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A Condition of Brauer–Cartan–Hua Type-II

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I. INTRODUCTION

This paper continues the investigation begun by one of the authors of the interrelationship between a semi-prime 2-torsion free associative ring, R with involution and a subset, T, with property $(H)^*$: namely, T is a self-adjoint Jordan subring (that is, T is closed under $t \circ u = tu + ut \in T$ for all $t, u \in T$ and $T = T^*$) and $xtx^* \in T$ for all $t \in T$, $x \in R$. Levitski and Jacobson semi-simplicity are shown to hold in T when they hold in R. As well, the relationship between the corresponding radicals is considered.

We also show that if certain ring properties hold in T then they also hold in R. In particular, we are able to prove analogous results to Britten and Montgomery when T satisfies either ascending or descending chain conditions on quadratic ideals.

We also indicate, by example, how a major portion of the results of I. N. Herstein, S. Montgomery, and others relating to regularity conditions on skew and symmetric elements hold, as well, for subsets with property $(H)^*$. In fact, we only prove three analogs of the many theorems related to these conditions that appear in the literature. Finally, we investigate the relationship between R and a subset T which is a self-adjoint Lie subring (that is, $[t, u] = tu - ut \in T$ for all $t, u \in T$ and $T = T^*$) and has the property that $xtx^* \in T$ for all $t \in T$, $x \in R$.

II. PRELIMINARY COMMENT

We wish to continue the investigation begun in [1] of particular Jordan subrings, T, of an (associative) ring R which is 2-torsion free, and has involution

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$x \to x^*$. We let S denote the set, $\{x \mid x = x^*\}$, (and K denotes the set $\{x \mid x = -x^*\}$), of symmetric and skew elements respectively.

The set T is said to be a self-adjoint Jordan subring of R if T is an additive group and, in addition, T is

(i) closed under the (linear) Jordan multiplication induced by R; that is, $t, u \in T$ implies $t \circ u = tu + ut \in T$.

(ii) closed under the involution, that is, $t \in T$ implies $t^* \in T$.

Furthermore, an additive subgroup A of T is said to be a Jordan ideal of T if $a \circ t \in A$ for all $a \in A$, $t \in T$.

We make the following definition which restricts us to the Jordan subrings of R which we wish to investigate.

Definition 1. We say that a set T has property $(H)^*$ if, and only if, T is a non-zero, self-adjoint Jordan subring of an associative ring R and

$$tU_x^* = xtx^* \in T$$

for all $x \in R$, $t \in T$.

We state the main theorem proved in [1] with immediate modifications which are necessary for further theorems in this paper.

Theorem 2. *Let R be a semiprime, 2-torsion-free ring with involution and let T have property $(H)^*$, then either*

(i) *T contains a self-adjoint (two-sided) ideal I ($I = I^*$) of R such that $0 \neq I \cap S$, or*

(ii) *there exists a self-adjoint ideal, I, such that $0 \neq I \cap S \subset T \subset S$.*

Furthermore,

(iii) *if $2R = R$ then $2I = I$*

and

(iv) *T is t-semiprime. That is, if A is a Jordan ideal of T and $AU_A = 0$ (that is, $aba = 0$ for all $a, b \in A$) then $A = 0$.*

This theorem allows one to extend the results of [2] which characterize the nil Jordan ideals of S of bounded index to nil Jordan ideals of T. This extension is

Lemma 3. *Let R be a semi-prime 2-torsion free ring with involution and let T have property $(H)^*$. Let A be a Jordan ideal of T such that $a^n = 0$ for all $a \in A$ (n a fixed positive integer). Then $A = 0$.*

Since R is a subdirect sum of 2-torsion free $*$-prime rings (see [1]) and since property H^* (and nil) carry to the homomorphic image we may assume without loss that R is a $*$-prime 2-torsion free ring.

Case 1

$$0 \neq I = I^* \subset T.$$

If $a^2 = 0$ for all $a \in A$, then $a(ai + ia)^2 = 0$ for all $i \in I$. Thus, $(ai)^3 = 0$. Thus, it follows that $a = 0$. Inductively, we see that the argument is the same. That is, $2a^2 \in A$ and hence $2^{k-1}a^k \in A$ for all $a \in A$. Letting $a^n = 0$ for all $a \in A$, and $2N = \{n \text{ or } n + 1\}$ we have

$$[2^{N-1}(a^N i + ia^N)]^{2N} = 0$$

for all $a \in A$, $i \in I$. Since R is two-torsion free, we have on premultiplying by a^N

$$(a^N i)^{2N+1} = 0.$$

Thus, we conclude that $a^N = 0$, completing the induction argument.

Case 2. $0 \neq I \cap S \subset T \subset S$.

If $a^2 = 0$ for all $a \in A$ then $ab + ba = 0$ and hence $aba = 0$ for all $a, b \in A$. That is, $AU_A = 0$, and as noted in Theorem 2, $A = 0$.

Now, in general, for $a \in A$, $i \in I$, $t \in T$, then $i + i^*$ and $it + ti^* \in I \cap S$ for all $i \in I$, $t \in T$. Furthermore, $2a^2 \in A$ for all $a \in A$ and hence $4ata = (2a) \circ (a \circ t) - 2a^2 \circ t \in A$. Hence, $(4a^2)i + i^*(4a^2) = 4a \circ (i^*a + ai) - 4a(i + i^*)a \in A$ for all $a \in A$, $i \in A$. Thus,

$$2^2(a^2 i + i^* a^2) \in A.$$

Replacing i by $a^{N-2}i$, we have

$$2^2(a^N i + i^* a^N) \in A$$

and hence letting $a^n = 0$ for all $a \in A$ and $2N = \{n \text{ or } n + 1\}$ we have

$$a^N(a^N i + i^* a^N)^{2N} = 0.$$

As before, by induction, we conclude $A = 0$.

