to that of the theorem holds with $S_{\mathscr{F}}$ instead of S.

COROLLARY 1.21. *Let $\mathscr{C}$ be a Grothendieck category with a projective generator and $\mathbf{T}_0$ a torsion class of $\mathscr{C}$ which is closed under products. Assume that there is an equivalence $F: \mathscr{C}/\mathbf{T}_0 \to S$-mod for some ring S. Then there exists an object M of $\mathscr{C}$ such that M is a finitely generated, quasiprojective, and CQF-3 object of $\mathscr{C}$, F is naturally equivalent to the functor* $\mathrm{Hom}_{\mathscr{C}}(M, -)$: $\mathscr{C}/\mathbf{T}_0 \to S$-mod, $S \cong \mathrm{End}_{\mathscr{C}}(M)$, *and* $\mathbf{T}_0 = \{X \mid \mathrm{Hom}_{\mathscr{C}}(M, X) = 0\}$.

Proof. It follows from Theorem 1.19 and the proof of Theorem 1.17.

2. *CQF*-3 OBJECTS

We keep the notations and general setting of the preceding section. As stated earlier, M is *CQF*-3 if and only if for every object X in $\mathscr{C}$, $X \in \mathbf{T}$ if and only if $\mathrm{Hom}_{\mathscr{C}}(M, X) = 0$. In particular, $\mathbf{T}$ is, in this case, closed under products and hence **T** is also a torsionfree class for a (not necessarily hereditary) torsion theory. However, **T** may be closed under products without M being CQF-3. For instance, if $\mathscr{C} = R$-mod, then **T** is closed under products if and only if the trace ideal T_M of M on R is idempotent [18, Proposition VI.6.12], while M is CQF-3 if and only if $T_M M = M$ (see [16]) and thus if $T_M = 0$ and $M \neq 0$, then **T** is closed under products but M is not CQF-3.

Suppose that **T** is closed under products and let (**D**, **T**) be the corresponding cohereditary torsion theory. **d** will denote the associated radical, which is epimorphism-preserving [15, Lemma 1.8]. Let $\mathscr{C}D_M$ be the full subcategory of $\mathscr{C}$ whose objects are precisely those which belong simultaneously to **D** and **F**. Then we have the following result.

PROPOSITION 2.1. *Let* **T** *be closed under products. Then the functor* $\mathscr{C}D_M \to \mathscr{C}_M$ *given by* $X \to \mathbf{a}(X)$ *is an equivalence of categories with inverse defined by* $Z \to \mathbf{d}(Z)$.

Proof. It is an easy exercise to verify that if X is in $\mathscr{C}D_M$, then $\mathbf{d}(\mathbf{a}(X)) = X$, and if Z is in $\mathscr{C}_M$, then $\mathbf{a}(\mathbf{d}(Z)) \cong Z$.

We are going to show next that the study of weakly **T**-closed objects M reduces to that of weakly **T**-closed CQF-3 objects M, when **T** is assumed to be closed under products.

PROPOSITION 2.2. *If* **T** *is closed under products, then* $\mathbf{d}(M)$ *is* CQF-3. *Moreover, if* M *is weakly* **T**-*closed, then* $\mathbf{d}(M)$ *is weakly* **T**-*closed and* $\mathrm{End}_{\mathscr{C}}(\mathbf{d}(M)) \cong \mathrm{End}_{\mathscr{C}}(M)$.

Proof. Take X in **D**. The canonical morphism $M^{(\mathrm{Hom}(M,X))} \to X$ is an epimorphism (because its cokernel must belong to both **T** and **D**) and hence it induces an epimorphism $\mathbf{d}(M)^{(\mathrm{Hom}(M,X))} \to X = \mathbf{d}(X)$. This proves that **D** consists precisely of all the objects generated by $\mathbf{d}(M)$, so that $\mathbf{d}(M)$ is CQF-3 [16, Lemma 2.2]. On the other hand, in view of the facts that $\mathbf{ad}(M) \cong \mathbf{a}(M)$ and $M/\mathbf{d}(M)$ is T-torsion, we have $\mathrm{End}_{\mathscr{C}}(\mathbf{d}(M)) \cong S \cong S' \cong \mathrm{End}_{\mathscr{C}}(\mathbf{a}(\mathbf{d}(M)))$, thus showing that $\mathbf{d}(M)$ is weakly **T**-closed.