III. Levitski and Jacobson Semi-simplicity in Subsets with Property $(H)^*$

In this section we show how Levitski semisimple (L.S.S) for a $*$-prime ring R (or how $*$-primitive for a ring R) transfers to subsets, T, with property $(H)^*$. A subset T with property $(H)^*$ is said to be L.S.S. as a Jordan ring if T has no nonzero locally nilpotent Jordan ideals while T, with property $(H)^*$ is Jacobson semi-simple if T has no non-zero quasi-regular Jordan ideals. The main theorem is:

THEOREM 4. *Let R be a $*$-prime, Levitski-semi-simple* (L.S.S.), *2-torsion free ring with $2R = R$ ($*$-primitive replaces $*$-prime,* L.S.S.). *Let T be a subset with property $(H)^*$, then T is* L.S.S., *t-$*$-prime Jordan ring (Jacobson semi-simple Jordan replaces* L.S.S. *Jordan ring.*)

Baxter [1] has shown that $R*$-prime implies T is t-$*$-prime. Hence, we need only prove that $\mathscr{L}(T)$, the maximal locally nilpotent ideal of T, and called the Levitski radical of T, is $0(\mathscr{J}(T)$, the maximal quasi-regular ideal of T, and called the Jacobson radical of T, is 0).

We prove this fact by a sequence of lemmas. *In each case, the conditions on R and T are stated in Theorem* 4.

LEMMA 5. *Let I be the self-adjoint ideal of Theorem* 2, *then* $\mathscr{L}(I \cap S) = 0(\mathscr{J}(I \cap S)) = 0)$.

This result follows immediately in the Levitsky case from Rich [12] where $\mathscr{L}(I \cap S) = \mathscr{L}(I) \cap S$. In the Jacobson case it follows from McCrimmon, [9], where $\mathscr{J}(I \cap S) = \mathscr{J}(I) \cap S$. As immediate, from this lemma, is the following lemma.

LEMMA 6. $F = \mathscr{L}(T \cap S) \cap (I \cap S) = 0$

$$(G = \mathscr{J}(T \cap S) \cap (I \cap S) = 0).$$

Introducing the notation $\{x, t, y)_* = xty + y^*tx^*$.

LEMMA 7. $\mathscr{L}(T \cap S) = 0\ (\mathscr{J}(T \cap S) = 0)$.

To prove this lemma, let $a \in \mathscr{L}(T \cap S)$ and $u \in I \cap S$, then $au + ua = 0 = uau$. Thus, $u^2a = 0$ for all $a \in \mathscr{L}(T \cap S)$, $u \in I \cap S$. Hence,

$$u^2(ta + at) = 0$$

or $u^2ta = 0$ for all $u \in I \cap S$, $a \in \mathscr{L}(T \cap S)$ and $t \in T \cap S$. In particular,

$$u^2[u^2U_x^*]a = 0 \qquad \text{for all} \quad x \in R.$$

Linearizing, we have

$$u^2\{x, u^2, x^*u^2y\}_*a = 0$$

which implies

$$u^2[u^2U_x^*]\, u^2Ra = 0.$$

Now R is $*$-prime ($*$-primitive implies $*$-prime) and thus $a = 0$ (that is $\mathscr{L}(T \cap S) = 0(\mathscr{J}(T \cap S) = 0)$ our desired conclusion), or

$$u^2[u^2 U_x^*]\, u^2 = 0.$$

It is easily seen that this latter implies $(u^2x)^4 = 0$, or $u^2 = 0$ for all $u \in I \cap S$. However, $I \cap S$ is a Jordan ideal of S. By Baxter [2], we have $I \cap S = 0$, a contradiction.

We are now prepared to prove Theorem 4. Suppose $\mathscr{L}(T) \neq 0$ (the argument is the same for $\mathscr{J}(T)$). Since T is self-adjoint, we have as we subsequently show in the associative case, that $\mathscr{L}(T)$ is self-adjoint. Since, $\mathscr{L}(T) \cap S \subset \mathscr{L}(T \cap S) = 0$, we have for all $x \in \mathscr{L}(T)$ that $x + x^* = 0$. Since $\mathscr{L}(T)$ is a Jordan ideal of T and R is 2-torsion free, we have $x^2 = 0$ for all $x \in \mathscr{L}(T)$. Thus, by Lemma 3, $\mathscr{L}(T) = 0$, completing the proof of Theorem 4.

IV. Levitski and Jacobson Radical of a Subset with Property $(H)^*$

In the last section we defined for a subset T with property $(H)^*$ the Levitski and Jacobson radicals, denoted by $\mathscr{L}(T)$ and $\mathscr{J}(T)$ respectively. One might ask, as we did in [1] relative to other radicals, what is the relation between these radicals and the corresponding radicals $\mathscr{L}(R)$ and $\mathscr{J}(R)$, the Levitski and Jacobson radicals of R respectively. Since $\mathscr{L}(R)$ is by definition the maximal locally nilpotent ideal of R and $\mathscr{J}(R)$ is by definition the maximal quasi-regular Jordan ideal of R, we immediately see that $\mathscr{L}(R) \cap T \subseteq \mathscr{L}(R)$ and $\mathscr{J}(R) \cap T \subseteq \mathscr{J}(T)$. We wish to show under suitable conditions that the latter is equality.

Lemma 9. *Let R be a 2-torsion-free ring with involution and with $2R = R$. Then $\mathscr{L}(R)$ is a 2-divisible, self-adjoint ideal. Furthermore, $\mathscr{L}(R) = \{\bigcap P^* \mid P^*$ is $*$-prime, 2-divisible ideal of R and $R/P^* = Q$ is* L.S.S.$\}$. *In addition, $2Q = Q$.*

That $\mathscr{L}(R)$ is closed under the involution follows from the remark that if $x \notin \mathscr{L}(R)$ then there exists a finitely generated subring M of R contained in (x), the principal ideal generated by x, such that M^n is not nilpotent for any n. Hence, $M^* \subset (x^*)$ has the same property and thus $(x^*) \not\subseteq \mathscr{L}(R)$; that is $x^* \notin \mathscr{L}(R)$

Next we note that $\mathscr{L}(R)$ is 2-divisible since if $2x \in \mathscr{L}(R)$, then the principal left ideal, $(2x)$, generated by $2x$ is locally nilpotent, That is, if $B = \{n_i(2x) + r_i(2x) \mid 1 \leqslant i \leqslant k,\ n_i$ an integer and $r_i \in R\}$ is a finite subset of $(2x)$ then there exists an n such that

$$2^n \prod_{i=1}^{n} (n_i x + r_i x) = 0$$

whenever each of $2(n_i x + r_i x) \in B$. Thus, since R is 2-torsion free, $\prod_{i=1}^{n}(n_i x + r_i x) = 0$, and the corresponding $B' = \{n_j x + r_j x \mid 1 \leqslant j \leqslant k\}$ generates a nilpotent subring of (x), that is, $x \in \mathscr{L}(R)$. The proof is now the same as that of Devinsky [5, p. 128] with only minor modification.

Let $\mathscr{M} = \{Q_\beta \mid Q_\beta$ is a 2-divisible ideal of R with $\mathscr{L}(R) \subset Q_\beta$ and $H^n \not\subseteq Q_\beta$ (H the finitely generated ring of [5])$\}$.

Then $\mathscr{M}$ has a maximal Q which is prime and 2-divisible (see Baxter's Remark, [1] Lemma 8 relative to this latter point). Thus, following the argument of Divinsky, R/Q is Levitski-semi-simple.

Since $2R = R$, it is immediate that $2Q = Q$ completing the proof of Lemma 9.

THEOREM 10. *Let R be a 2-torsion free semi-prime ring with involution. Let $2R = R$. Let T be a subset with property $(H)^*$, Then $\mathscr{L}(T) = \mathscr{L}(R) \cap T$.*

We have observed that $\mathscr{L}(R) \cap T \subseteq \mathscr{L}(T)$. Therefore, we must justify equality. Consider $R/P^* = Q$, where P^* is *-prime, 2-divisible ideal of R with R/P^* L.S.S. and where $T \not\subseteq P^*$. Then, $R/P^* = Q$ contains $T + P^*/P^*$. By Theorem 4, $T + P^*/P^*$ is t-$*$-prime and L.S.S. (as a Jordan ring). However, $T + P^*/P^* \simeq T/T \cap P^*$. Thus, the latter, $T/T \cap P^*$ is t-$*$-prime and L.S.S. as a Jordan ring.

Now, by Tsai [13], $\mathscr{L}(T) = \{\bigcap W_\alpha \mid W_\alpha$ is a Jordan ideal of T and T/W_α is t-$*$-prime, L.S.S$\}$. Therefore, $\mathscr{L}(T) \subset T \cap P^*$. Thus, $\mathscr{L}(T) \subseteq \bigcap W_\alpha \subseteq \bigcap_{T \not\subseteq P^*} (T \cap P^*) \subseteq T \cap (\bigcap_{T \not\subseteq P^*} P^*)$. But, $T \subseteq P^*$ certainly means that $\mathscr{L}(T) \subseteq T \subseteq \bigcap_{T \subseteq P^*} P^*$. Therefore, $\mathscr{L}(T) \subseteq T \cap (\bigcap_{P^*} P^*) = T \cap \mathscr{L}(R)$, completing the argument.

The following lemma is a consequence of Baxter and Martindale [3].

LEMMA 11. *Let R be a 2 torsion free semi-prime ring with involution and with the property that $2R = R$. Then $\mathscr{J}(R) = \bigcap \{P \mid P$ is a $*$-primitive, 2-divisible ideal$\}$.*

From this and as a consequence of Theorem 4 we obtain

THEOREM 12. *Let R be a 2-torsion free, semi-prime ring with involution and let $2R = R$. Let T be a subset with property $(H)^*$ and the added property: $2T = T$. Suppose, either R has an identity or $T \subset S$. Then, $\mathscr{J}(T) = \mathscr{J}(R) \cap T$.*

The role of the additional hypothesis is to assure that the quasi-inverse, b, of an element $a \in T$ is also in T. To observe this fact we note $a + b - ab = 0$, or $(1 - a)(1 - b) = 1$ (in a formal sense if $1 \notin R$, in a real sense $1 \in R$). Thus $b = (-a)(1 - b) = -a + a[(-a)(1 - b)] = -a - a^2(1 - b) = -a - (1 - b^*)(1 - a^*)\, a^2(1 - b)$. Now if $a \in T$, then $(1 - a^*)\, a^2 \in T$ (recall $2T = T$ and T is a Jordan subring). Hence, if $1 \in R$ then $b \in T$. As well, if $a \in S$, then $b \in S$ and, hence one concludes that $b \in T$.

We offer finally in this section without a formal argument the immediate result:

COROLLARY 13. *Let R be a 2-torsion free, semi-prime ring with involution. Let $2R = R$. Let T be a subset with property $(H)^*(T \not\subset \mathscr{J}(R))$ and let either $1 \in R$ or $T \subset S$, then*

(i) *If R is primary (that is, R is a ring with identity, 1, and $R/\mathscr{J}(R)$ is a simple ring), then T is primary (that is, $1 \in T$, and $T/\mathscr{J}(T)$ is simple Jordan);*

(ii) *If R is semi-primary then T is semi-primary.*

V. CONDITIONS ON A SET WITH PROPERTY $(H)^*$ WHICH IMPLY CONDITIONS ON R

This section *assumes that T is a subset with property $(H)^*$ in a $*$-prime, 2-torsion-free ring R.* The key lemma is:

LEMMA 14. *$I \cap T \neq 0$ for any non-zero, self-adjoint ideal I.*

The lemma is proved if we can show that $I \cap T = 0$ leads to $I = 0$.
Note that if $I \cap T = 0$ then for all $a \in I$, $t \in T$

$$tU_a^* \in I \cap T = 0.$$

Hence, $\{a, t, b^*\}_* = 0$ for all $a, b \in I$, $t \in T$. Thus, $0 = a^*t\{a, t, b^*\}_* = a^*tbta^*$. Hence, $(at)^3 = 0$ for all $a \in I$, $t \in T$. In a semi-prime ring, this means that $at = 0$. Hence, since R is $*$-prime we have $a = 0$, a contradiction.
We use this lemma to immediately conclude

THEOREM 15. *Let R be $*$-prime and 2-torsion-free. Let T be a subset of R with property $(H)^*$. Then R is semi-simple whenever T is a semi-simple Jordan ring.*

We make the following definition.

DEFINITION 16. We say $e \in T$ is an identity for T if, and only if, $et + te = 2t$ for all $t \in T$.
Using this definition we conclude

THEOREM 17. *Let R be a 2-torsion free ring with involution. Let T be a subset with property $(H)^*$. It then follows that whenever e is an identity for T, then e is a central symmetric idempotent for R.*

In proving this theorem we note that $2e^2 = 2e$, or $e^2 = e$. Hence, e is an idempotent. Moreover, since T is self-adjoint, both e and e^* have the identity property. Thus,

$$2e^* = e^*e + ee^* = 2e$$

or $e = e^*$. Hence, e is a symmetric idempotent. Now, for all $t \in T$, $2t = et + te$ implies that

$$et = te$$

(post and premultiplying by e yields this fact).