According to [19, Theorem 1.8], if M is CQF-3 and $\mathscr{C}$ has enough projectives, then every object X of $\mathscr{C}$ has a colocalization with respect to (**D**, **T**) (that is, a morphism $f\colon Q \to X$ such that Q is **D**-codivisible, $Q \in \mathbf{D}$, and $\mathrm{Ker} f$ and $\mathrm{Coker} f$ are in **T**). In fact, if M is CQF-3 and has a colocalization $f\colon Q \to M$, then every object of $\mathscr{C}$ has a colocalization (by the same argument of the proof of [16, Theorem 2.6(i) $\Rightarrow$ (ii)]). Now, we will see that if we want to study endomorphism rings of CQF-3 and weakly **T**-closed objects M such that M has a colocalization, we may already suppose that M is **D**-codivisible.

PROPOSITION 2.3. *If M is CQF-3 and $f: Q \to M$ is a colocalization of M, then Q is also CQF-3 and Q is weakly* **T**-*closed. Moreover, if M is weakly M-distinguished, then* $\mathrm{End}_{\mathscr{C}}(Q) \cong S$.

Proof. Since M is Q-generated and $Q \in \mathbf{D}$, it is clear that **D** consists of all the Q-generated objects of $\mathscr{C}$, so that Q is CQF-3 [16, Lemma 2.2]. Q is weakly **T**-closed, because it is **D**-codivisible. Also, a direct argument shows that $\bar{M} \cong \bar{Q}$ and $\mathrm{End}_{\mathscr{C}}(Q) \cong \mathrm{End}_{\mathscr{C}}(\bar{M})$. Therefore, if M is weakly M-distinguished, then clearly $\mathrm{End}_{\mathscr{C}}(Q) \cong S$.

Let us assume that **T** is closed under products. We denote by $\mathscr{C}^M$ the full subcategory of $\mathscr{C}$ whose objects are all the **D**-torsion and **D**-codivisible objects of $\mathscr{C}$. It is shown in [19] that, if every object of $\mathscr{C}$ has a colocalization with respect to $(\mathbf{D}, \mathbf{T})$, then the inclusion functor $\mathbf{u}: \mathscr{C}_M \to \mathscr{C}$ has an exact right adjoint **c**, which assigns to each object X of $\mathscr{C}$ its colocalization object. Moreover, we recall the following result, which was proved in [19, Proposition 4.4].

PROPOSITION 2.4. *Assume that* **T** *is closed under products and that each object of $\mathscr{C}$ has a colocalization with respect to* $(\mathbf{D}, \mathbf{T})$. *Then, the restrictions to $\mathscr{C}_M$ and $\mathscr{C}^M$ of the functors* **c** *and* **a**, *respectively, are inverse equivalences of categories between $\mathscr{C}_M$ and $\mathscr{C}^M$.*

If, in particular, M is CQF-3 and **D**-codivisible, then the hypotheses of Proposition 2.4 are fulfilled. In this case, an easy check shows that the category $\mathscr{C}_M$ consists of all those objects X of $\mathscr{C}$ such that there exists an exact sequence of the form $M^{(I)} \to M^{(J)} \to X \to 0$. These are precisely the objects which have M-codominant dimension $\geqslant 2$, according to the terminology of [17].

Under the hypotheses of Proposition 2.4, we have that each of the categories $\mathscr{C}_M, \mathscr{C}^M$, and $\mathscr{C}D_M$ is equivalent to $(S, \mathscr{F})$-mod. With an additional assumption, we obtain up to six equivalent categories, as shown below.