Furthermore

$$2t = et + te = 2et$$

implies that $t = et = te$ for all $t \in T$. We note in passing that these facts about e are consistent with the definition of McCrimmon [10] relative to a unity in a quadratic Joedan algebra.

Now, for all $x, y \in R$ and $t \in T$ we have

$$e\{x, t, y\}_* = \{x, t, y\}_* e.$$

In particular, let $x = t = e$ and we conclude that

$$ey + ey^*e = eye + y^*e$$

for all $y \in R$. Pre and post-multiplying by e yields

$$ey = eye = ye$$

for all $y \in R$. That is, $e \in Z(R)$. Hence we have shown that e is a symmetric, central, idempotent.

With this result we are able to conclude.

LEMMA 18. *If R is $*$-prime and e is an identity for T then e is an identity for R.*

We know that $e = e^* = e^2$ and $e \in Z(R)$.

Now, $I = \{x - ex \mid x \in R\}$ is a $*$-ideal of R. Moreover, if $x - ex \in T$ then $x - ex = e(x - ex) = 0$. Thus, $I \cap T = 0$. By Lemma 14, $I = 0$. Thus $x = ex$ for all $x \in R$.

With this lemma we are able to prove the following theorem.

THEOREM 19. *Let R be 2-torsion free $*$-prime ring and let T be a subset with property $(H)^*$. Let e be an identity for T and let T be a $*$-simple Jordan ring (that is, T contains no self-adjoint Jordan ideals). Then R is a$*$-simple associative ring.*

By Lemma 14, $I \cap T = T$ for all non-zero self adjoint ideals I of R. Thus, $e \in I$. However, by Lemma 18, e is an identity of R. Hence, $R = I$; that is, R is $*$-simple.

One notes that if one takes the classical example of a vector space with countable basis over a field, and the primitive ring of matrices acting on this vector space with involution, transposition, then it is well-known that the matrices which are 0 except in an $n \times n$ block satisfy the conditions for T. In this situation, T has no identity, and the theorem is false.

VI. Chain conditions on a Subset with Property $(H)^*$

In this section we prove analogous results to those of Britten [4] and Montgomery [11] for a subset T with property $(H)^*$ of a $*$-prime, 2-torsion free ring, R.

We are considering those situations where T satisfies ascending chain conditions (acc) or descending chain condition (dcc) on quadratic ideals. Q is a quadratic ideal of T if Q is an additive subgroup of T which is closed under the quadratic multiplication; that is, $tU_q = qtq \in Q$ for all $t \in T$, $q \in Q$. Recall that R satisfies Goldie's Theorem for associative rings: R has a ring of quotients which is semi-simple, Artinian if, and only if, R is semi-prime (satisfied by our hypothesis), contains no infinite direct sum of left ideals, and satisfies a.c.c. on left annihilators.

We prove:

THEOREM 20. *Let R be $*$-prime, 2-torsion free, and let T be a subset with property $(H)^*$. Suppose that T satisfies either* acc *or* dcc *on quadratic ideals, then R is a Goldie ring.*

The proof is separated into the 2 cases of Theorem 2.

Case 1. $0 \neq I = I^* \subset T$. Since T has acc (or dcc) on quadratic ideals, T contains no infinite direct sum of quadratic ideals (See Lemma E of [4].) Let $A \neq 0$ be a left ideal of R. Then, $A \cap I \neq 0$. If not, then $IA = 0$ (and R is $*$-prime), hence $A = 0$ which is a contradiction. In fact, for any $a \neq 0 \in A$, $Ia \neq 0$. Now, let $\mathscr{M} = \{A_i\}$, an infinite sequence of left ideals of R whose sum is direct.

Hence, $\{I \cap A_i \neq 0\}$ is an infinite sequence of quadratic ideals of T whose sum is direct, a contradiction. Thus R contains no infinite direct sum of left ideals.

Let $\{A_i\}$ be an ascending chain of left ideals which are left annihilators and such that $A_i \neq A_{i+1}$. Let $B_i = \mathscr{O}\!\ell_R(A_i) = \{b \mid A_i b = 0\}$. Then $A_i = \mathscr{O}\!\ell_L(B_i) = \{c \mid cB_i = 0\}$. Now, since $\{0 \neq I \cap A_i\}$ is an ascending chain of quadratic ideals of T and every such chain breaks off, we have a positive integer N such that $I \cap A_N = I \cap A_{N+i}$ for all positive integers i.

Thus, $(I \cap A_{N+i})\, B_N = 0$. In particular, $IA_{N+i}B_N = 0$. Hence, $A_{N+i}B_N = 0$ since $I = I^*$. Thus, $A_{N+i} \subsetneq A_N$, a contradiction. Thus, every ascending chain breaks off. Hence, R is a Goldie ring.

Case 2. $0 \neq I \cap S \subset T \subset S$ for some self-adjoint ideal I.

We observe that if A is a left ideal of R then $A \cap T$ is a quadratic ideal of T. Moreover, we observe that the argument that $A \cap I \neq 0$, for $A \neq 0$, still holds. Thus, suppose that $A \cap T = 0$, then we claim $A \cap S = 0$. If not, for $a \neq 0 \in A \cap S$, $i \in I \cap S \subset T$, then $aia = 0$ and hence $iaisiai = 0$ for all $s \in S$. Since R is $*$-prime, S is t-semi-prime (see [6]) and hence $iai = 0$ for all $i \in I \cap S$. Therefore, $ax^*[ia(ix + x^*i) + (ix + x^*i)ai] = 0$ for all $x \in R$, $i \in I \cap S$. Thus

$$ax^*iax^*i = 0$$

or $ia = 0$ for all $i \in I \cap S$. Hence, $(ix + x^*i)a = 0$ and thus $a = 0$, a contradiction.

We have just seen that $A \cap S = 0$. Thus, $a^*ta = 0$ for all $t \in T \subset S$, $a \in A$. Now, $ata^* \in T \subset S$ and

$$ata^*sata^* = 0 \qquad \text{for all} \quad a \in A, t \in T, s \in S$$

since, $a^*sa \in A \cap S$. Therefore, $ata^* = 0$, as well.

Now consider for $t \in T$, $x \in A \cap I$;

$$tU_{x+x^*} = (x + x^*)\, t(x + x^*) = xtx + x^*tx^* = y + y^*$$
$$\text{where} \quad y = xtx \in A \cap I.$$

Furthermore, if $Q = \{x + x^* \mid x \in A \cap I\} = 0$, then $A \cap I \subset K$ and, hence $a^2 = 0$ for all $a \in A \cap I$. This means that $A \cap I$ is a nil left ideal of bounded index, from which it follows that $A \cap I = 0$, a contradiction.

Note that we have shown that $Q = \{x + x^* \mid x \in A \cap I\}$ is a non-zero quadratic ideal of T.

Now, continuing with the argument, in this case, we have the same situation as Montgomery [11], Theorem 1 and hence we can argue similarly with only the following modification: instead of choosing $r \in R$, as Montgomery does, choose $i \in I$. Hence, R has no infinite direct sum of right ideals.

We now wish to show that R satisfies acc on left annihilators. Let $A_1 \subset A_2 \subset \cdots \subset A_i \cdots$ be a proper ascending chain of left annihilators, and $B_i = \mathcal{Cl}_R(A_i)$ be the corresponding descending chain of right annihilators. First, assume $A_i \cap T \neq 0$ for some i. Then without loss, we may assume $A_i \cap T \neq 0$ for all i.

Now, $\{A_i \cap T\}$ is an ascending chain of T. Thus, $A_N \cap T = A_{N+1} \cap T$ for some N. Let $a \neq 0 \in A_N \cap T$, $b \in A_{N+1}$, $b \notin A_N$, $j \in T$.

Then,

$$b^*ja + aj^*b \in A_{N+1} \cap T = A_N \cap T.$$

Hence, $(b^*ja + aj^*b)\, B_N = 0$, or

$$aj^*bB_N = 0 \qquad \text{for all } j \in I(= I^*).$$

Since $a \neq 0 \in T \subset S$, we have $bB_N = 0$. Thus, $b \in A_N$ and we have a contradiction. Hence, we may assume $A_i \cap T = 0$ for all i.

Now, as noted, for each i, $Q_i = \{x + x^* \mid x \in A_i \cap I\}$ is a non-zero quadratic ideal of T and we argue as above and in Montgomery, Theorem 2 and conclude that the chain breaks off, a contradiction.

This completes the proof of Theorem 20.

In certain situations we can characterize R more completely. To do this, we observe the following lemma.

LEMMA 21. *Let $t \in T \cap S$ be regular in $T \cap S$ (that is, $uU_t = 0$, $u \in T \cap S$, implies $u = 0$) then t is regular in R (that is, t has no non-zero left or right zero divisors).*

Consider $\mathcal{O}\!\ell_L(t) = \{x \mid xt = 0\}$, the set of left annihilators. Then since $xt = 0$, we have $tx^* = 0$. Therefore, for all $i \in I$, we have $(ix + x^*i^*)\, U_t = 0$. However, t is regular in $T \cap S$. Thus, $ix + x^*i^* = 0$ for all $i \in I$. That is, $I\mathcal{O}\!\ell_L(t) \subset K$ and so $ixix = 0$. That is, Rix is a nil left ideal of bounded index. Hence, $ix = 0$ and since I is self-adjoint, we conclude that $\mathcal{O}\!\ell_L(t) = 0$, the desired consequence.

Now if a is regular in R and t is regular in $T \cap S$, then, by Lemma 21, ata^* is regular in R. Hence we are prepared to state the following corollary to Lemma 21.

COROLLARY 22. *Let R be $*$-prime, 2 torsion free, and let T be a subset with property $(H)^*$. Suppose further that T satisfies either* acc *or* dcc *on quadratic ideals and that $T \cap S$ has a regular element. If W is the ring of quotients of R and $v \in S(W)$, the symmetric elements of W, then $v = U_{x^{-1}}b$ where $x \in T \cap S$, $b \in S$.*

Furthermore, if I has a regular element then both x and b can be chosen in $I \cap S$; every element of W is $c^{-1}d$, c, $d \in I$.

Let $a^{-1}b \in W$, then a is regular in R. Thus a^*ta is regular if $t \in T \cap S$ is regular. Furthermore,

$$a^{-1}b = (a^*ta)^{-1}(a^*tb).$$

Now assume $a^{-1}b \in S(W)$ then

$$(a^*ta)^{-1}(a^*tb) = (b^*ta)(a^*ta)^{-1}$$

or $a^*tba^*ta = a^*tab^*ta$. Letting $x = a^*ta$ and $y = a^*tb$, then $yx \in S$ and $U_x^{-1}(yx) = x^{-1}(yx)\,x^{-1}$. Hence, $x^{-1}y = U_x^{-1}b$ where $x \in T \cap S$ and $b \in S$.

The remarks on I are self-evident.

VI. SPECIAL LIE SUBRINGS OF R

In this section we investigate an additive subgroup, T, of a semi-prime 2-torsion free (associative) ring R with involution which have the properties that

(i) T is closed under the Lie multiplication; that is, $t, u \in T$ implies $[t, u] = tu - ut \in T$ (T is said to be a Lie subring) and

(ii) $tU_x^* = xtx^* \in T$ for all $x \in R$, $t \in T$.

For such a Lie subring we are able to prove

THEOREM 23. *Let R be a 2-torsion-free, semi-prime ring with involution. Let T be a self-adjoint Lie subring and suppose further that $tU_x^* \in T$ for all $x \in R$, $t \in T$. Then either T contains a non-zero self-adjoint ideal I of R or $t + t^* \in Z$, the center of R, for all $t \in T$.*

As in [1], one observes that since the elements

$$\{x, t, uy\}_* \qquad \text{and} \qquad \{xt, u, y\}_*$$

are in T for all $x, y \in R$, $t, u \in T$, then $R(u^*t - ut^*)R \subset T$. Letting $u^*t - ut^* = w$, we also have

$$Rw^*R \subset T$$

and hence, $R(w \pm w^*)R \subset T$. That is, T contains a non-zero self-adjoint ideal I unless

$$w + w^* = w - w^* = 0.$$

That is, if T does not contain a non-zero self-adjoint ideal then

$$u^*t = ut^* \qquad \text{for all} \quad u, t \in T.$$

Now, since T is a Lie subring, we have

$$\{x, [t, u], y\}_* \in T$$

and

$$\{x, t, uy\}_* - \{xu, t, y\}_* \in T$$

for all $x, y \in R$, $u, t \in T$.