PROPOSITION 2.5. *Assume that $\mathscr{C}$ is a locally finitely generated Grothendieck category with a projective generator U and that the object M is such that* **T** *is closed under products. Let J be the left ideal of S consisting of all the endomorphisms f of M which factor in the form $f = h \circ g$, where $h: U^n \to M$ and $g: M \to U^n$ verify that* $\mathrm{Im}\, h \subseteq \mathbf{d}(M)$ *and* $\mathrm{Im}\, g$ *is contained in a finitely generated subobject U' of U^n. Then, the left Gabriel filter $\mathscr{F}$ of S consists of all left ideals I such that $J \subseteq I$.*

Proof. The same methods used in the proof of Theorem 1.7 show, in this case, that $J \in \mathscr{F}$. On the other hand, if $I \in \mathscr{F}$, then there is M' such that $\mathbf{d}(M) \subseteq M' \subseteq M$ and an epimorphism $\pi: M^{(I)} \to M'$, with $f = \pi \circ q_f$ for each $f \in I$ (where, as usual, q_f are the canonical injections). If $\alpha \in J$, $\alpha = h \circ g$, with $\operatorname{Im} h \subseteq \mathbf{d}(M) \subseteq M'$, and $\operatorname{Im} g \subseteq U'$, U' being a finitely generated subobject of some U^n. The projectivity of U^n implies that $h\colon U^n \to M$ factors through π, and it is then clear that α factors through some canonical injection $M^{(F)} \to M^{(I)}$, F being a finite set, from which it follows that $\alpha \in I$. Therefore, $\mathscr{F} = \{I \leqslant_S S \mid J \subseteq I\}$.

By Proposition 2.5, J is an idempotent two-sided ideal and the torsion theory of S-mod associated to J verifies that its torsion class is closed under products and that each object of S-mod has a colocalization [19, Theorem 1.8]. Thus the quotient category $(S, \mathscr{F})$-mod is equivalent, by Propositions 2.1 and 2.4, to the full subcategories of S-mod consisting of: (i) the J-generated $\mathscr{F}$-torsionfree S-modules; and (ii) the J-generated and codivisible S-modules (this latter category is the category ${}_J\mathscr{C}$ of [11]). Thus we have the following corollary.

COROLLARY 2.6. *In the hypotheses of Proposition* 2.5, *the following six categories are equivalent.* (i) $\mathscr{C}_M$, (ii) $\mathscr{C}^M$, (iii) $\mathscr{C}D_M$, (iv) $(S, \mathscr{F})$-mod, (v) ${}_J\mathscr{C}$, *and* (vi) *the category of all J-generated and $\mathscr{F}$-torsionfree left S-modules.*

When one takes a Σ-quasiprojective module M and $\mathscr{C} = \sigma[M]$ then we have in particular [9, Theorem 1.3]. On the other hand, the equivalence between $\mathscr{C}^M$ and $(S, \mathscr{F})$-mod is given in [16, Theorem 2.5] under the more general assumption that $\mathscr{C}$ be a cocomplete abelian category with exact direct limits. In fact, [16, Theorem 2.5] identifies also the colocalization and localization functors $\mathbf{c}$ and $\mathbf{a}$. This we do now in a shorter way ($\mathbf{j}$ and $\mathbf{b}$ below denote the inclusion functor from $(S, \mathscr{F})$-mod to S-mod and its left adjoint, respectively).

PROPOSITION 2.7. *If M is CQF*-3 *and* $\mathbf{D}$*-codivisible, then the colocalization functor* $\mathbf{u} \circ \mathbf{c}\colon \mathscr{C} \to \mathscr{C}$ *is equivalent to the composition of the functors* $H = \operatorname{Hom}_{\mathscr{C}}(M, -)\colon \mathscr{C} \to S\text{-mod}$ *followed by* $G = M \otimes_S -\colon S\text{-mod} \to \mathscr{C}$. *On the other hand, the localization functor* $\mathbf{j} \circ \mathbf{b}\colon S\text{-mod} \to S\text{-mod}$ *is equivalent to the composition* $H \circ G$.

Proof. By Propositions 1.4 and 2.4, and the fact that $S' \cong S$ in this case, we see that the functor $F\colon \mathscr{C}^M \to (S, \mathscr{F})$-mod given on objects by $F(Z) = \operatorname{Hom}_{\mathscr{C}}(M, \mathbf{iau}(Z))$ is an equivalence, whose inverse is given by $F'\colon (S, \mathscr{F})\text{-mod} \to \mathscr{C}^M$, with $F'(X) = \mathbf{c}(M \otimes_S X)$. Thus for each object Z of $\mathscr{C}^M$ one has a canonical isomorphism between Z and $\mathbf{c}(M \otimes_S \operatorname{Hom}_{\mathscr{C}}(M, \mathbf{iau}(Z)))$. But it follows from the fact that M is *CQF*-3 and $\mathbf{D}$-codivisible that $\operatorname{Hom}_{\mathscr{C}}(M, \mathbf{iau}(Z)) \cong \operatorname{Hom}_{\mathscr{C}}(M, \mathbf{u}(Z))$ and that $M \otimes_S Y$ belongs to $\mathscr{C}^M$ for every ${}_S Y$ (because it has M-codominant dimension ≥ 2), so that $\mathbf{u}(Z) \cong M \otimes_S \operatorname{Hom}_{\mathscr{C}}(M, \mathbf{u}(Z))$. This gives that, for each X in $\mathscr{C}$, $\mathbf{uc}(X) \cong M \otimes_S \operatorname{Hom}_{\mathscr{C}}(M, \mathbf{uc}(X))$. But again the conditions on M clearly imply that $\operatorname{Hom}_{\mathscr{C}}(M, \mathbf{uc}(X)) \cong \operatorname{Hom}_{\mathscr{C}}(M, X)$, so that there is a natural isomorphism $M \otimes_S \operatorname{Hom}_{\mathscr{C}}(M, X) \cong \mathbf{uc}(X)$.

To prove the second part of the proposition, note that one can easily show that the equivalence of Theorem 1.6 gives in this case that for each object X of $(S, \mathscr{F})$-mod there is a canonical isomorphism $X \cong \operatorname{Hom}_{\mathscr{C}}(M, M \otimes_S X)$. It follows that if M is *CQF*-3 and weakly **T**-closed, the localization functor assigns to each left S-module Y the S-module $\operatorname{Hom}_{\mathscr{C}}(M, \mathbf{ia}(M \otimes_S Y))$. The **D**-codivisibility of M now implies the result.

3. Applications to the Study of Endomorphism Rings

In this section M will be a left R-module, $S = \operatorname{End}({}_R M)$. As suggested by Proposition 1.16, we shall take $\mathscr{C} = \sigma[M]$ from now on. We need the following definition, due to Brodskii [2].

Definition 3.1. A left R-module M will be called intrinsically projective when for every natural number n and every epimorphism $p: M^n \to L$, where L is a submodule of M, the induced homomorphism $p_*: \operatorname{Hom}_R(M, M^n) \to \operatorname{Hom}_R(M, L)$ is surjective.

From [2, Lemma 2] it follows that M is intrinsically projective if and only if every finitely generated left ideal I of S verifies $I = \{f \in S \mid \operatorname{Im} f \subseteq MI\}$, so that I can be identified, in this case, with $\operatorname{Hom}_R(M, MI)$. We have the following result.

Theorem 3.2. *Let M be a left R-module which is weakly* **T**-*closed as an object of $\sigma[M]$. The following conditions are equivalent.*

(i) *S is left semihereditary.*

(ii) *M is intrinsically projective and for every finitely M-generated submodule N of M, $M \otimes_S \operatorname{Hom}_R(M, N)$ is a direct summand of M^n for some integer n.*

(iii) *M is intrinsically projective and for every finitely M-generated submodule N of M there exists an exact sequence $0 \to K \to L \to N \to 0$, where K is* **T**-*torsion and L is a direct summand of M^n for some integer n.*

Proof. (i) $\Rightarrow$ (ii) By hypothesis, every finitely generated left ideal I of S is a direct summand of some S^r and hence I is $\mathscr{F}$-closed, in view of Proposition 1.14. By [18, Proposition IX.4.2], I is $\mathscr{F}$-saturated in S. Now, if $J = \mathrm{Hom}_R(M, MI) \leqslant S$, we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & M\otimes_S I & \longrightarrow & M\otimes_S J & \longrightarrow & M\otimes_S (J/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & MI & \xrightarrow{1} & MI & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the Ker–Coker lemma, $\mathrm{Ker}\, g \cong M\otimes_S (J/I)$ is isomorphic to a quotient of $\mathrm{Ker} f$, which is T-torsion by Theorems 1.6 and 1.7. Therefore, J/I is $\mathscr{F}$-torsion and hence $J = I$, because S/I is $\mathscr{F}$-torsionfree. This means that M is intrinsically projective.