That is, on subtracting we have

$$R[t, (u + u^*)]R \subset T.$$

Likewise

$$R[t^*, (u + u^*)]R \subset T$$

and, as before, T contains a non-zero self-adjoint ideal unless

$$[t, u + u^*] = 0 \qquad \text{for all} \quad t, u \in T.$$

Hence, we have shown that if T does not contain a nonzero self-adjoint ideal I then

$$\text{(I)} \qquad u^*t = ut^* \qquad \text{and} \qquad \text{(II)} \quad [t, u + u^*] = 0$$

for all $t, u \in T$. We wish to show that these conditions imply $t + t^* \in Z$ for all $t \in T$.

Since a semi-prime 2-torsion free ring with involution is the subdirect sum of $*$-prime, 2-torsion free rings, and since the properties on T will carry to the homomorphic image, *we henceforth assume that R is $*$-prime, 2-torsion free and that* (I) *and* (II) *hold for all elements of T.*

Rewriting the expression (II), we have

$$\text{(III)} \qquad [t, u] = [u^*, t] = ut^* - tu^* \in K$$

for all $t, u \in T$.

In particular, $t^*[u, v] = t[u, v]^* = t[v^*, u^*]$. (III) implies that $t[v^*, u^*] = t[u, v^*] = t[v, u]$. Therefore, $t^*[u, v] = -t[u, v]$ or

$$(t + t^*)(uv - vu) = 0$$

for all $t, u, v \in T$. Since T is self-adjoint $(t + t^*)\, U_x^* \in T$ for all $x \in R$, $t \in T$ and thus

$$\{x, (t + t^*), x^*y\}_*(uv - vu) = 0$$

for all $x, y \in R$, $t, u, v \in T$. That is,

$$(t + t^*)\, U_x^* R(uv - vu) = 0.$$

Thus, either $t + t^* = 0$ (this follows from $(t + t^*)U_x^* = 0$ and the fact that R is semi-prime) and hence $t + t^* \in Z$ for all $t \in T$ or

$$\text{(IV)} \qquad [u, v] = uv - vu = 0$$

for all $u, v \in T$.

We therefore assume the latter holds. Then from (I) we conclude

$$\text{(V)} \qquad u^*t = t^*u \qquad \text{for all } u, t \in T.$$

Defining $M = \{\sum xtx^* \mid x \in R, t \in T\} \subset T$, we conclude that $ma + a^*m \in M$ for all $m \in M$, $a \in R$. By (IV) we have

$$[m, m \circ s] = 0 \qquad \text{for all} \quad m \in M, s \in S.$$

That is, $[m^2, s] = 0$ for all $m \in M, s \in S$. Thus, $[m^2 + (m^2)^*, s] = 0$ for all $m \in M$, $s \in S$. In a 2-torsion free semiprime ring we conclude that

$$m^2 + (m^*)^2 \in Z.$$

Linearizing, we conclude that $(m \circ n) + (m \circ n)^* \in Z$ for all $m, n \in M$. However, $[m, n] = [m, n]^* = 0$. Therefore, for all $m, n \in M$

$$mn + n^*m^* \in Z. \tag{VI}$$

Noting that M is self-adjoint we have. in particular,

$$mm^* \in Z \qquad \text{for all} \quad m \in M.$$

Linearizing, we conclude

$$mn^* + nm^* \in Z \qquad \text{for all } m, n \in M.$$

From (V), we have

$$mn^* \in Z \qquad \text{for all } m, n \in M.$$

Therefore, $mn \in Z$ for all $m, n \in M$, since M is self-adjoint. In particular, making frequent use of $m^2 \in Z$ for $m \in M$, we have

$$m(n + n^*)\, n(a + a^*) = m[n(na + na^*) + (na + na^*)^*n] \in Z$$

for all $m, n \in M$, $a \in R$.

Furthermore, $[mn + nm^*]n \in Z$, which by the use of (V) yields

$$m(n + n^*)n \in Z \qquad \text{for all} \quad m, n \in M.$$

Thus, $m(n + n^*)\, n((a + a^*)b - b(a + a^*)) = 0$ for all $m, n \in M$, $a, b \in R$. Therefore,

$$m(n + n^*)\, nR[a + a^*, b] = 0.$$

Since $m(n + n^*)n \in S$ and R is $*$-prime, we have either $a + a^* \in Z$ for all $a \in R$ (in particular, $m + m^* \in Z$) or

$$m(n + n^*)n = 0 \qquad \text{for all} \quad m, n \in M.$$

Since $m(n + n^*) \in Z$, we have

$$m(n + n^*)\, Rn^* = 0.$$

Since $m(n + n^*) \in S$, we conclude that either $M = 0$ (and hence $m + m^* \in Z$) or

$$m(n + n^*) = 0$$

for all $m, n \in M$. Therefore,

$$mU_x^*(n + n^*) = 0$$

for all $x \in R$. Thus,

$$\{x, m, x^*y\}_*(n + n^*) = 0.$$

That is, $xmx^* = 0$ (which implies $M = 0$ and so $m + m^* \in Z$) or $n + n^* = 0 \in Z$.

Thus, in all cases

$$m + m^* \in Z \qquad \text{for all} \quad m \in M.$$

That is,

$$(t + t^*)\, U_x^* \in Z$$

for all $t \in T$, $x \in R$. Therefore,

$$\{x, t + t^*, x^*y\}_* \in Z$$

for all $x, y \in R$, $t \in T$.

That is,

$$[(t + t^*)\, U_x^*](y + y^*) \in Z.$$

As before, we are able to conclude that either $t + t^* = 0$ for all $t \in T$ or $y + y^* \in Z$ for $y \in R$. In either case, $t + t^* \in Z$ for all $t \in T$. This completes the proof of the theorem.

We are now in a position to prove the analog of Theorem 2 of Herstein [7].