Let now $M^n \xrightarrow{p} N \to 0$ be exact, with N a submodule of M. Inasmuch as M is intrinsically projective, we have that $p_*: \mathrm{Hom}_R(M, M^n) \to \mathrm{Hom}_R(M, N)$ is surjective. Since $\mathrm{Hom}_R(M, N)$ is a finitely generated left ideal of S, we have that p_* splits. By tensoring with M we get that $M\otimes_S \mathrm{Hom}_R(M, N)$ is a direct summand of $M\otimes_S \mathrm{Hom}_R(M, M^n) \cong M^n$.

(ii) ⇒ (iii) If N is a finitely M-generated submodule of M, then the canonical homomorphism $\phi: M\otimes_S \mathrm{Hom}_R(M, N) \to M$ verifies that $\mathrm{Ker}\, \phi$ is T-torsion, by Theorems 1.6 and 1.7, $\mathrm{Im}\, \phi = N$, and $M\otimes_S \mathrm{Hom}_R(M, N)$ is a direct summand of M^n for some n.

(iii) ⇒ (i) Let I be a finitely generated left ideal of S. By (iii) there is an exact sequence $0 \to K \to L \to MI \to 0$, where K is T-torsion and L is a direct summand of M^n. Since $\mathrm{Hom}_R(M, \mathrm{t}(M)) = 0$ by the hypothesis, $\mathrm{Hom}_R(M, \mathrm{t}(M^n)) = 0$ and hence $\mathrm{Hom}_R(M, K) = 0$. On the other hand, taking into account that M is intrinsically projective, $\mathrm{Hom}_R(M, L) \to \mathrm{Hom}_R(M, MI)$ is an epimorphism and, in fact, an isomorphism. Since $\mathrm{Hom}_R(M, L)$ is a direct summand of $\mathrm{Hom}_R(M, M^n) \cong S^n$, $I = \mathrm{Hom}_R(M, MI)$ is a projective left S-module.

COROLLARY 3.3. *If M is* T-*M-injective and M-distinguished in $\sigma[M]$, then S is left semihereditary if and only if M is intrinsically projective and every finitely M-generated submodule of M is a direct summand of some M^n.*

COROLLARY 3.4. *If M is a Σ-quasiprojective left R-module, then S is left semihereditary if and only if every finitely M-generated submodule of $\bar{M}$ is a direct summand of some $\bar{M}^n$.*

Note that [7, Theorem 7] is a consequence of Corollary 3.4.

In [3] a module M is called a CS-module when every essentially closed submodule of M is a direct summand of M, and a ring R is a left CS-ring when ${}_RR$ is a CS-module. Clearly, M is a CS-module if and only if every submodule of M is essential in a direct summand. We have the following result.

THEOREM 3.5. *Let M be weakly M-distinguished in $\sigma[M]$. Then S is a left CS-ring if and only if $\bar{M}$ is a CS-module.*

Proof. Assume first that S is left CS and let $X \leqslant \bar{M}$. It is easy to see that there exists an M-generated submodule X_0 of M such that $p(X_0) \subseteq X$ and $X/p(X_0) \in \mathbf{T}$, p being the canonical projection of M onto $\bar{M}$. If $I = \{f \in S \mid \operatorname{Im} f \subseteq X_0\}$, then, by hypothesis, I is essential in Se for some idempotent e of S. Let $N = \operatorname{Im} e$; thus we have that $Se = \{f \in S \mid \operatorname{Im} f \subseteq N\}$. Then $p(N)$ is a direct summand of $\bar{M}$, $p(X_0) \subseteq p(N)$, and we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & p(X_0) & \longrightarrow & X & \longrightarrow & X/p(X_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 0 & \longrightarrow & p(N) & \xrightarrow{u} & \bar{M} & \longrightarrow & \bar{M}/p(N) & \longrightarrow & 0 \end{array}$$

where the vertical arrow on the right must be zero since $X/p(X_0)$ is $\mathbf{T}$-torsion and $\bar{M}/p(N) \in \mathbf{F}$. Thus there exists $v\colon X \to p(N)$ such that $u \circ v = j$ and so $X \subseteq p(N)$. To prove that $\bar{M}$ is a CS-module it will be enough to show that $p(X_0)$ is essential in $p(N)$. To see this, let $0 \neq Y \subseteq p(N)$. By the same reasoning made above, there exists an M-generated submodule Y_0 of M such that $0 \neq p(Y_0) \subseteq Y$. Let J be the left ideal of S, $J = \{f \in S \mid \operatorname{Im} f \subseteq Y_0\} \neq 0$. Since $Y_0 \subseteq N$, we have $J \subseteq Se$ and, inasmuch as I is essential in Se, $I \cap J \neq 0$. But if $f \in I \cap J$ and $f \neq 0$, then $\operatorname{Im} f \subseteq Y_0 \cap X_0$. Therefore $Y_0 \cap X_0$ is not $\mathbf{T}$-torsion (because $\operatorname{Hom}_R(M, \mathbf{t}(M)) = 0$ by hypothesis) and thus $p(Y_0) \cap p(X_0) \neq 0$.

Conversely, assume that $\bar{M}$ is a CS-module and let I be an essentially closed left ideal of S. Calling $\bar{S} = \operatorname{End}({}_R\bar{M}) \cong S$, I may also be considered as a left ideal of $\bar{S}$. Since, as it was seen in the proof of Theorem 1.6, the torsion ideal of S, $\mathbf{t}_{\mathscr{F}}(S)$, is just $\operatorname{Hom}_R(M, \mathbf{t}(M))$, S is $\mathscr{F}$-torsionfree and hence the essentially closed left ideals of S are precisely the essentially closed elements of the lattice $\operatorname{Sat}_{\mathscr{F}}(S)$. By the equivalence of categories between $\mathscr{C}_M$ and $(S, \mathscr{F})$-mod, this lattice is isomorphic to the lattice $\operatorname{Sat}_{\mathbf{T}}(\bar{M})$, by means of the mapping $J \to \psi_{\bar{M}}^{-1}(\mathbf{ia}(\bar{M}J))$, and hence in this isomorphism the left ideal I corresponds to $X = \psi_{\bar{M}}^{-1}(\mathbf{ia}(\bar{M}I))$, which is an essentially closed submodule of $\bar{M}$, for $\bar{M}$ is $\mathbf{T}$-torsionfree. By hypothesis, there is an idempotent e in $\bar{S}$ such that $\bar{M}e = X$ and thus $\bar{S}e = \mathrm{Hom}_R(\bar{M}, X)$. By using the equivalence of categories of Theorems 1.6 and 1.7, it is easily seen that the localization functor $\mathbf{b}$: S-mod $\to (S, \mathscr{F})$-mod is given by $\mathbf{b}(N) \cong \mathrm{Hom}_R(M, \mathbf{ia}(M \otimes_S N))$. Therefore $I_{\mathscr{F}}$ is isomorphic to $\mathrm{Hom}_R(M, \mathbf{ia}(M \otimes_S I)) \cong \mathrm{Hom}_R(M, \mathbf{ia}(MI)) \cong \mathrm{Hom}_R(\bar{M}, \mathbf{ia}(\bar{M}I))$ and, since S is $\mathscr{F}$-torsionfree, I is an essential $\bar{S}$-submodule of $\mathrm{Hom}_R(\bar{M}, \mathbf{ia}(\bar{M}I))$. Now, $\mathrm{Hom}_R(\bar{M}, \bar{M}I) \subseteq \mathrm{Hom}_R(\bar{M}, X) \subseteq \mathrm{Hom}_R(\bar{M}, \mathbf{ia}(\bar{M}I))$ and hence the left ideal $\mathrm{Hom}_R(\bar{M}, X)$ of $\bar{S}$ is an essential extension of I, so that $I = \mathrm{Hom}_R(\bar{M}, X) = \bar{S}e$ is a direct summand of $\bar{S} \cong S$.

The following corollary generalizes [3, Corollary 3.6].

COROLLARY 3.6. *If M is M-distinguished in $\sigma[M]$, then S is a left CS-ring if and only if M is a CS-module.*

A particular class of left CS-rings is that of left continuous rings. Recall that a ring R is said to be left continuous when R is a left CS-ring such that if a left ideal I of R is isomorphic to a direct summand of R, then I is also a direct summand of R. The concept of a continuous module is analogous.