COROLLARY 24. *Let R be a $*$-prime, 2-torsion free ring and suppose that T is a non-zero self-adjoint Lie subring of R with the properties*:

(i) *$xtx^* \in T$ for all $t \in T$, $x \in R$ and*

(ii) *T does not contain a non-zero self-adjoint ideal of R. Then $T \subset K$ or $T \subset Z \cap S$ and R satisfies a standard identity in 4 variables.*

Since $t + t^* \in Z$ for all $t \in T$ and $ut^* = u^*t$ for all $u, t \in T$, we conclude that $(u - u^*)\, R(t + t^*) = 0$.
Hence either $t + t^* = 0$ for all $t \in T$ (that is, $T \subset K$), or $u - u^* = 0$ for all $u \in T$ (that is, $T \subset S \cap Z$). Now since $xtx^* \in S \cap Z$ for all $t \in T$, we conclude that $xx^* \in Z$ for all $x \in R$. Hence, as Herstein remarks, R satisfies a standard identity in 4-variables.

COROLLARY 25. *Let R be $*$-simple, not satisfy a standard identity in 4 variables, and T as above, then either $T = R$ or $T = K$.*

$T = R$ if T contains a non-zero self-adjoint ideal. Hence, assume T does not contain a self-adjoint ideal.

Next consider the case where $T \subset K$. Now, recall that $2R = R$. Hence, if we define $M = \{\sum xtx^* \mid x \in R, t \in T\}$ and $V = \{\sum ma - a^*m \mid m \in M, a \in R\}$

then $0 \neq M \subset K$, $V \subset S$ and $M \oplus V$ is an ideal of R. [This remark is justified below.] Moreover, $M \subset T$. Thus, $M = K \subset T \subset K$. Therefore, $K = T$. Now, if $T \subset S \cap Z$; then we have seen that R satisfies a standard identity in 4-variables which is excluded.

Thus, the lemma is proved if we show $M \oplus V$ is an ideal. It is sufficient, because of a symmetric argument, to show $2ma$ and $2va \in M \oplus V$ for all $m \in M$, $v \in V$ and $a \in R$. Now, $2ma = (ma + a^*m) + (ma - a^*m) \in M \oplus V$. Likewise, $2va = 2(mb - b^*m)a$ where $v = mb - b^*m$.

Thus, $2va = 2mba - [b^*ma + a^*mb] - [b^*ma - a^*mb]$. The first two terms on the right are in $M \oplus V$. Thus, we are done if we conclude the latter term is in V. However, this is true since

$$b^*ma - a^*mb = b^*(ma + a^*m) - (ma + a^*m)b + m(ab) - (ab)^*m.$$

VII. Theorems of Positive Definiteness Type

In this section we show that the so-called "positive-definiteness theorems" (see [8], pg. 73-80) which appear in the literature have their analog in sets with property $(H)^*$. We do not prove all the theorems or even state them but rather select three for proof. We emphasize that the other theorems have their corresponding statement and proof.

The first of these theorems is essentially Theorem 2.1.7 ([8], pg. 62). The second of these theorems is the analog of Theorem 2.2.1 ([8], p. 73) and is due to I. N. Herstein.

THEOREM 26. *Let R be a 2-torsion-free semi-prime ring with involution. Let T be a subset with property $(H)^*$ in which every element of $T \cap S$ is invertible in R. Then $T \cap S = S$ and R is*

(1) *a division ring, or*

(2) *the direct sum of a division ring and its opposite, relative to the exchange involution $(x, y)^* = (y, x)$ or*

(3) *the 2×2 matrices over a field, relative to the symplectic involution; namely, $\begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}^* = \begin{pmatrix} \sigma & -\beta \\ -\gamma & \alpha \end{pmatrix}$.*

Now, since T is a subset with property $(H)^*$, there exists a non-zero self-adjoint ideal, I, such that $a \neq I \cap S \subset T \cap S$. Now the ideal I can contain no invertible elements if it is proper, thus we must conclude that $I \cap S = S = T \cap S$, and hence $T \cap S$, and every element of S is invertible in R. Thus, the conditions of Theorem 2.1.7 of [8] are satisfied and thus the desired conclusion.

THEOREM 27. *Let R be a 2-torsion free prime ring with involution. Let T be a subset with property $(H)^*$ and with no non-zero nilpotent elements in $T \cap S$. Then either*

(i) *$xx^* \neq 0$ in R if $x \neq 0$ in R or*

(ii) *$S \subset Z(R)$ and R is an order in*

$$F_2 = R_{T \cap S} = \left\{ \frac{r}{z} \,\middle|\, r \in R \text{ and } z \neq 0 \in T \cap S \right\}.$$

Suppose (i) does not hold, then $xx^* = 0$ for some $x \neq 0$. Thus, $x^*(T \cap S)x \subset T \cap S$ and moreover $(x^*tx)^2 = 0$ for all $t \in T \cap S$. Thus, by hypothesis, $x^*tx = 0$ for all $t \in T \cap S$. In particular, $x^*(i + i^*)x = 0$ for all $i \in I$ (I is the $*$-ideal such that $0 \neq I \cap S \subset T \cap S$). Thus, $x^*ix \in K$ for all $i \in I$. In particular, $x^*ix = 0$ for all $i \in I \cap S$. Now if $x^*ix = 0$ for all $i \in I$, then it follows since R is prime, that $x = 0$, a contradiction. Therefore, there exists an $i \in I$ such that $x^*ix = k \neq 0 \in I$ and $k^2 = 0$. Moreover, $ksk \in I \cap S \subset T \cap S$ and $(ksk)^2 = 0$. By hypothesis, $ksk = 0$. Now if $s \in S$ then $ks - sk \in I \cap S$ and $(ks - sk)^2 = 0$. Hence, $[k, s] = 0$ for all $s \in S$. Now by Theorem 2.1.5, Herstein [8] either $S \subset Z$ or S contains a non-zero ideal, G, of R. Now if the second case, $k \in Z(R)$ and $k^2 = 0$, a contradiction in a prime ring, therefore $S \subset Z$.

Moreover, since R is prime, Z has no zero divisors. Thus we can localize R at $T \cap S \subset S \subset Z$ and obtain the $R_{T \cap S} = \{r/z \mid r \in R, z \neq 0 \in T \cap S\}$. $R_{T \cap S}$ is a prime ring with involution. Its non-zero symmetric elements are all invertible Moreover, $k/z \in R_{T \cap S}$ is a non-zero nilpotent element. Hence, by Theorem 26, $R_{T \cap S} = F_2$. But every element of $R_{T \cap S}$ is of the form rz^{-1}, $r \in R$, $z \neq 0 \in T \cap S$.

Therefore, if r is regular in R, it is regular in $R_{T \cap S}$. Thus, we conclude that R is an order in F_2, completing the proof.

We also prove the analogous theorem to Theorem 2.2.4, Herstein [8], (p. 79) by invoking Corollary 24.

THEOREM 28. *Let R be a 2-torsion free prime ring with involution and suppose T is a non-zero self-adjoint Lie subring of R with the properties*

(i) *xtx^* T for all $t \in X$, $x \in R$*

(ii) *no non-zero element, $t - t^* \in T$, for $t \in T$, is nilpotent.*

Then,

(i) *$xx^* \neq 0 \in R$ if $x \neq 0$ or*

(ii) *R is an order in F_2, R a field.*

Corollary 24 tells us that either T contains a non-zero self-adjoint ideal I or $T \subset K$ or $T \subset Z$.

Now, if $T \subseteq I$ and $I \cap K = 0$, then it follows that $I \subset S$ and hence is commutative. From which it follows that R is a prime, commutative ring, hence a commutative integral domain and thus i) holds for all $x \in R$.

Thus we need to examine the Theorem under the assumption R is not commutative.

If $T \subset Z$ then $xtx^* \in Z$ for all $t \in T$, $x \in R$. Thus, $txx^* \in Z$ and so $t[xx^*, a] = 0$ for all $a \in R$. Now, this implies $tR[xx^*, a] = 0$. That is $xx^* \in Z$ for all $x \in R$. Hence, R satisfies a standard identity of 4 variables. Therefore, by a theorem of Posner ([8], Theorem 1.3.4) either (i) or (ii) holds.

Therefore, $T \subset K$ and either T contains a non-zero ideal I, $0 \neq I \cap K \subset T$ or $T \subset K$. This latter possibility also forces the existence of an ideal, I, such that $I \cap K \neq 0 \subset T$. Indeed, if $T \cap K \neq 0$ then $I = R(2m)R$, $m \neq 0 \in T \cap K$ is the desired ideal. For if, $xmy + y^*mx^* = 0$ for all $x, y \in R$ then $(x^*m)^3 = 0$ or $m = 0$, a contradiction. Hence, without loss, $0 \neq I \cap K \subset T$ for a self-adjoint ideal, I.

Now let $xx^* = 0$, $x \neq 0 \in R$. Thus, $x^*(t - t^*)x = x^*tx - (x^*tx)^* = l - l^*$, $l \in T$. Moreover, $[x^*(t - t^*)x]^2 = 0$. Thus, by hypothesis, $x^*(t - t^*)x = 0$ for all $t \in T$ or $x^*tx = 0$ for all $t \in T$. Now if $a \in I$ then $a - a^* \in I \cap K$. That is, $x^*(a - a^*)x = 0$. Hence, $x^*ax \in S \cap I$ for all $a \in I$. Moreover, since R is prime, $s = x^*a_0x \neq 0$ for some $a_0 \in I$, while $s^2 = 0$. Now, $s(t - t^*)s = 0$ for all $t \in T$ as before. Thus, $sts = 0$ for all $t \in T \cap K$. Therefore, $sis = 0$ for all $i \in I \cap K$. Since, $si + is \in I \cap K$ for all $i \in I \cap K$, we have $(si + is)^2 = si^2s$ and $(si + is)^4 = 0$ for all $i \in I \cap K$. Therefore, by hypothesis, $si + is = 0$. That is, $si = -is$ for all $i \in I \cap K$.

Thus, s centralizes $(I \cap K)^2$. Now the latter is a Lie ideal of R. Therefore, the subring, L, generated by $(I \cap K)^2$ either contains a non-zero ideal of R or $(I \cap K)^2 \subset Z$ (see [8], Theorem 2.1.2). Since $s \in S$ is nilpotent, it is non-central. Hence, s cannot centralize a non-zero ideal of R. In consequence, $(I \cap K^2) \subset Z$, that is, $w \in (I \cap K)^2$ implies $w^2 \in Z$. In particular, if $u \in I \cap K$ then $u^4 \neq 0 \in Z$. Thus $z \neq 0$. Hence, $Z^+ = Z \cap S \neq 0$ and we can localize R at Z^+, calling it Q. Moreover, if $k \neq 0 \in I \cap K$ then $k^2 \neq 0 \in Z_1 = Z(Q)$. Thus, k is invertible in Q.

We claim Q is simple. For if $V \neq 0$ is an ideal of Q, then $W = VV^* \neq 0 \subset V$ and is self-adjoint in Q. If $W \cap (T \cap K) = 0$ then $wi - i^*w^* = 0$ for all $i \in I$, $w \in W$. That is, $wi \in S$. Hence, $qwi \in S$ for all $q \in Q$. That is, $qwi = (qwi)^* = wiq^*$. Hence, for $p, q \in Q$, $pqwi = wiq^*p^* = qwip^* = qpwi$. That is, $[p, q]wi = 0$.

Thus, $Cwi = 0$, where C is the ideal generated by $[Q, Q]$. Since we are in a prime ring, $C = 0$; that is, Q is commutative and hence R, a contradiction. Thus, $wi = 0$; that is, $W = 0$ which is also false. Hence, $W \cap (T \cap K) \neq 0$. But, every element of $T \cap K$ is invertible. Therefore, $1 \in W$ and so $V = Q$; that is Q is simple.

Now $IZ^+ = \{iz^{-1}\ i \in I, z \in Z^+\}$ is an ideal of W. Hence, $Q = IZ^+$. Therefore, $K(Q)$ (the set of skew elements of Q) $= K(IZ^+) = (K \cap I)Z^+ = (T \cap K)Z^+$. Furthermore, if $u \neq 0$, $v \neq 0 \in T \cap K$ then $uv \neq 0 \in Z_1$, and since Z_1 is a field,

$$u = \alpha v, \qquad \alpha \in Z_1$$

follows. But then the skews are 1 dimension over the center of $Z(Q)$ and hence they do not generate Q. This is sufficient to conclude that Q is 4 dimension over a field and that R is an order in F_2.

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