PROPOSITION 3.7. *Let M be weakly M-distinguished in $\sigma[M]$. Then S is a left continuous ring if and only if $\bar{M}$ is a continuous module.*

Proof. Let S be left continuous and $\bar{S} = \mathrm{End}_R(\bar{M}) \cong S$. In view of Theorem 3.5, we only have to show that if L and N are isomorphic submodules of $\bar{M}$ and N is a direct summand of $\bar{M}$, then so is L. Put $I = \{f \in \bar{S} \mid \mathrm{Im} f \subseteq L\}$ and $J = \{f \in \bar{S} \mid \mathrm{Im} f \subseteq N\}$. It is clear that the isomorphism $L \cong N$ induces an isomorphism between I and J. Now, J is a direct summand of $\bar{S}$ and, by hypothesis, so is I. Since $L \cong N$ is M-generated, $L = L_M = \bar{M}I$ is a direct summand of $\bar{M}$.

To prove the converse, let e be an idempotent of $\bar{S}$ with $N = \bar{M}e$ and p: $\bar{S}e \to I$ an isomorphism between $\bar{S}e$ and a left ideal I of $\bar{S}$. If $p(e) = h \in I$, then the annihilator $\mathrm{ann}_{\bar{S}}(e)$ is just $\mathrm{ann}_{\bar{S}}(h)$ and, since $\mathrm{ann}_{\bar{S}}(e) = \mathrm{Hom}_R(\bar{M}, \mathrm{Ker}\, e)$ and $\mathrm{ann}_{\bar{S}}(h) = \mathrm{Hom}_R(\bar{M}, \mathrm{Ker}\, h)$, we have $\mathrm{Ker}\, e = (\mathrm{Ker}\, h)_M$. Now, $\mathrm{Ker}\, h/\mathrm{Ker}\, e$ is isomorphic to a submodule of N, which is $\mathbf{T}$-torsionfree and hence $\mathrm{Ker}\, e = \mathrm{Ker}\, h$. If we call u_1 and u_2 to the canonical injections of N and $L = \bar{M}h$, respectively, into $\bar{M}$ and $e_1 : \bar{M} \to N$ and $h_1 : \bar{M} \to L$ are such that $u_1 \circ e_1 = e$, $u_2 \circ h_1 = h$, then the above equation gives an isomorphism θ: $N \to L$ such that $\theta \circ e_1 = h_1$. Let f: $\bar{M} \to L$ be an arbitrary homomorphism. Then the image of $\theta^{-1} \circ f$ is contained in N and thus there exists $s \in \bar{S}$ with $\theta^{-1} \circ f = e_1 \circ s$, so that $f = (\theta \circ e_1) \circ s = h_1 \circ s$ and hence the left ideal $\mathrm{Hom}_R(\bar{M}, L)$ is contained in $\bar{S}h = I$, and both ideals coincide. Since $\bar{M}$ is continuous and $L \cong N$, L is a direct summand of $\bar{M}$

and therefore I is a direct summand of $\bar{S}$. Finally, Theorem 3.5 completes the proof.

Recall that a ring R is said to be left Kasch [18, Chap. XIV] when $E({}_R R)$ is a cogenerator of the category R-mod. A module M is called an RZ-module if every simple quotient of M is isomorphic to a submodule of M. In [9, Theorem 3.1] it is shown that the endomorphism ring of a Σ-quasiprojective module M is left Kasch if and only if $\bar{M}$ is a finitely generated RZ-module. More generally, we have the following result.

PROPOSITION 3.8. *Suppose that* $\mathrm{Hom}_R(M, \mathbf{t}(M)) = 0$. *Then S is a left Kasch ring if and only if M is a finitely generated quasiprojective module and $\bar{M}$ is an RZ-module.*

Proof. By our assumption, S is $\mathscr{F}$-torsionfree. The condition of S being left Kasch implies that every simple quotient of S is isomorphic to a left ideal and hence $\mathscr{F}$-torsionfree. Thus there is no proper left ideal of S in $\mathscr{F}$ and $\mathscr{F}$ is the trivial filter, $\mathscr{F} = \{S\}$. By Theorem 1.17, M is a finitely generated and quasiprojective module, so that M is a Σ-quasiprojective module [1, Proposition I.1.8]. Now, [9, Theorem 3.1] achieves the proof.

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